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A Study of Geostrophic Waves around the Equator

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A Study of Geostrophic Waves around the Equator

M.Sc.-Thesis

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Abstract

Atmospheric waves play an important role in the dynamics of the atmosphere. In this report we investigate atmospheric waves in a special region: the equator. The fact that the Coriolis force vanishes at the equator leads to the existence of an unique group of fast and slow waves. We are interested in the slow westward moving waves and we present here a theoretical study to describe these slow waves by *not* assuming a standard formulation of geostrophic balance between Coriolis force and pressure gradient force. Instead we use a more general approach, assuming geostrophic motions are nondivergent.

To describe these waves we introduce a simplified set of linear equations in Chapter 2. We then demonstrate in Chapter 3, after assuming an isentropic background state, that these equations can be reduced to a set of equations that is analogous to the shallow-water model. This model is used in Chapter 4 to describe the various different equatorial waves. Here we focus on deriving an approximate solution for slow westward waves, assuming zero divergence of the velocity field.

The results prove that especially the slow equatorial Rossby (planetary) waves can be described very accurately with this method. The maximum error in the dispersion relation is not larger than a few percent. However, for the westward branch of the mixed Rossby-gravity wave this method of zero divergence is only valid to some degree. The approximated solution only holds if the wave numbers are large, i.e. if the mixed wave simply behaves like a slow planetary wave.

1 Introduction

The atmosphere and oceans of our planet are in constant motion, which is driven by energy received from the sun. This energy is not distributed equally (Hartmann, 1994). Excess of energy at the equatorial region produces temperature gradients in the meridional (north-south) direction. The motions that take place in the atmosphere and oceans tend to reduce these gradients. Among the different atmospheric motions, waves occupy a special place. An atmospheric wave is a periodic disturbance resulting from the action of a restoring force. This restoring force may be due to compressibility, electromagnetic effects, gravity or rotation (Holton, 1992). Especially the earth's rotation has a predominant effect on the large scale atmospheric dynamics. First of all, it tends to establish a quasi-geostrophic balance, i.e. an approximate balance between the Coriolis force and pressure-gradient force. Secondly, it creates a class of waves that owe their existence to the variation of the vertical component of the rotation rate f. These waves are called *Rossby waves*, or planetary waves. At mid-latitudes, the Rossby waves are often approximated by using a mid-latitude beta-plane, a simplified system where the Coriolis parameter deviates *linearly* around a constant value ($f = f_0 + \beta y$), and is characterized by a near balance between Coriolis force and pressure-gradient force:

$$f_0 \mathbf{k} \times \mathbf{v} \approx \frac{1}{\rho} \nabla_z p.$$

However, to describe similar waves near the equator, this formulation of the quasi-geostrophic theory is less appropriate because, despite the fact that β is largest there, the Coriolis parameter vanishes. The objective of this study is to describe the slow equatorial waves using a more general approach to utilize the quasi-geostrophic theory. *Not* a near balance of the Coriolis force with the pressure-gradient force, but the assumption that slow waves are approximately nondivergent. The assumption of zero divergence is valid globally and this general approach is previously discussed by Verkley (2005).

1.1 Equatorial Region

The dynamics of the equatorial region are rather different from the higher latitudes. To approximate the equatorial waves we will construct a simple linearized barotropic model on the equatorial beta-plane. This system will illustrate that the equator acts like a waveguide, i.e. a spectrum of equatorial waves propagating along the equator with amplitude falling sharply away in the tropical region. These waves have simple or complex structures depending on the meridional mode number. In addition, the equatorial waves can either be fast moving (high-frequency) or slow moving (low-frequency). The eastward moving *equatorial Kelvin* *wave* is of the simplest kind. It is somewhat special because motions are everywhere parallel to the equator (meridional wind speed v is zero) and geostrophically unbalanced (Bouchut et al., 2005). The high-frequency *equatorial gravity waves* (also known as Poincaré waves) and the fast moving branch of the *mixed Rossby-gravity wave* (also known as Yanai wave) are likewise geostrophically unbalanced and thus not attractive for this study. Therefore, we investigate specifically the westward slow moving waves, i.e. the *equatorial Rossby waves* and the slow branch of the mixed Rossby-gravity wave. These slow waves tend to establish a balance between the velocity field and the pressure-gradient. Hopefully, we are able to approximate these waves using the notion of nondivergence.

1.2 This Study

We want to know if the slow equatorial waves can be described with the general formulation of nondivergence. To investigate this problem we introduce a simplified system that can be solved around the equator analytically. The thesis will follow systematically the steps to create such a barotropic system. In Chapter 2 we start with introducing the basic governing equations for momentum, mass, and energy that portray the state and evolution of the atmosphere. By linearizing these equations a perturbation part is obtained around a background state of no motion. The last part of this chapter illustrates, using sound waves as an example, how the linearized system can be transformed into a wave equation for a single variable. If solutions are represented by a simple sinusoidal wave, a dispersion relation between frequency and wave number can be obtained. This technique is used to derive the more complex equatorial waves in Chapter 4.

In Chapter 3, the linearized system is simplified even more by applying a background state of uniform potential temperature (isentropic). After some manipulations the system takes a barotropic form that appears to be analogous with the shallow-water equations. Subsequently in Chapter 4, this linearized barotropic model is used to describe equatorial waves. In the first part a general solution is deduced that describes both the fast and the slow equatorial waves. The second part will focus on our main objective; deriving a solution to approximate only the slow equatorial waves using nondivergence. In addition, we will investigate the new solution in more detail by comparing it with the general solution. Chapter 5 answers the research question and draws conclusions from the obtained results.

2 Governing Equations

2.1 Introduction

This chapter introduces the basic physical laws that describe the dynamics of the atmosphere. The three basic laws that govern atmospheric motions are: conservation of momentum, conservation of mass, and conservation of energy. First, section 2.2 introduces these important equations in our basic Cartesian coordinate system, with height z as the vertical coordinate. In section 2.3, the primitive equations are linearized to obtain a form that can be handled analytically. Each field variable is divided into two parts; a background state and a perturbation part. The perturbation equations with a background state of no motion will be the main tool for describing atmospheric waves. Section 2.4 introduces some of the wave characteristics, like for instance the wave equation and the dispersion relation.

2.2 Conservation Laws

The basic conservation laws that are important for our study are introduced here in the general form (i.e. without applying linearization). These laws are: conservation of momentum, conservation of mass, and conservation of energy. To make the equations tractable, several basic assumptions are discussed to simplify the set of equations.

2.2.1 Conservation of Momentum

The momentum equation (based on Newton's second law) is used to describe the evolving motions in atmosphere. Its general form is derived in well-known textbooks like Holton (1992). Newton's second law of motion, relative to a reference frame rotating with the earth, can be written as

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} + \rho 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p + \rho\mathbf{g}}, \tag{2.2.1}$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the three-dimensional velocity vector, p is the air pressure, ρ is the air density and $\mathbf{\Omega}$ is the angular velocity of the earth. The gravity force $\mathbf{g} = -g\mathbf{k}$ combines the centrifugal force with the gravitational force. Any other curvature forcing terms and friction terms are ignored. Scale analysis by Holton (1992) and other authors show that these terms are not important for our purpose. The term $2\mathbf{\Omega} \times \mathbf{u}$ arises due to the rotation of the reference frame. It can be simplified by $f\mathbf{k} \times \mathbf{v}$, where $f = 2\Omega \sin\phi$ is the Coriolis parameter at latitude ϕ and $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ is the horizontal velocity vector. The term is called the Coriolis force and must be regarded as a pseudo-force, constructed to make it appear that Newton's second law is holding despite the rotation of the reference frame (James, 1995).

Taking these assumptions into account, for each component the primitive momentum equation is approximated as

$$\rho\frac{Du}{Dt} - \rho f v = -\frac{\partial p}{\partial x}, \tag{2.2.2}$$

$$\rho\frac{Dv}{Dt} + \rho f u = -\frac{\partial p}{\partial y}, \tag{2.2.3}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g, \tag{2.2.4}$$

where u is the zonal wind speed, v is the meridional wind speed and w is the vertical wind speed. The total derivative D/Dt is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2.2.5}$$

2.2.2 Conservation of Mass

The second important law is the conservation of mass. It states that mass cannot be created or destroyed. Consider the Lagrangian derivative, where a variable control volume $\delta V = \delta x \delta y \delta z$ with fixed mass ($\delta M = \rho \delta V$) is followed during the motion:

$$\frac{1}{\delta M}\frac{D\delta M}{Dt} = \frac{1}{\rho\delta V}\frac{D\rho\delta V}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta V}\frac{D\delta V}{Dt} = 0. \tag{2.2.6}$$

The term containing the volume element can be written as

$$\frac{1}{\delta V}\frac{D\delta V}{Dt} = \frac{1}{(\delta x\delta y\delta z)}\frac{D(\delta x\delta y\delta z)}{Dt} = \frac{1}{\delta x}\frac{D\delta x}{Dt} + \frac{1}{\delta y}\frac{D\delta y}{Dt} + \frac{1}{\delta z}\frac{D\delta z}{Dt}.$$

If one takes the limit $\delta x\delta y\delta z \to 0$, this term becomes

$$\lim_{\delta x\delta y\delta z\to 0}\left[\frac{1}{\delta V}\frac{D\delta V}{Dt}\right] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla\cdot\mathbf{u},$$

so that (2.2.6) leads to the continuity equation

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0}. \tag{2.2.7}$$

It means that the three-dimensional divergence field is in balance with the evolution of density.

2.2.3 Conservation of Energy

The first law of thermodynamics expresses the principle of energy conservation. It states that energy cannot be created or destroyed. This is important because atmospheric motions are driven by the transformation process of heat energy (from the sun) into various forms of mechanical energy. If dry air is considered, pressure p, temperature T, and density ρ are related by the *ideal gas law* (James, 1995):

$$p = \rho RT. \tag{2.2.8}$$

In this equation of state, R is the gas constant for dry air (287 $\text{JK}^{-1}\text{kg}^{-1}$). James (1995) argues that the law is perfectly adequate over the range of temperatures and pressure encountered in earth's atmosphere. Temperature can be interpreted as the measure for internal energy U i.e. the energy related with random motions of molecules. Change of internal energy implies change of temperature, and its relationship can be written as,

$$\delta U = c_v \delta T, \tag{2.2.9}$$

where c_v is the specific heat of dry air at constant volume (717 $\text{JK}^{-1}\text{kg}^{-1}$).

If a particular volume of air is heated with heat energy Q, it will increase its internal energy, and/or it may be transformed to mechanical energy W. A small amount of heat must be balanced by a small change of internal energy and mechanical energy, i.e.

$$\delta Q = \delta U + \delta W. \tag{2.2.10}$$

Mechanical energy is related with the work that has been done to change the volume of air, i.e.

$$\delta W = p\delta\alpha, \tag{2.2.11}$$

where $\alpha = \rho^{-1}$. Using equation of state (2.2.8), the first law of thermodynamics takes the form,

$$\delta Q = c_p \delta T - \alpha \delta p, \tag{2.2.12}$$

where $c_p = c_v + R$ is the specific heat of dry air at constant pressure (1004 $\text{JK}^{-1}\text{kg}^{-1}$). Taking the time-derivative of (2.2.12), substituting the equation of state in the second term on the right-hand side, and dividing the whole expression by T, we obtain the entropy form of the thermodynamic energy equation, i.e.

$$\frac{c_p}{T}\frac{DT}{Dt} - \frac{R}{p}\frac{Dp}{Dt} = \frac{Q}{T} \equiv \frac{Ds}{Dt}, \tag{2.2.13}$$

where s is the specific entropy. By defining the potential temperature θ by

$$\theta = T\left(\frac{p_r}{p}\right)^{R/c_p}, \tag{2.2.14}$$

we may write

$$\frac{c_p}{T}\frac{DT}{Dt} - \frac{R}{p}\frac{Dp}{Dt} = \frac{c_p}{\theta}\frac{D\theta}{Dt}.$$

We thus obtain the entropy form as

$$\frac{c_p}{\theta}\frac{D\theta}{Dt} = \frac{c_p}{T}\frac{DT}{Dt} - \frac{R}{p}\frac{Dp}{Dt} = \frac{Q}{T} \equiv \frac{Ds}{Dt}. \tag{2.2.15}$$

One can notice that change in potential temperature is proportional with change of entropy. Therefore, an air parcel that conserves entropy must follow an isentropic surface (constant θ-surface) (Holton, 1992).

To eliminate the disturbance part of the density from our set of linearized equations, discussed in the next section, we work with an alternative expression for the thermodynamic energy equation. It can be derived by assuming no heat exchange ($Q = 0$, adiabatic process), and temperature T in (2.2.14) being characterized by $p/\rho R$. Its full derivation is discussed in Appendix A and it yields the following form,

$$\boxed{\frac{c_s^2\rho}{\theta}\frac{D\theta}{Dt} = \left[\frac{Dp}{Dt} - c_s^2\frac{D\rho}{Dt}\right] = 0}, \tag{2.2.16}$$

where the speed of sound is given by $c_s^2 = \frac{c_p}{c_v}RT$. It is not regarded as infinitely large. Hence, the atmosphere is considered to be compressible and it implies that the velocity field is divergent.

2.3 Linearized Conservation Laws

To study atmospheric waves analytically we have to simplify our set conservation laws (2.2.1), (2.2.7), and (2.2.16). These primitive equations are linearized in this section, using the method of small perturbations. This method is based on the principle that field variables can be divided into two parts: a background state and a disturbance part, where the disturbance part is small compared to the background state. An important assumption of the perturbation method is that the background state should satisfy the primitive equations when there is no disturbance. Yet, the disturbance part must be considered small enough, so that products of these perturbations are negligible (Holton, 1992). After linearizing the primitive equations, five prognostic perturbation equations are obtained that approximate our system. In the second part of this section, the set of equations is reduced to four after eliminating the density disturbance.

2.3.1 Linearization of the Conservation Equations

The primitive conservation equations, introduced in section 2.2, consist of seven different field variables: u, v, w, p, ρ, T, and θ. For now, assume that the velocity background state is zero. The pressure, density and temperature background states only depend on height and are marked by a subscript '$_0$'. The perturbation part is usually symbolized with a single prime. Utilizing the perturbation method, one can separate the seven variables as follows

$$
\begin{aligned}
u(x,y,z,t) &= u'(x,y,z,t),\\
v(x,y,z,t) &= v'(x,y,z,t),\\
w(x,y,z,t) &= w'(x,y,z,t),\\
p(x,y,z,t) &= p_0(z)+p'(x,y,z,t),\\
\rho(x,y,z,t) &= \rho_0(z)+\rho'(x,y,z,t),\\
T(x,y,z,t) &= T_0(z)+T'(x,y,z,t),\\
\theta(x,y,z,t) &= \theta_0(z)+\theta'(x,y,z,t).
\end{aligned}
$$

The momentum equation (2.2.1) can be linearized by substituting the variables given above. The vector $\mathbf{u}$ represents the scalars u, v, and w. Thus,

$$
(\rho_0+\rho')\left[\frac{\partial \mathbf{u}'}{\partial t}+\mathbf{u}'\cdot\nabla\mathbf{u}'\right]+2\,(\rho_0+\rho')\,\boldsymbol{\Omega}\times\mathbf{u}'=-\nabla\,(p_0+p')+(\rho_0+\rho')\,\mathbf{g}.
$$

Neglecting products of primed quantities leaves us with the following perturbation expression

$$
\rho_0\frac{\partial \mathbf{u}'}{\partial t}+2\rho_0\boldsymbol{\Omega}\times\mathbf{u}'=-\nabla p'+\rho'\mathbf{g}. \tag{2.3.1}
$$

Here it has been assumed that the background state is in hydrostatic equilibrium, i.e.

$$
-\nabla p_0+\rho_0\mathbf{g}=0 \text{ or } -\frac{\partial p_0}{\partial z}+\rho_0 g=0.
$$

The same procedure is also carried out for the continuity and the thermodynamic energy equation. Linearization of the continuity equation (2.2.7) gives

$$
\left[\frac{\partial}{\partial t}(\rho_0+\rho')+\mathbf{u}'\cdot\nabla\,(\rho_0+\rho')\right]+(\rho_0+\rho')\,\nabla\cdot\mathbf{u}'=0.
$$

Again, if one assumes no products of primed quantities, the perturbation part gives

$$
\frac{\partial \rho'}{\partial t}+w'\frac{\partial \rho_0}{\partial z}+\rho_0\nabla\cdot\mathbf{u}'=0. \tag{2.3.2}
$$

And finally, linearization of the thermodynamic energy equation (2.2.16) gives

$$
\frac{c_{0s}^2\rho_0}{\theta_0}\left[\frac{\partial \theta'}{\partial t}+w'\frac{\partial \theta_0}{\partial z}\right]=\left[\frac{\partial p'}{\partial t}+w'\frac{\partial p_0}{\partial z}\right]-c_s^2\left[\frac{\partial \rho'}{\partial t}+w'\frac{\partial \rho_0}{\partial z}\right]=0. \tag{2.3.3}
$$

where $c_{0s}^2=\frac{c_p}{c_v}RT_0$, i.e. the background speed of sound squared.

In addition, it is convenient to simplify the momentum equations (2.3.1) by replacing $2\boldsymbol{\Omega}\times\mathbf{u}'$ with $f\mathbf{k}\times\mathbf{v}'$, already introduced in section 2.2. Our simplified linearized system, including the background state of the diagnostic expressions, is then described by the following equations:

Background state hydrostatic equilibrium

$$\frac{\partial p_0}{\partial z} = -\rho_0 g, \tag{2.3.4}$$

Background state ideal-gas

$$p_0 = \rho_0 R T_0, \tag{2.3.5}$$

Background state potential temperature

$$\theta_0 = T_0 \left(\frac{p_r}{p_0}\right)^{R/c_p}, \tag{2.3.6}$$

Momentum equations

$$\rho_0 \frac{\partial u'}{\partial t} - \rho_0 f v' + \frac{\partial p'}{\partial x} = 0, \tag{2.3.7}$$

$$\rho_0 \frac{\partial v'}{\partial t} + \rho_0 f u' + \frac{\partial p'}{\partial y} = 0, \tag{2.3.8}$$

$$\rho_0 \frac{\partial w'}{\partial t} + \frac{\partial p'}{\partial z} + \rho' g = 0, \tag{2.3.9}$$

Continuity equation

$$\frac{\partial \rho'}{\partial t} + w' \frac{\partial \rho_0}{\partial z} + \rho_0 \nabla \cdot \mathbf{u}' = 0, \tag{2.3.10}$$

Thermodynamic equation

$$\frac{c_{0s}^2 \rho_0}{\theta_0} \left[\frac{\partial \theta'}{\partial t} + w' \frac{\partial \theta_0}{\partial z}\right] = \frac{\partial p'}{\partial t} + w' \frac{\partial p_0}{\partial z} - c_{0s}^2 \left[\frac{\partial \rho'}{\partial t} + w' \frac{\partial \rho_0}{\partial z}\right] = 0. \tag{2.3.11}$$

2.3.2 Elimination of Density Perturbation

By eliminating density perturbation ρ', the perturbation equations (2.3.7)-(2.3.11) are reduced from five to four. It requires some manipulation and rearrangements. First, we combine the continuity equation with the thermodynamic energy equation. Substitute (2.3.10) in (2.3.11) and take the hydrostatic background state (2.3.4) into account, i.e.

$$\frac{\partial p'}{\partial t} + w' \frac{\partial p_0}{\partial z} + c_{0s}^2 \left(\rho_0 \nabla \cdot \mathbf{u}'\right) = 0,$$

$$\boxed{\frac{\partial p'}{\partial t} - w' \rho_0 g + c_{0s}^2 \left(\rho_0 \nabla \cdot \mathbf{u}'\right) = 0}. \tag{2.3.12}$$

Next step: remove perturbation of density in the z-momentum equation (2.3.9). We carry out accordingly

$$\rho_0 \frac{\partial w'}{\partial t} + \rho' g + \frac{\partial p'}{\partial z} = 0,$$

$$\frac{\partial w'}{\partial t} + \frac{\rho'}{\rho_0} g + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} = 0,$$

$$\frac{\partial^2 w'}{\partial t^2} + \frac{g}{\rho_0}\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial z}\right) = 0.$$

Now, substitute (2.3.10):

$$\frac{\partial^2 w'}{\partial t^2} - \frac{g}{\rho_0}\left(w'\frac{\partial \rho_0}{\partial z} + \rho_0 \nabla \cdot \mathbf{u}'\right) + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial z}\right) = 0,$$

$$\frac{\partial^2 w'}{\partial t^2} - \frac{g w'}{\rho_0}\frac{\partial \rho_0}{\partial z} - g\nabla \cdot \mathbf{u}' + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial z}\right) = 0.$$

Subsequently, we use the Brunt-Väisälä frequency N_0, which is a measure of the static stability (Holton, 1992). It can be written in different forms. Holton (1992) writes it as background state

$$N_0^2 = g\frac{\partial \ln \theta_0}{\partial z} = \frac{g}{\theta_0}\frac{\partial \theta_0}{\partial z}. \tag{2.3.13}$$

With some mathematical manipulation done in Appendix B, this equation can be transformed to

$$N_0^2 = \frac{-g}{\rho_0}\frac{\partial \rho_0}{\partial z} - \frac{g^2}{c_{0s}^2}, \tag{2.3.14}$$

using the definition for potential temperature (2.2.14), the equation of state (2.2.8) and the hydrostatic approximation. Continue by substituting the new definition for N_0^2 in the z-momentum equation, i.e.

$$\frac{\partial^2 w'}{\partial t^2} + N_0^2 w' + \frac{w' g^2}{c_{0s}^2} - g\nabla \cdot \mathbf{u}' + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial z}\right) = 0. \tag{2.3.15}$$

Rewrite (2.3.12) to

$$\frac{1}{\rho_0}\frac{\partial}{\partial t}\frac{g p'}{c_{0s}^2} = \frac{w' g^2}{c_{0s}^2} - g\nabla \cdot \mathbf{u}',$$

and substitute it in (2.3.15). Hence,

$$\frac{\partial^2 w'}{\partial t^2} + N_0^2 w' + \frac{1}{\rho_0}\frac{\partial}{\partial t}\frac{g p'}{c_{0s}^2} + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial z}\right) = 0.$$

It gives the final result:

$$\boxed{\left(\frac{\partial^2}{\partial t^2} + N_0^2\right) w' + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z} + \frac{g}{c_{0s}^2}\right) p' = 0}. \tag{2.3.16}$$

To summarize; five basic perturbation equations (2.3.7)-(2.3.11) are rearranged to four basic perturbation equations (2.3.7), (2.3.8), (2.3.12), and (2.3.16). Perturbation of density is eliminated. The four equations here below are rearranged a bit so they are now in agreement with Gill (1982) [see (6.14.6), (6.14.15), (7.12.1), and (7.12.2)]:

$$\frac{\partial u'}{\partial t} - fv' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0, \tag{2.3.17}$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} = 0, \tag{2.3.18}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} + \frac{1}{\rho_0 c_{0s}^2}\left(\frac{\partial p'}{\partial t} - \rho_0 g w'\right) = 0, \tag{2.3.19}$$

$$\frac{\partial^2 w'}{\partial t^2} + N_0^2 w' + \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial z} + \frac{g}{c_{0s}^2}p'\right) = 0, \tag{2.3.20}$$

where $N_0^2 = g\frac{\partial \ln \theta_0}{\partial z}$, and $c_{0s}^2 = \frac{c_p}{c_v}RT_0$.

2.4 Wave Characteristics

In order to deal with the dynamics of atmospheric waves, this section introduces briefly its basic physical characteristics. First question that comes up is; what is a wave? Pedlosky (2003) argues that there is no simple definition of a wave. He states:

> A wave is a moving signal, typically at a rate distinct from the motion of the medium. These signals (or waves) are disturbances from an equilibrium state that propagate through space.

Generally, two types of waves can be distinguished: the mechanical wave, that travels through a medium, and the electromagnetic wave, that can travel through vacuum. Mechanical waves that travel through a medium can be categorized into two different waves: longitudinal and transverse waves (Figure 2.1). With a longitudinal wave (or compressible wave), the molecules oscillate in the direction parallel with the propagation. Sound waves and some of the seismic waves are examples of longitudinal waves. In a transverse wave, molecules oscillate in perpendicular direction of the propagation. Like a guitar string that moves up and down around an equilibrium state, the wave itself travels at a 90 degree angle of the disturbance (AMS-Glossary, 2007).

Observing large scale meteorological currents, periodic change in zonal to meridional velocity led to the notion of troughs and ridges as part of a planetary wave. On smaller scales we can observe internal gravity waves, where fluctuations occur between two different homogeneous density layers (Kroon, 2007).

Utilizing the linearized system, derived in the previous section, wave motions can be described analytically. Holton (1992) argues that the representation of perturbations as a simple sinusoidal wave can be an oversimplification since disturbances in the atmosphere are never purely sinusoidal. However, let us assume it is indeed sufficient to represent disturbances sinusoidally, a particular disturbance can be described by Fourier's theorem (Gill,

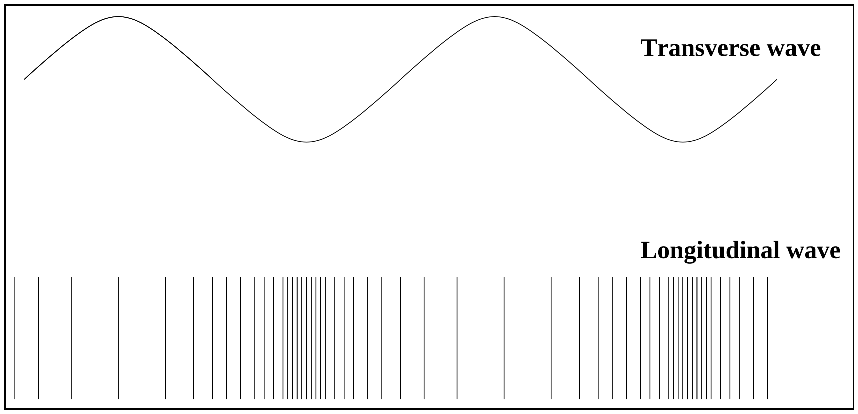


Figure 2.1: *Two kinds of waves; the transverse wave and the longitudinal wave. [Reproduced from Kroon (2007)]*

1982). A Fourier component can be written more compactly by using complex exponential notation according to the formula of Euler (Holton, 1992), i.e.

$$e^{i(\boldsymbol{\mu}\cdot\mathbf{r}-\omega t)} = \cos(\boldsymbol{\mu}\cdot\mathbf{r}-\omega t) + i\sin(\boldsymbol{\mu}\cdot\mathbf{r}-\omega t), \tag{2.4.1}$$

where $\boldsymbol{\mu}$ is the wave number, $\mathbf{r}$ is the spatial vector, and ω is the angular frequency. A simple sinusoidal traveling wave is deduced by taking the real part of exponential expression (2.4.1); the imaginary part does not have a physical meaning. So, for a particular fluctuating variable Θ', it follows that

$$\Theta'(\mathbf{r},t) = \hat{\Theta} Re\left[e^{i(\boldsymbol{\mu}\cdot\mathbf{r}-\omega t)}\right] = \hat{\Theta}\cos(\boldsymbol{\mu}\cdot\mathbf{r}-\omega t), \tag{2.4.2}$$

where $\hat{\Theta}$ represents the amplitude. It is straight forward to derive that (2.4.2) is the solution of

$$\boxed{\frac{\partial^2\Theta'}{\partial t^2} = c^2\nabla^2\Theta'}, \tag{2.4.3}$$

where $c^2 = \frac{\omega^2}{\mu^2}$, i.e. the phase speed squared. This equation is also known as the *wave equation*, a partial differential equation that describes evolution of a particular variable (AMS-Glossary, 2007).

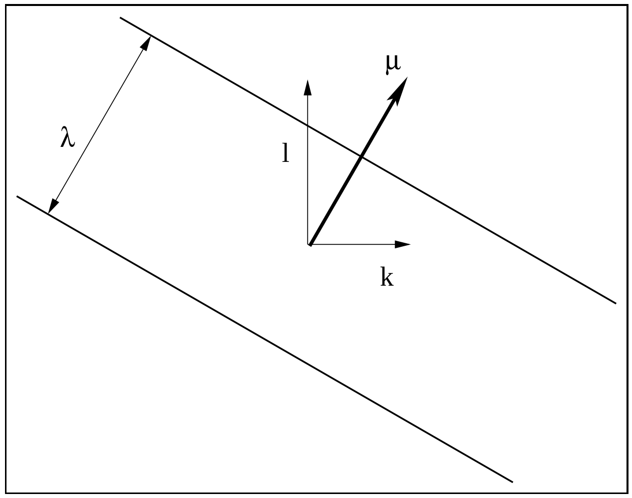


Figure 2.2: *Two-dimensional schematic representation of a plane wave. It illustrates two crests separated by single wave length λ and propagates in the direction of $\boldsymbol{\mu}$. [Reproduced from Pedlosky (2003)]*

Figure 2.2 gives a two-dimensional schematic representation of a plane wave illustrating the crests and other quantities. The variables and quantities identified in equations (2.4.2)-(2.4.3) and Figure 2.2 are summarized here below:

- Θ' represents an arbitrary perturbation variable. For instance: velocity, air pressure or air density.
- $\hat{\Theta}$ embodies the amplitude of the variable. The amplitude is the distance between equilibrium and maximum perturbation value.
- $\boldsymbol{\mu} = k\mathbf{i} + l\mathbf{j} + m\mathbf{k}$ is the wave number (radians m^{-1}). It represents the amount of complete wave cycles per meter. The scalar wave number can be written as $\mid \boldsymbol{\mu} \mid = \mu = \frac{2\pi}{\lambda}$, where λ is the wave length (Pedlosky, 1987). Thus, wave propagation is in the direction of the wave vector $\boldsymbol{\mu}$. The magnitude of the scalar wave number is given by $\mu^2 = k^2 + l^2 + m^2$.
- $\lambda = \frac{2\pi}{\mu}$ is the three-dimensional wave length (m), i.e. the length of a single wave with one crest and one trough.
- $\omega = \frac{2\pi}{\tau}$ is the angular frequency (radians s^{-1}), where τ is the time period. It is a measure of the oscillation rate.
- τ is time period (s) that is needed to complete one single wave cycle.
- $c = \frac{\omega}{\mu}$ is the phase speed (ms^{-1}) of a traveling crest or trough. It can be written as the ratio of the angular frequency and wave number. In case of an one-dimensional wave in the x-direction, the phase speed is defined as $c = \frac{\omega}{k}$.
- $\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the spatial vector.

As an example, a wave equation for sound waves is derived here using the linearized equations. Sound waves are longitudinal, which means that oscillations are parallel to the direction of propagation. Sound travels by adiabatic compression and expansion of the atmosphere. Our perturbation equations are appropriate because they are valid for compressible fluids. In addition, assume that the atmosphere is at rest and perturbations only propagate in the x-direction. Furthermore, it is reasonable to exclude gravity and Coriolis terms. With these assumptions, the momentum equation (2.3.17) and the continuity equation (2.3.19) are reduced as follows:

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{2.4.4}$$

$$\frac{\partial p'}{\partial t} = -\rho_0 c_{0s}^2 \frac{\partial u'}{\partial x}, \tag{2.4.5}$$

These two equations can be reduced to one wave equation for pressure. First, take the time derivative of (2.4.5), thus,

$$\frac{\partial^2 p'}{\partial t^2} = -\rho_0 c_{0s}^2 \frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial t}\right). \tag{2.4.6}$$

Substitution of (2.4.4) in (2.4.6) gives

$$\frac{\partial^2 p'}{\partial t^2} = -\rho_0 c_{0s}^2 \frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\right). \tag{2.4.7}$$

It leads finally to the standard wave equation for pressure,

$$\frac{\partial^2 p'}{\partial t^2} = c_{0s}^2 \frac{\partial^2 p'}{\partial x^2}, \tag{2.4.8}$$

which has the familiar form as (2.4.3). According to Appendix A, the phase (sound) speed c_{0s} is equal to $\sqrt{\frac{c_p}{c_v}RT_0}$. Eventually, perturbation of pressure (with amplitude $\hat{p}$) can be described by a sinusoidal function, i.e.

$$p'(x,t) = \hat{p}Re[e^{ik(x-c_{0s}t)}] = \hat{p}Re[e^{i(kx-\omega t)}] = \hat{p}\cos(kx-\omega t). \tag{2.4.9}$$

By substituting the solution (2.4.9) in the wave equation (2.4.8), one may obtain a dispersion relation. It illustrates the relationship between frequency ω and a given wave number k, i.e.

$$\omega = kc_{0s}. \tag{2.4.10}$$

Substituting the appropriate quantities, the magnitude of speed of sound is

$$c_{0s} = \sqrt{\frac{c_p}{c_v}RT_0} = \sqrt{\frac{1004}{717}287\cdot 273} = 331\text{m s}^{-1}, \tag{2.4.11}$$

where c_p is the specific heat of dry air at constant pressure, c_v is the specific heat of dry air at constant volume, R is the gas constant for dry air, and T_0 is the absolute background temperature.

3 System Linearized around an Isentropic Background State

3.1 Introduction

To understand complex atmospheric processes, like for instance equatorial waves, we will derive a simplified barotropic system: a system in which density only depends on pressure and vice-versa (Holton, 1992). To obtain such a system, our linearized equations (2.3.17), (2.3.18), (2.3.19), and (2.3.20) are simplified using a background state of uniform potential temperature, a reasonable approximation of the troposphere. Section 3.2 introduces for such a background state the temperature, pressure, and density. Furthermore, we recall the four prognostic perturbation equations obtained in section 2.3. It appears after applying isentropic conditions (plus another additional assumption) that one of the perturbation equations turns out to be a first order differential equation for p'. By solving p', the horizontal momentum equations (explained in section 3.3) and continuity equation (explained in section 3.4) gain a very simple barotropic form. Finally, section 3.5 explains that the equations of this barotropic system are analogous to those of a shallow-water system. These shallow-water equations will be the basis for describing equatorial waves in Chapter 4.

3.2 Basic Dynamics in an Isentropic Layer

In this section we introduce the basic linearized diagnostic and prognostic equations valid in an isentropic layer, where potential temperature is uniform with height. In Appendix C we derived, for such a layer, expressions for absolute temperature $T(z)$, air pressure $p(z)$, and air density $\rho(z)$. The background state is given by the following expressions:

Absolute temperature

$$T_0(z) = \theta_0 - z\frac{g}{c_p}, \tag{3.2.1}$$

Pressure

$$p_0(z) = p_r\left(\frac{T_0}{\theta_0}\right)^{\frac{1}{\kappa}}, \tag{3.2.2}$$

Density

$$\rho_0(z) = \rho_r\left(\frac{T_0}{\theta_0}\right)^{\left(\frac{1}{\kappa}-1\right)}, \tag{3.2.3}$$

where $\kappa = R/c_p$, p_r is a reference pressure, and ρ_r is a reference density. The background temperature at surface level is equal to the background potential temperature, i.e. $T_{0s} = \theta_0$.

The first three prognostic perturbation equations (2.3.17), (2.3.18), and (2.3.19), derived in section 2.3, are

Momentum equations

$$\frac{\partial u'}{\partial t} - fv' = \frac{-1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{3.2.4}$$

$$\frac{\partial v'}{\partial t} + fu' = \frac{-1}{\rho_0}\frac{\partial p'}{\partial y}, \tag{3.2.5}$$

Continuity equation

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} + \frac{1}{\rho_0 c_{0s}^2}\left(\frac{\partial p'}{\partial t} - \rho_0 g w'\right) = 0. \tag{3.2.6}$$

The fourth perturbation equation (2.3.20) contains the Brunt-Väisälä frequency N_0 which is zero in an isentropic layer, i.e.

$$N_0^2 = g\frac{d\ln\theta_0}{dz} = 0. \tag{3.2.7}$$

We can simplify (2.3.20) even further by neglecting the term $\partial^2 w'/\partial t^2$. Justification of this assumption must prove itself yet, but it leads to a very simple form of (2.3.20):

$$\frac{\partial p'}{\partial z} + \frac{g}{c_{0s}^2}p' = 0, \tag{3.2.8}$$

where c_{0s}^2 (speed of sound squared) is equal to

$$c_{0s}^2 = \frac{c_p}{c_v}RT_0. \tag{3.2.9}$$

Equation (3.2.8) is the linearized hydrostatic approximation, valid for adiabatic perturbations for which $\theta' = 0$.

Momentum and continuity form a set of three prognostic equations with four unknowns, i.e. u', v', w', and p'. However, the linearized hydrostatic equation (3.2.8) can be solved to obtain an expression for p'. The next sections illustrate that this notion leads to the momentum and continuity equations having a form that is analogous to the shallow-water equations.

3.3 Linearized Momentum Equations for an Isentropic Layer

In this section we derive an expression for pressure p' using the hydrostatic approximation (3.2.8). After some algebra we obtain the desired expression, where p' depends on the surface pressure perturbation p'_s and background density ρ_0. Eventually, the two momentum equations (3.2.4)-(3.2.5) obtain a very simple barotropic form.

First, the linearized hydrostatic approximation is given by (3.2.8),

$$\frac{\partial p'}{\partial z} = \frac{-g}{c_{0s}^2} p'.$$

Substituting (3.2.9) gives

$$\frac{\partial p'}{\partial z} = \frac{-g c_v}{c_p R T_0} p',$$

which can also be written as

$$\frac{\partial p'}{\partial z} = -\frac{(1-\kappa)}{\kappa} \frac{g}{c_p T_0} p'.$$

This is an homogeneous linear differential equation that can be solved with help of (3.2.1):

$$\frac{1}{p'} \partial p' = -\frac{(1-\kappa)}{\kappa} \frac{g}{c_p T_0} \partial z,$$

$$\int_{p'_s}^{p'} \frac{1}{p'} dp' = -\frac{(1-\kappa)}{\kappa} \frac{g}{c_p} \int_{T_{0s}}^{T_0} \frac{1}{T_0} dz,$$

$$\ln\left(\frac{p'}{p'_s}\right) = -\frac{(1-\kappa)}{\kappa} \frac{g}{c_p} \ln\left(\frac{T_0}{T_{0s}}\right)\left(\frac{-c_p}{g}\right),$$

$$\ln\left(\frac{p'}{p'_s}\right) = \frac{(1-\kappa)}{\kappa} \ln\left(\frac{T_0}{T_{0s}}\right),$$

where p'_s is the surface pressure perturbation and $T_{0s} = \theta_0$ is the background surface temperature. Removing the natural logarithm on the left-hand side gives

$$\frac{p'}{p'_s} = e^{\frac{(1-\kappa)}{\kappa} \ln\left(\frac{T_0}{\theta_0}\right)},$$

and eventually it can be written as

$$p' = p'_s \left(\frac{T_0}{\theta_0}\right)^{\frac{1}{\kappa}-1}.$$

Substituting the background state of density (3.2.3) yields the final expression for pressure perturbation, i.e.

$$\boxed{p'(x,y,z,t) = p'_s \frac{\rho_0}{\rho_r}}. \tag{3.3.1}$$

From this expression it is clear that p' depends, in the horizontal direction, on the surface pressure perturbation p'_s and the height dependency is related to the background density ρ_0.

Before we substitute this new expression in the momentum equations, we first derive the other perturbation expressions for temperature and density. Assuming the hydrostatic approximation and $\theta' = 0$, perturbation equation for temperature follows after linearizing (C-9) in Appendix C:

$$T' = -z'\frac{g}{c_p} = \left(\frac{p'}{\rho_0 g}\right)\frac{g}{c_p}, \tag{3.3.2}$$

Substituting (3.3.1) gives

$$\boxed{T'(x, y, t) = \frac{p'_s}{\rho_r c_p}}. \tag{3.3.3}$$

Perturbation equation for density follows after linearization of (C-13) in Appendix C:

$$\rho' = \rho_r\left(\frac{1}{\kappa} - 1\right)\left(\frac{T_0}{\theta_0}\right)^{\left(\frac{1}{\kappa}-1\right)}\left(\frac{\theta_0}{T_0}\right)\frac{T'}{\theta_0}.$$

Substituting the background state of density (3.2.3) and the expression for T' (3.3.3) gives

$$\boxed{\rho'(x, y, z, t) = \left(\frac{1}{\kappa} - 1\right)\rho_0\frac{p'_s}{\rho_r c_p T_0}}. \tag{3.3.4}$$

As for pressure, T' and ρ' depend in the horizontal direction on the surface pressure perturbation p'_s. The height dependency of ρ' is related with background states of both temperature T_0 and density ρ_0.

The new expression for pressure perturbation can be substituted in the momentum equations (3.2.4)-(3.2.5). That eliminates ρ_0 and leads to the important conclusion that the velocity field is independent of height, i.e. barotropic:

$$\boxed{\frac{\partial u'}{\partial t} - fv' = \frac{-1}{\rho_r}\frac{\partial p'_s}{\partial x}}, \tag{3.3.5}$$

$$\boxed{\frac{\partial v'}{\partial t} + fu' = \frac{-1}{\rho_r}\frac{\partial p'_s}{\partial y}}. \tag{3.3.6}$$

The only unknown in the set prognostic equations (3.2.4), (3.2.5), and (3.2.6) is the perturbation of vertical velocity. The next section deduces an expression for w' that leads to a new barotropic equation for continuity.

3.4 Linearized Continuity Equation for an Isentropic Layer

The vertical velocity perturbation remains the only unknown in the set conservation equations (3.2.4), (3.2.5), and (3.2.6). The continuity equation (3.2.6) is a linear differential equation of the first order because it contains function $w'(x, y, z, t)$ and its z-derivative $\partial w'/\partial z$. This can be solved, but the terms are first rearranged to obtain the following desired form,

$$\frac{\partial w'}{\partial z} - \frac{g}{c_s^2}w' = -\nabla_z \cdot \mathbf{v}' - \frac{1}{\rho_0 c_{0s}^2}\frac{\partial p'}{\partial t}. \tag{3.4.1}$$

An expression for p' (3.3.1), depending on p'_s and ρ_0, is derived in the previous section. Substitution in continuity equation gives

$$\frac{\partial w'}{\partial z} - \frac{g}{c_s^2} w' = -\nabla_z \cdot \mathbf{v}' - \frac{1}{\rho_r c_{0s}^2} \frac{\partial p'_s}{\partial t}. \tag{3.4.2}$$

An expression for the vertical velocity perturbation can be found by calculating a homogeneous w'_h and a particular w'_p solution. The general equation for the vertical velocity is the sum of these two solutions, i.e.

$$w' = w'_h + w'_p. \tag{3.4.3}$$

First the homogeneous solution is derived. To do that, the right-hand side of (3.4.2) must be zero. That gives

$$\frac{\partial w'}{\partial z} = \frac{g}{c_{0s}^2} w'.$$

Substitute (3.2.9) and separate the variables, thus,

$$\frac{1}{w'} \frac{\partial w'}{\partial z} = \frac{g c_v}{c_p R T_0}.$$

Applying integration gives

$$\int_{w'_s}^{w'} \frac{1}{w'} dw' = \frac{g c_v}{c_p R} \int_{T_{0s}}^{T_0} \frac{1}{T_0} dz,$$

$$\ln\left(\frac{w'}{w'_s}\right) = \frac{g c_v}{c_p R} \ln\left(\frac{T_0}{T_{0s}}\right) \frac{-c_p}{g},$$

$$\ln\left(\frac{w'}{w'_s}\right) = \frac{-c_v}{R} \ln\left(\frac{T_0}{T_{0s}}\right),$$

$$\ln\left(\frac{w'}{w'_s}\right) = \frac{-(1-\kappa)}{\kappa} \ln\left(\frac{T_0}{T_{0s}}\right).$$

Eliminating the natural logarithm and using $T_{0s} = \theta_0$ gives

$$\frac{w'}{w'_s} = \left(\frac{T_0}{\theta_0}\right)^{\left(1-\frac{1}{\kappa}\right)}.$$

After substituting the inverse of (3.2.3) we obtain the homogeneous solution for w', i.e.

$$w'_h = w'_s \frac{\rho_r}{\rho_0}, \tag{3.4.4}$$

where w'_s is the vertical velocity perturbation at the surface.

Seeking a particular solution usually involves some trial and error but gives eventually

$$w'_p = \frac{R T_0}{g} \nabla_z \cdot \mathbf{v}' + \frac{1}{\rho_r g} \frac{\partial p'_s}{\partial t}. \tag{3.4.5}$$

It can be proven by substituting the expression for w'_p in the continuity equation (3.4.2). The left-hand side of (3.4.2) gives

$$\frac{\partial}{\partial z}\left(\frac{RT_0}{g}\nabla_z\cdot\mathbf{v}'+\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\right)-\frac{gc_v}{c_pRT_0}\left(\frac{RT_0}{g}\nabla_z\cdot\mathbf{v}'+\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\right)=$$

$$\frac{R}{g}\nabla_z\cdot\mathbf{v}'\frac{\partial T_0}{\partial z}-\frac{c_v}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{1}{\rho_r c_{0s}^2}\frac{\partial p'_s}{\partial t}.$$

Under isentropic conditions $\frac{\partial T_0}{\partial z}=\frac{-g}{c_p}$. Which gives for the expression above

$$-\frac{R}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{c_v}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{1}{\rho_r c_{0s}^2}\frac{\partial p'_s}{\partial t}.$$

Gas constant R is equal to c_p-c_v, so this becomes

$$-\frac{(c_p-c_v)}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{c_v}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{1}{\rho_r c_{0s}^2}\frac{\partial p'_s}{\partial t}=$$

$$-\nabla_z\cdot\mathbf{v}'+\frac{c_v}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{c_v}{c_p}\nabla_z\cdot\mathbf{v}'-\frac{1}{\rho_r c_{0s}^2}\frac{\partial p'_s}{\partial t}.$$

The second and third term of the final expression cancel. So the expression turns out to be the right-hand side of (3.4.2), i.e.

$$-\nabla_z\cdot\mathbf{v}'-\frac{1}{\rho_r c_{0s}^2}\frac{\partial p'_s}{\partial t}. \tag{3.4.6}$$

The general solution can be obtained taking the sum of the homogeneous solution (3.4.4) and the particular solution (3.4.5). That gives

$$w'=w'_s\frac{\rho_r}{\rho_0}+\frac{RT_0}{g}\nabla_z\cdot\mathbf{v}'+\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}. \tag{3.4.7}$$

The next step is to solve the variable w'_s. At the surface of the layer ($\rho_0=\rho_r$ and $T_{0s}=\theta_0$) there is a boundary condition $w'=0$:

$$0=w'_s+\frac{R\theta_0}{g}\nabla_z\cdot\mathbf{v}'+\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}.$$

In that case w'_s gives

$$w'_s=-\frac{R\theta_0}{g}\nabla_z\cdot\mathbf{v}'-\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}.$$

The general solution (3.4.7) takes then the following form:

$$w'=-\left(\frac{R\theta_0}{g}\nabla_z\cdot\mathbf{v}'+\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\right)\frac{\rho_r}{\rho_0}+\frac{RT_0}{g}\nabla_z\cdot\mathbf{v}'+\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}.$$

However, in a more convenient way it gives

$$\boxed{w'(x,y,z,t) = \frac{R\theta_0}{g}\nabla_z \cdot \mathbf{v}'\left(\frac{T_0}{\theta_0} - \frac{\rho_r}{\rho_0}\right) + \frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\left(1 - \frac{\rho_r}{\rho_0}\right)}. \tag{3.4.8}$$

At the upper boundary ($z = H$), this expression obtains the following appearance:

$$w'(H) = \frac{R\theta_0}{g}\nabla_z \cdot \mathbf{v}'\left(\frac{T_H}{\theta_0} - \frac{\rho_r}{\rho_H}\right) + \frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\left(1 - \frac{\rho_r}{\rho_H}\right), \tag{3.4.9}$$

where T_H and ρ_H are respectively background temperature and density at top of the layer. Let us assume that the temperature in stratosphere remains constant with height and is always equal to the temperature at top of our layer. Remember, the isentropic layer represents the troposphere. This situation is visualized in Figure 3.1.

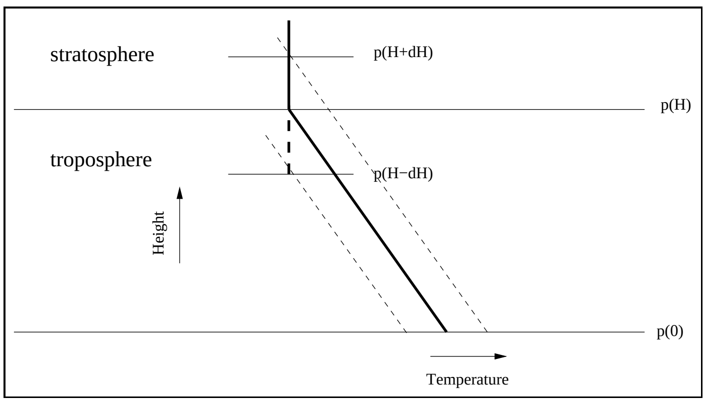


Figure 3.1: *The relationship between temperature perturbation and height of tropopause. Background temperature decreases dry-adiabatically in the isentropic troposphere and is uniform in the stratosphere. If there is a positive temperature perturbation, a positive height displacement dH must take place to keep temperature in stratosphere constant. Vice-versa if there is a negative temperature perturbation.*

A temperature perturbation at upper boundary has to correspond with a vertical displacement to keep the temperature of upper boundary and stratosphere in-sync (or equal). The relationship between vertical displacement and T' is given by (3.3.2). If vertical velocity is recognized as the rate of change of vertical displacement, we can write w' at average depth H as

$$w'(H) = \frac{c_p}{g}\frac{\partial T'}{\partial t} = \frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}. \tag{3.4.10}$$

After substitution in (3.4.9) it yields

$$\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t} = \frac{R\theta_0}{g}\nabla_z \cdot \mathbf{v}'\left(\frac{T_H}{\theta_0} - \frac{\rho_r}{\rho_H}\right) + \frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\left(1 - \frac{\rho_r}{\rho_H}\right). \tag{3.4.11}$$

Equation (3.4.11) can be rearranged accordingly

$$\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t}\frac{\rho_r}{\rho_H} = \frac{R\theta_0}{g}\nabla_z \cdot \mathbf{v}'\left(\frac{T_H}{\theta_0} - \frac{\rho_r}{\rho_H}\right),$$

$$\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t} = \frac{R\theta_0}{g}\nabla_z \cdot \mathbf{v}'\left(\frac{\rho_H T_H}{\rho_r \theta_0} - 1\right),$$

$$\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t} = \nabla_z \cdot \mathbf{v}'\left(\frac{\rho_H R T_H}{\rho_r g} - \frac{R\theta_0}{g}\right).$$

Take $p_H = \rho_H R T_H$ and $R\theta_0 = \frac{p_r}{\rho_r}$,

$$\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t} = \nabla_z \cdot \mathbf{v}'\left(\frac{p_H}{\rho_r g} - \frac{p_r}{\rho_r g}\right),$$

where p_r and ρ_r are a reference pressure and density. The quantities p_H and ρ_H represents background pressure and density at the upper boundary. The thickness of layer H is defined by $\frac{p_r}{\rho_r g} - \frac{p_H}{\rho_r g}$. The linearized continuity equation is eventually reduced to a barotropic form

$$\boxed{\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t} + H\left(\nabla_z \cdot \mathbf{v}'\right) = 0}. \tag{3.4.12}$$

3.5 Shallow-water System

To summarize our previous efforts, we started with the four prognostic perturbation equations that where derived in Chapter 2. To simplify the set of equations two assumptions were made: consideration of isentropic conditions and neglecting the second-order time-derivative of w'. Sections 3.3 and 3.4 lead us to conclude that, due to utilization of these assumptions, our physical system is reduced to a set of barotropic equations. The two momentum equations (3.2.4)-(3.2.5) are reduced to

$$\frac{\partial u'}{\partial t} - f v' = \frac{-1}{\rho_r}\frac{\partial p'_s}{\partial x}, \tag{3.5.1}$$

$$\frac{\partial v'}{\partial t} + f u' = \frac{-1}{\rho_r}\frac{\partial p'_s}{\partial y}. \tag{3.5.2}$$

As already mentioned before, in this simplified system, the velocity field remains constant with height. The continuity equation (3.2.6) becomes

$$\frac{1}{\rho_r g}\frac{\partial p'_s}{\partial t} + H\left(\nabla_z \cdot \mathbf{v}'\right) = 0. \tag{3.5.3}$$

These conservation laws form a system that can be solved. It assumes the same form as the corresponding laws in the shallow-water system with uniform density ρ_r (Verkley, 2000). The shallow-water system contains a set of equations that describe the flow below a horizontal free-moving interface (see Figure 3.2). The main advantage is that shallow-water equations have a much easier form than the original linearized equations. Experience has shown that the shallow-water model is capable of describing important aspects of atmospheric and oceanic motions (Pedlosky, 1987).

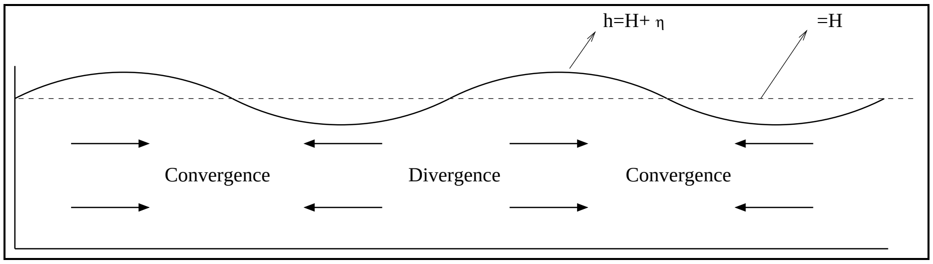


Figure 3.2: *Schematic representation shallow-water layer. Divergence and convergence zones emerge under the oscillating interface, with mean depth H. [Reproduced from Holton (1992)]*

In Figure 3.2, the interface height $h(x, y, t)$ can be separated in two parts: background state layer depth H and a disturbance part $\eta(x, y, t)$. Using the assumptions mentioned in section 3.2, the pressure perturbations are seen to be hydrostatic [see (3.2.8)]. Therefore, a small change of interface height will express itself as a fluctuation of surface pressure, i.e.

$$p'_s = \rho_r g\eta. \tag{3.5.4}$$

From this point onwards, primes are dropped. Unless otherwise noted, unprimed variables are the fluctuating quantities. By substituting the relationship (3.5.4), our equations become

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}, \tag{3.5.5}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}, \tag{3.5.6}$$

$$\frac{\partial \eta}{\partial t} + H\left(\nabla_z \cdot \mathbf{v}\right) = 0. \tag{3.5.7}$$

These equations form the well-known shallow-water equations. They form a closed system as we have three equations and three unknowns (u, v, and η). In the next chapter we apply these equations to describe the equatorial waves.

Besides the three prognostic equations, the considerations in sections 3.3 and 3.4 also led to expressions for pressure, temperature, density, and vertical velocity. A very useful characteristic is that all these variables depend on the surface pressure and horizontal velocity perturbations. After substituting (3.5.4), these expressions can be written in terms of η, i.e.

$$p(x, y, z, t) = \rho_0 g\eta, \tag{3.5.8}$$

$$T(x, y, t) = \frac{g\eta}{c_p}, \tag{3.5.9}$$

$$\rho(x, y, z, t) = \left(\frac{1}{\kappa} - 1\right)\rho_0\frac{g\eta}{c_p T_0}, \tag{3.5.10}$$

$$w(x, y, z, t) = \frac{-R\theta_0}{gH}\frac{\partial \eta}{\partial t}\left(\frac{T_0}{\theta_0} - \frac{\rho_r}{\rho_0}\right) + \frac{\partial \eta}{\partial t}\left(1 - \frac{\rho_r}{\rho_0}\right). \tag{3.5.11}$$

A nice feature is that if the evolution of interface height is known, we can calculate each quantity at every position in space and time.

The importance of rotation in the shallow-water equations can be demonstrated by taking the $\partial/\partial x$ of (3.5.5) plus $\partial/\partial y$ of (3.5.6), and combining with (3.5.7). For f is constant, this gives

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) + fH\zeta = 0, \tag{3.5.12}$$

where $c = \sqrt{gH}$ is the shallow-water phase speed and $\zeta = \partial v/\partial x - \partial u/\partial y$ is the relative vorticity perturbation. In case of no rotation, i.e. $f = 0$, (3.5.12) is simply a wave equation in one variable only. However, when f is nonzero, this equation is coupled to the vorticity (Gill, 1982).

In the 19th century, William Thomson (1st Baron Kelvin) derived from the shallow-water equations an equation that is of fundamental importance in the theory of rotating fluids (Gill, 1982): the conservation of potential vorticity. It can be required by taking first $\partial/\partial y$ of (3.5.5) minus $\partial/\partial x$ of (3.5.6) where we assume for the moment f is a constant. It gives the vorticity equation:

$$\frac{\partial \zeta}{\partial t} + f\nabla_z \cdot \mathbf{v} = 0. \tag{3.5.13}$$

Or in other words, rate of change of ζ/f equals minus divergence,

$$\frac{\partial}{\partial t}\frac{\zeta}{f} = -\nabla_z \cdot \mathbf{v}. \tag{3.5.14}$$

Subtraction of the continuity equation (3.5.7) gives the important linearized form of the conservation of potential vorticity, i.e.

$$\frac{\partial}{\partial t}\left(\zeta - f\frac{\eta}{H} \right) = 0. \tag{3.5.15}$$

Gill (1982) argues that the fact that this equation can be integrated easily with respect to time is very powerful. We can identify the quantity between the parenthesis as Q':

$$Q' = \frac{\zeta}{H} - f\frac{\eta}{H^2}. \tag{3.5.16}$$

The unlinearized form of the conservation equation of potential vorticity gives

$$\frac{DQ}{Dt} = 0, \tag{3.5.17}$$

where

$$Q = \frac{f + \zeta}{H + \eta}. \tag{3.5.18}$$

In a state of rest, the background state of Q is given by

$$Q_0 = \frac{f}{H}, \tag{3.5.19}$$

and is constant if f and H are constants. In a state of rest the perturbation equation becomes

$$\frac{DQ'}{Dt} = \frac{\partial Q'}{\partial t} + u\frac{\partial Q_0}{\partial x} + v\frac{\partial Q_0}{\partial y} + w\frac{\partial Q_0}{\partial z} = 0. \tag{3.5.20}$$

Since the x, y and z-derivatives are zero (assuming that Q_0 is constant), we obtain a form that is parallel with (3.5.15).

The fact that Q' remains constant with time is exploited in the next chapter. It is used to find an equilibrium solution for slow equatorial waves for a particular initial state without using any details of the motion at finite times (Gill, 1982). However, it becomes apparent that we cannot take the Coriolis parameter f as a constant. It leads to the understanding that slow waves are not in perfect geostrophic state but in *quasi-geostrophic* state.

4 Describing Equatorial Waves

4.1 Introduction

The rotation of the earth has important consequences for atmospheric dynamics, among which the establishment of geostrophic balance. However, at the equator the Coriolis parameter vanishes, as a result of which a special dynamic region is established around the equator. Linear waves exist with unusually strong signals, which are involved with atmospheric and oceanic phenomena such as the Quasi-Biennial Oscillation, El Niño (Pedlosky, 2003), and phenomena associated with convection (Wheeler et al., 2000). These waves are trapped around the equator, i.e. they decay away from this region (Holton, 1992). To better understand these waves a general dispersion equation is derived in section 4.2. We can use this relationship to distinguish the different fast and slow waves. For this we use the shallow-water equations that are derived in Chapter 3, however now utilized on an equatorial beta-plane. In section 4.3 we derive an approximate dispersion relation that is applicable for slow westward waves only. To keep the motion in a quasi-geostrophic state we do not use an equilibrium state between Coriolis force and pressure gradient force. This assumption is only valid at higher latitudes and not at the equator. Hence, to describe the slow waves we use in section 4.3 a more general approach to formulate the quasi-geostrophic theory; the notion that slow movements must be nondivergent. Finally, in section 4.4 we compare the approximated solution with the original solution. We want to know how good the slow westward waves are described with the approximated equations.

4.2 Equatorial Wave Guide

This section tries to describe the different kind of equatorial waves by deriving and analyzing the dispersion relation. This relation gives the relationship between wave frequency ω, zonal wave number k and meridional mode number n (Pedlosky, 2003). To describe these wave motions we continue with the set of equations that are derived in the previous chapter. There we derived a set of perturbation equations in an isentropic layer, which are analogous to the shallow-water equations. For the equatorial region we impose another assumption; the equatorial beta-plane approximation. This assumption takes into account the importance of a variable Coriolis term. Gill (1982) argues that the maximum error of such approximation in the tropical latitudes (less than 30^o) is only 14%. The Coriolis parameter f is thus approximated by

$$f = \beta y, \tag{4.2.1}$$

where β is a constant given by

$$\beta = \frac{2\Omega}{a} = 2.3 \times 10^{-11}\mathrm{m}^{-1}\ \mathrm{s}^{-1}. \tag{4.2.2}$$

Here is y the poleward distance from the equator and a is the mean radius of the earth (6.37×10^6 m). Substituting (4.2.1) in the shallow-water momentum equations (3.5.5)-(3.5.6) gives

$$\frac{\partial u}{\partial t} - \beta y v = -g\frac{\partial \eta}{\partial x}, \tag{4.2.3}$$

$$\frac{\partial v}{\partial t} + \beta y u = -g\frac{\partial \eta}{\partial y}. \tag{4.2.4}$$

The shallow-water continuity equation (3.5.7) remains unchanged, i.e.

$$\frac{\partial \eta}{\partial t} + H\left(\nabla_z \cdot \mathbf{v}\right) = 0. \tag{4.2.5}$$

As done in the previous chapter we can derive an equation for the conservation of potential vorticity. First, we obtain the vorticity equation by taking $\partial/\partial y$ of (4.2.3) minus $\partial/\partial x$ of (4.2.4), i.e.

$$\frac{\partial \zeta}{\partial t} + \beta v + \beta y \nabla_z \cdot \mathbf{v} = 0. \tag{4.2.6}$$

Then we multiply the vorticity equation with the layer depth H and the continuity equation with βy, i.e,

$$\frac{\partial H\zeta}{\partial t} + H\beta v + H\beta y \nabla_z \cdot \mathbf{v} = 0,$$

$$\frac{\partial \beta y \eta}{\partial t} + H\beta y \nabla_z \cdot \mathbf{v} = 0.$$

The next step is to eliminate the horizontal divergence by subtraction of these equations. It gives

$$\frac{\partial}{\partial t}\left(H\zeta - \beta y \eta\right) + H\beta v = 0.$$

Finally, we divide the expression by H. That gives the following form of the potential vorticity equation:

$$\frac{\partial}{\partial t}\left(\zeta - \beta y \frac{\eta}{H}\right) + \beta v = 0. \tag{4.2.7}$$

In contrast to the potential vorticity equation (3.5.15) in Chapter 3, the latter one involves an extra term with β because we allowed the Coriolis parameter to vary linearly with latitude. The unlinearized conservation equation, the background state, and the perturbation of potential vorticity Q are defined likewise (3.5.17), (3.5.19), and (3.5.16), i.e.

$$\frac{DQ}{Dt} = \frac{D}{Dt}\left(\frac{\zeta + \beta y}{H + \eta}\right) = 0,$$

$$Q_0 = \frac{\beta y}{H},$$

$$Q' = \frac{\zeta}{H} - \frac{\beta y \eta}{H^2},$$

where f is now replaced by βy. In this case Q_0 is not a constant anymore but varies with latitude. The slow equatorial waves, which we discuss in the next section, propagate along the contours of Q_0.

Now we continue to derive a dispersion equation to describe the infinite set of equatorial waves. First, we have to deduce a form of the wave equation for velocity v alone. Pedlosky (2003) argues that it turns out to be far simpler to pose the problem in terms of meridional velocity v than u or η. By looking for solutions in form of a plane wave we can deduce the general dispersion relation.

To derive an equation for velocity v alone, we take Gill (1982) as our main guide. However, in this study the steps are carried out without big leaps. First we take $-(\beta y/c^2)\partial/\partial t$ of (4.2.3), $(1/c^2)\partial^2/\partial t^2$ of (4.2.4), $-(1/H)\partial^2/\partial y \partial t$ of (4.2.5), and $-\partial/\partial x$ of (4.2.7):

$$-\frac{\beta y}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{\beta^2 y^2}{c^2}\frac{\partial v}{\partial t} - \frac{\beta y g}{c^2}\frac{\partial^2 \eta}{\partial x \partial t} = 0, \tag{4.2.8}$$

$$\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{\beta y}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{g}{c^2}\frac{\partial^3 \eta}{\partial y \partial t^2} = 0, \tag{4.2.9}$$

$$\frac{-1}{H}\frac{\partial^3 \eta}{\partial y \partial t^2} - \left(\frac{\partial^3 u}{\partial x \partial y \partial t} + \frac{\partial^3 v}{\partial y^2 \partial t}\right) = 0, \tag{4.2.10}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial t} - \frac{\partial^3 v}{\partial x^2 \partial t} + \frac{\beta y}{H}\frac{\partial^2 \eta}{\partial x \partial t} - \beta\frac{\partial v}{\partial x} = 0. \tag{4.2.11}$$

Remember from (3.5.12) that $c = \sqrt{gH}$ is the shallow-water phase speed. Now substitute (4.2.8) in (4.2.9):

$$\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{\beta^2 y^2}{c^2}\frac{\partial v}{\partial t} - \frac{\beta y g}{c^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{g}{c^2}\frac{\partial^3 \eta}{\partial y \partial t^2} = 0. \tag{4.2.12}$$

Secondly, substitute (4.2.11) in (4.2.10):

$$\frac{-1}{H}\frac{\partial^3 \eta}{\partial y \partial t^2} - \left(\frac{\partial^3 v}{\partial x^2 \partial t} + \frac{\partial^3 v}{\partial y^2 \partial t} - \frac{\beta y}{H}\frac{\partial^2 \eta}{\partial x \partial t} + \beta\frac{\partial v}{\partial x}\right) = 0. \tag{4.2.13}$$

Finally, substitution of (4.2.13) in (4.2.12) gives

$$\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{\beta^2 y^2}{c^2}\frac{\partial v}{\partial t} - \frac{\beta y g}{c^2}\frac{\partial^2 \eta}{\partial x \partial t} - \left(\frac{\partial^3 v}{\partial x^2 \partial t} + \frac{\partial^3 v}{\partial y^2 \partial t}\right) + \frac{\beta y}{H}\frac{\partial^2 \eta}{\partial x \partial t} - \beta\frac{\partial v}{\partial x} = 0. \tag{4.2.14}$$

The third and sixth term cancel, and after rearranging the other terms one obtains an equation for v alone, i.e.

$$\frac{\partial}{\partial t}\left(\nabla_z^2 v - \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} - \frac{\beta^2 y^2}{c^2}v\right) + \beta\frac{\partial v}{\partial x} = 0. \tag{4.2.15}$$

It has some similarities with the general non-dispersive wave equation but now it includes extra terms due to variation of the Coriolis parameter. We can continue to deduce the dispersion relation. Assuming that free oscillations are possible, a solution may be sought in the form of a plane wave in x and t (see section 2.4), i.e.

$$v(x,y,t) = \hat{v}(y)e^{i(kx-\omega t)}, \tag{4.2.16}$$

where $\hat{v}(y)$ is meridional velocity amplitude at distance y away from the equator. The first-order derivatives with respect to x and t are

$$\frac{\partial v}{\partial x} = ik\hat{v}e^{i(kx-\omega t)}, \tag{4.2.17}$$

$$\frac{\partial v}{\partial t} = -i\omega\hat{v}e^{i(kx-\omega t)}, \tag{4.2.18}$$

The third-order derivative with respect to t gives

$$\frac{\partial^3 v}{\partial t^3} = -i\omega^3\hat{v}e^{i(kx-\omega t)}. \tag{4.2.19}$$

The second-order derivatives with respect to x and y are

$$\frac{\partial^2 v}{\partial x^2} = -k^2\hat{v}e^{i(kx-\omega t)},$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 \hat{v}}{\partial y^2}e^{i(kx-\omega t)}.$$

Thus,

$$\frac{\partial}{\partial t}\nabla_z^2 v = -i\omega\left(\frac{\partial^2\hat{v}}{\partial y^2} - k^2\hat{v}\right)e^{i(kx-\omega t)}. \tag{4.2.20}$$

The derivatives (4.2.17), (4.2.18), (4.2.19), and (4.2.20) can be substituted in (4.2.15). That yields

$$-i\omega\left(\frac{\partial^2\hat{v}}{\partial y^2} - k^2\hat{v}\right)e^{i(kx-\omega t)} - \frac{i\omega^3\hat{v}}{c^2}e^{i(kx-\omega t)} + \frac{\beta^2y^2}{c^2}i\omega\hat{v}e^{i(kx-\omega t)} + \beta ik\hat{v}e^{i(kx-\omega t)} = 0.$$

Dividing this expression by $-i\omega e^{i(kx-\omega t)}$ gives

$$\frac{\partial^2\hat{v}}{\partial y^2} - k^2\hat{v} + \frac{\omega^2}{c^2}\hat{v} - \frac{\beta^2y^2}{c^2}\hat{v} - \frac{k\beta}{\omega}\hat{v} = 0.$$

After some rearrangements, the second-order differential equation becomes

$$\frac{\partial^2\hat{v}}{\partial y^2} + \frac{\omega^2}{c^2}\hat{v} - \frac{\beta^2y^2}{c^2}\hat{v} - k\left(k + \frac{\beta}{\omega}\right)\hat{v} = 0. \tag{4.2.21}$$

To obtain a standard form of this equation it is more convenient to replace y by a dimensionless meridional coordinate ξ (Pedlosky, 2003), where ξ is defined as

$$\xi = Ay.$$

Substituting the scaled coordinate in (4.2.21) gives

$$\frac{\partial^2 \hat{v}}{\partial \xi^2} A^2 + \frac{\omega^2}{c^2}\hat{v} - \frac{\beta^2}{c^2 A^2}\xi^2 \hat{v} - k\left(k + \frac{\beta}{\omega}\right)\hat{v} = 0.$$

Divide by A^2:

$$\frac{\partial^2 \hat{v}}{\partial \xi^2} + \frac{\omega^2}{c^2 A^2}\hat{v} - \frac{\beta^2}{c^2 A^4}\xi^2 \hat{v} - k\left(k + \frac{\beta}{\omega}\right)\frac{\hat{v}}{A^2} = 0.$$

If we assume that it is chosen such that

$$\frac{\beta^2}{c^2 A^4} = 1,$$

it follows that

$$A = \left(\frac{\beta}{c}\right)^{1/2},$$

and thus, the meridional scaled coordinate becomes

$$\xi = \left(\frac{\beta}{c}\right)^{1/2} y. \tag{4.2.22}$$

Substituting the expression for A, the differential equation is written in terms of ξ:

$$\frac{\partial^2 \hat{v}}{\partial \xi^2} - \xi^2 \hat{v} + \frac{\omega^2}{\beta c}\hat{v} - \frac{c}{\beta}k\left(k + \frac{\beta}{\omega}\right)\hat{v} = 0. \tag{4.2.23}$$

This equation has the same form as the differential equation that describes a quantum mechanical oscillator (Arfken, 1970),

$$\frac{\partial^2 \hat{v}}{\partial \xi^2} - \xi^2 \hat{v} + (2n+1)\,\hat{v} = 0, \tag{4.2.24}$$

if

$$2n + 1 = \frac{\omega^2}{\beta c} - \frac{c}{\beta}k\left(k + \frac{\beta}{\omega}\right). \tag{4.2.25}$$

Its solutions are

$$\hat{v} = \hat{v}_n(\xi) = \hat{v}_0 2^{-n/2} e^{-\xi^2/2} H_n(\xi), \tag{4.2.26}$$

where $\hat{v}_0$ is the meridional velocity amplitude at the equator (ms^{-1}), n $(= 0, 1, 2, 3...)$ is the meridional mode number and $H_n(\xi)$ is a Hermite polynomial. The number n corresponds with the number of nodes in meridional velocity profile in domain $|y| < \infty$. Notice that the trapped behavior of equatorial waves is caused by the exponential function, a Gaussian distribution that vanishes for higher meridional coordinates. The Hermite polynomial $H_n(\xi)$ can be written as (Arfken, 1970)

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n}\left(e^{-\xi^2}\right).$$

The first three solutions of this function are:

$$H_0(\xi) = (-1)^0 e^{\xi^2} \frac{d^0}{d\xi^0}\left(e^{-\xi^2}\right) = e^{\xi^2} \cdot e^{-\xi^2} = 1,$$

$$H_1(\xi) = (-1) e^{\xi^2} \frac{d}{d\xi}\left(e^{-\xi^2}\right) = -e^{\xi^2} \cdot -2\xi e^{-\xi^2} = 2\xi,$$

$$H_2(\xi) = (-1)^2 e^{\xi^2} \frac{d^2}{d\xi^2}\left(e^{-\xi^2}\right) = e^{\xi^2} \cdot 4\xi^2 e^{-\xi^2} - 2e^{\xi^2} \cdot e^{-\xi^2} = 4\xi^2 - 2.$$

Recalling (4.2.16), the meridional velocity can be written as

$$v = \hat{v}_n(\xi) e^{i(kx-\omega t)} = \hat{v}_0 2^{-n/2} e^{-\xi^2/2} H_n(\xi) \cos(kx - \omega t). \tag{4.2.27}$$

Or in terms of y-coordinate:

$$v = \hat{v}_n\left[(\beta/c)^{1/2} y\right] e^{i(kx-\omega t)} = \hat{v}_0 2^{-n/2} e^{-\beta y^2/2c} H_n\left[(\beta/c)^{1/2} y\right] \cos(kx - \omega t). \tag{4.2.28}$$

Derivation of zonal velocity and height perturbation is less apparent. Gill (1982) states that to relate v to other variables it is better to work with definitions q and r. These are defined as

$$q = \frac{g\eta}{c} + u, \tag{4.2.29}$$

$$r = \frac{g\eta}{c} - u. \tag{4.2.30}$$

These quantities are obtained after taking the sum and difference between the x-momentum equation (4.2.3) and g/c times the continuity equation (4.2.5):

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + c\frac{\partial v}{\partial y} - \beta y v = 0, \tag{4.2.31}$$

$$\frac{\partial r}{\partial t} - c\frac{\partial r}{\partial x} + c\frac{\partial v}{\partial y} + \beta y v = 0. \tag{4.2.32}$$

If v takes the form of (4.2.28), the corresponding solutions for q and r are

$$q = \hat{v}_0 \frac{(2\beta c)^{1/2}}{(ck - \omega)} 2^{-(n+1)/2} e^{-\beta y^2/2c} H_{n+1}\left[(\beta/c)^{1/2} y\right] \sin(kx - \omega t), \tag{4.2.33}$$

$$r = \hat{v}_0 \frac{(2\beta c)^{1/2}}{(ck + \omega)} 2^{-(n-1)/2} e^{-\beta y^2/2c} n H_{n-1}\left[(\beta/c)^{1/2} y\right] \sin(kx - \omega t), \tag{4.2.34}$$

and is proven in Appendix D by solving each term of (4.2.31) and (4.2.32). The corresponding zonal velocity and height perturbation follow using definitions (4.2.29) and (4.2.30), i.e.

$$u = \frac{1}{2}(q - r), \tag{4.2.35}$$

$$\eta = \frac{c}{2g}(q + r). \tag{4.2.36}$$

The general (or original) dispersion relation for infinite trapped equatorial waves follows after rearranging the terms of (4.2.25), i.e

$$\boxed{\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} = (2n+1)\frac{\beta}{c}}. \tag{4.2.37}$$

To construct a dispersion diagram like Figure 4.1, it is more convenient to write it as the quadratic for k in terms of ω (Pedlosky, 2003):

$$\boxed{k = -\frac{\beta}{2\omega} \pm \frac{1}{2}\left[\left(\frac{\beta}{\omega} - \frac{2\omega}{c}\right)^2 - 8n\frac{\beta}{c}\right]^{1/2}}. \tag{4.2.38}$$

In the dispersion diagram (Figure 4.1) all possible types of equatorial waves are plotted. One can observe that for meridional modes $n \geq 1$, waves can be divided in two groups; high-frequency waves and low-frequency waves. For the high-frequency waves, Gill (1982) argues term $\beta k/\omega$ is relatively small. In that case, the dispersion relation for these fast waves are approximated by

$$\omega^2 = c^2 k^2 + (2n+1)\beta c. \tag{4.2.39}$$

These waves are analogous to the linear Poincaré gravity waves in shallow layer:

$$\omega^2 = c^2\left(k^2 + l^2\right) + f^2. \tag{4.2.40}$$

For the equatorial region they are called appropriately equatorial trapped gravity waves. As one can see in Figure 4.1, these waves can propagate eastward (for positive wave numbers) and westward (for negative wave numbers). Gravity waves are geostrophically unbalanced (Bouchut et al., 2005), and are not considered further in this study.

The lower branch of waves with $n \geq 1$ represent the slow (low-frequency) equatorial waves. These waves are called the equatorially trapped planetary waves or equatorially trapped Rossby waves. It is close to geostrophic balance and its phase speed is westward. The wave has been related to variations of convection in the Pacific convergence zones (Wheeler et al., 2000).

For $n = 0$ there is a special kind of wave. It can behave like a fast gravity wave if k is large and positive or like a slow planetary wave if k is large and negative. For that reason it is called the mixed Rossby-gravity wave or Yanai wave (Gill, 1982). The phase speed can either be east or west. Its dispersion relation is given by

$$\frac{\omega}{c} - k - \frac{\beta}{\omega} = 0. \tag{4.2.41}$$

The slow Rossby waves and the slow branch of mixed Rossby-gravity wave are the focus point of this study and are discussed further in the next section. There we try to deduce directly the dispersion relation for slow waves by considering zero divergence in the velocity field.

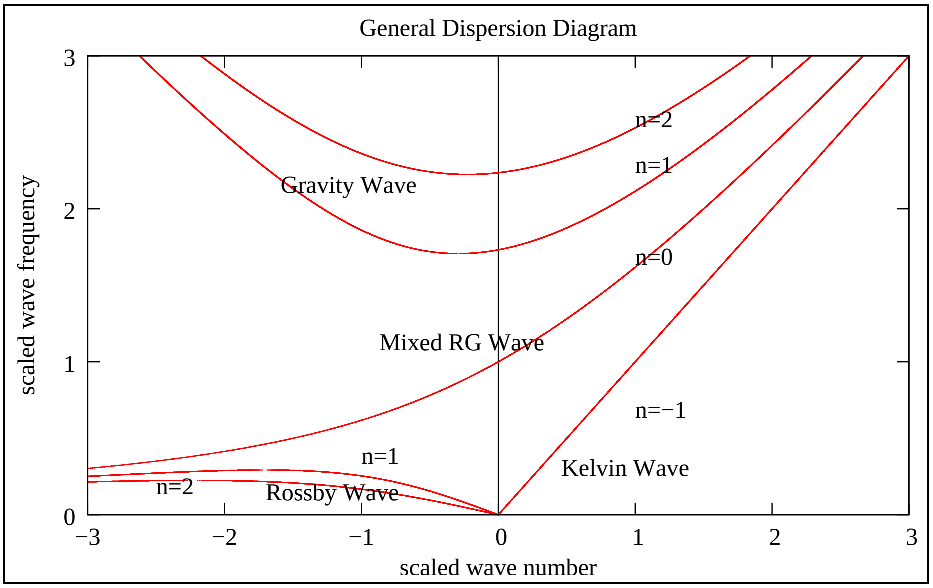


Figure 4.1: *Dispersion relation for equatorially trapped waves. The general dispersion equation (4.2.38) is used to calculate each type of wave. The horizontal axis represents the scaled wave number in units of $(\beta/c)^{1/2}$ and the vertical axis represents the scaled wave frequency in units of $(\beta c)^{1/2}$. Westward traveling waves have negative wave numbers and eastward traveling waves have positive wave numbers. The equatorial trapped Rossby and gravity waves are plotted for meridional mode numbers 1 and 2. The mixed Rossby-gravity wave is solved for mode number 0 and the Kelvin wave is satisfied for mode number -1.*

There is one other type of wave in equatorial wave guide present; the Kelvin wave. In our barotropic system it is generally a nondispersive (AMS-Glossary, 2007) shallow-water gravity wave and propagates only eastward parallel along the equator (Holton and Lindzen, 1968). Like the Poincaré gravity waves, we do not discuss the Kelvin wave further due to its unbalanced nature (Bouchut et al., 2005). Its meridional velocity is zero ($v = 0$), so it does not appear explicitly in our analysis. Its dispersion relation is obtained by taking $n = -1$ (Gill, 1982), i.e.

$$\omega = ck. \tag{4.2.42}$$

4.3 Describing Slow Equatorial Waves Using Zero Divergence

In this section the slow westward propagating Rossby waves are investigated. As we have already discussed, at mid-latitudes these waves are characterized by an equilibrium between Coriolis force and pressure-gradient force. However, at the equator such approximation is less appropriate. Instead, we use the more general notion that these motions are approximately nondivergent. As already mentioned in section 4.2, by rewriting the shallow-water equations on equatorial plane, we may derive a potential vorticity equation. Using the assumption of zero divergence, the potential vorticity equation can be reduced to an equation in a single variable. This equation can be solved and leads to an approximate dispersion relation for slow equatorial Rossby waves.

Now it is more appropriate to write the momentum equations in terms of vorticity and divergence. The vorticity equation has been derived already and is given by (4.2.6). The divergence equation follows after taking the sum of $\partial/\partial x$ of (4.2.3) and $\partial/\partial y$ of (4.2.4). Hence, the equations of the shallow-water system become

$$\frac{\partial \zeta}{\partial t} + \beta v + \beta y D = 0, \tag{4.3.1}$$

$$\frac{\partial D}{\partial t} + \beta u - \beta y \zeta + g \nabla_z^2 \eta = 0, \tag{4.3.2}$$

$$\frac{\partial \eta}{\partial t} + HD = 0, \tag{4.3.3}$$

where $D = \partial u/\partial x + \partial v/\partial y$ is the horizontal divergence perturbation. In the previous section we combined (4.3.1) with (4.3.3) to obtain a form of the potential vorticity equation for a shallow-water homogeneous layer,

$$\frac{\partial}{\partial t}\left(\zeta - \beta y \frac{\eta}{H}\right) + \beta v = 0, \tag{4.3.4}$$

which can also be written as $DQ'/Dt = 0$.

The seemingly important potential vorticity equation does not constitute a closed system, nor a definite result can be obtained from it yet. However, if we assume nondivergence, one can see that the potential vorticity equation is reduced from three variables (ζ, η, and v) to an equation in a single variable. Let us suppose the evolution of the height field is very slow, i.e. $\partial \eta/\partial t = 0$. In that case, considering (4.3.3), it is appropriate to approximate the perturbation of horizontal divergence as

$$D = 0, \tag{4.3.5}$$

and its evolution as

$$\frac{\partial D}{\partial t} = 0. \tag{4.3.6}$$

The dynamical system is now closed. It allows us to approximate velocity and vorticity in terms of a streamfunction ψ, i.e.

$$u = -\frac{\partial \psi}{\partial y}, \tag{4.3.7}$$

$$v = \frac{\partial \psi}{\partial x}, \tag{4.3.8}$$

$$\zeta = \nabla_z^2 \psi. \tag{4.3.9}$$

After substituting the new expressions for ζ and v, potential vorticity equation becomes

$$\frac{\partial}{\partial t}\left(\nabla_z^2 \psi - \beta y \frac{\eta}{H}\right) + \beta \frac{\partial \psi}{\partial x} = 0. \tag{4.3.10}$$

At this point potential vorticity is reduced to an equation of two variables, i.e. ψ and η. Recall the divergence equation (4.3.2) which, under assumption (4.3.6), reduces to

$$\beta u - \beta y\zeta + g\nabla_z^2\eta = 0.$$

The expressions for u and ζ in terms of ψ are known, thus it may be written as

$$-\beta\frac{\partial\psi}{\partial y} - \beta y\nabla_z^2\psi + g\nabla_z^2\eta = 0.$$

It is further approximated as

$$-\nabla_z^2\beta y\psi + g\nabla_z^2\eta = -\nabla_z^2\left(\beta y\psi - g\eta\right) = 0.$$

Then the term between parenthesis must be a constant, i.e.

$$\beta y\psi - g\eta = C.$$

We assume that the constant is zero. Then the height perturbation can be expressed in terms of the streamfunction:

$$\eta = \frac{\beta y}{g}\psi. \tag{4.3.11}$$

In that case the velocity components can be written in terms of η:

$$u = -\frac{\partial\psi}{\partial y} = -g\frac{\partial}{\partial y}\left(\frac{\eta}{\beta y}\right) = \frac{g\eta}{\beta y^2} - \frac{g}{\beta y}\frac{\partial\eta}{\partial y}, \tag{4.3.12}$$

$$v = \frac{\partial\psi}{\partial x} = \frac{g}{\beta y}\frac{\partial\eta}{\partial x}. \tag{4.3.13}$$

Observe the extra term in (4.3.12). It makes clear that the velocity field is indeed not in a state of perfect geostrophic balance.

Substituting (4.3.11) in (4.3.10) yields the approximated quasi-geostrophic potential vorticity equation in terms of one variable ψ:

$$\boxed{\frac{\partial}{\partial t}\left(\nabla_z^2\psi - \frac{\beta^2 y^2}{c^2}\psi\right) + \beta\frac{\partial\psi}{\partial x} = 0}. \tag{4.3.14}$$

Notice that if this expression is differentiated with respect to x, equation (4.2.15) for v reappears with exception of term $(1/c^2)\partial^2 v/\partial t^2$. Apparently, for slow equatorial waves this term has no importance.

The main question is now: how good is this equation for describing the westward slow equatorial waves in a shallow-water system? A solution for (4.3.14) may be sought in the form of a plane wave (see section 2.4), i.e.

$$\psi(x, y, t) = \hat{\psi}(y)e^{i(kx-\omega t)}, \tag{4.3.15}$$

The first-order derivatives with respect to x and t are given by

$$\frac{\partial \psi}{\partial x} = ik\hat{\psi}e^{i(kx-\omega t)}, \tag{4.3.16}$$

$$\frac{\partial \psi}{\partial t} = -i\omega\hat{\psi}e^{i(kx-\omega t)}. \tag{4.3.17}$$

The second-order derivatives with respect to x and y give

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2\hat{\psi}e^{i(kx-\omega t)},$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \hat{\psi}}{\partial y^2}e^{i(kx-\omega t)}.$$

Thus,

$$\frac{\partial}{\partial t}\nabla_z^2\psi = -i\omega\left(\frac{\partial^2 \hat{\psi}}{\partial y^2} - k^2\hat{\psi}\right)e^{i(kx-\omega t)}. \tag{4.3.18}$$

The derivatives (4.3.16), (4.3.17), and (4.3.18) can be substituted in the potential vorticity equation (4.3.14). That yields

$$-i\omega\left(\frac{\partial^2 \hat{\psi}}{\partial y^2} - k^2\hat{\psi}\right)e^{i(kx-\omega t)} + \frac{\beta^2 y^2}{c^2}i\omega\hat{\psi}e^{i(kx-\omega t)} + \beta ik\hat{\psi}e^{i(kx-\omega t)} = 0.$$

After dividing the whole expression by $-i\omega e^{i(kx-\omega t)}$ we then obtain

$$\frac{\partial^2 \hat{\psi}}{\partial y^2} - k^2\hat{\psi} - \frac{\beta^2 y^2}{c^2}\hat{\psi} - \frac{k\beta}{\omega}\hat{\psi} = 0,$$

and after some rearrangements the expression becomes

$$\frac{\partial^2 \hat{\psi}}{\partial y^2} - \frac{\beta^2 y^2}{c^2}\hat{\psi} - k\left(k + \frac{\beta}{\omega}\right)\hat{\psi} = 0. \tag{4.3.19}$$

Again, consider a scaled meridional coordinate ξ to obtain a standard form. As in section 4.2, ξ is defined by

$$\xi = \left(\frac{\beta}{c}\right)^{1/2} y.$$

After substituting the expression for ξ, the differential equation in scaled coordinates becomes

$$\frac{\partial^2 \hat{\psi}}{\partial \xi^2} - \xi^2\hat{\psi} - \frac{c}{\beta}k\left(k + \frac{\beta}{\omega}\right)\hat{\psi} = 0. \tag{4.3.20}$$

It has the same form as (4.2.23) and can be written as

$$\frac{\partial^2 \hat{\psi}}{\partial \xi^2} - \xi^2\hat{\psi} + (2n+1)\,\hat{\psi} = 0, \tag{4.3.21}$$

with

$$2n+1 = -\frac{c}{\beta}k\left(k+\frac{\beta}{\omega}\right). \tag{4.3.22}$$

Its solution is

$$\hat{\psi} = \hat{\psi}_n(\xi) = \hat{\psi}_0 2^{-n/2} e^{-\xi^2/2} H_n(\xi), \tag{4.3.23}$$

where $\hat{\psi}_0$ is the amplitude of the streamfunction at the equator (m^2s^{-1}). The term $2^{-n/2}e^{-\xi^2/2}$ is multiplied by one of an infinite set of Hermite polynomial functions H_n (see section 4.2). The full solution for ψ follows after combining (4.3.15) with (4.3.23):

$$\psi = \hat{\psi}_0 2^{-n/2} e^{-\xi^2/2} H_n(\xi)\cos(kx-\omega t) \tag{4.3.24}$$

With (4.3.24) and expressions for velocity and height perturbations [see (4.3.12), (4.3.13), and (4.3.11)] we can solve the corresponding expressions for u, v, and η, i.e.

$$u = -\frac{\partial\psi}{\partial y} = -\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial y} = -\left(\frac{\beta}{c}\right)^{1/2}\hat{\psi}_0 2^{-n/2}e^{-\xi^2/2}\left[-\xi H_n(\xi)+2nH_{n-1}(\xi)\right]\cos(kx-\omega t),$$

$$v = \frac{\partial\psi}{\partial x} = -k\hat{\psi}_0 2^{-n/2}e^{-\xi^2/2}H_n(\xi)\sin(kx-\omega t),$$

$$\eta = \frac{\beta y}{g}\psi = \frac{\beta\xi}{g}\left(\frac{c}{\beta}\right)^{1/2}\hat{\psi}_0 2^{-n/2}e^{-\xi^2/2}H_n(\xi)\cos(kx-\omega t).$$

Assume $-k\hat{\psi}_0 = \hat{v}_0$ and make sure the expressions are in the same phase as the expressions in section 4.2. Subsequently, the set of equations for u, v, and η becomes

$$u = \frac{\hat{v}_0}{k}\left(\frac{\beta}{c}\right)^{1/2} 2^{-n/2}e^{-\xi^2/2}\left[-\xi H_n(\xi)+2nH_{n-1}(\xi)\right]\sin(kx-\omega t), \tag{4.3.25}$$

$$v = \hat{v}_0 2^{-n/2}e^{-\xi^2/2}H_n(\xi)\cos(kx-\omega t), \tag{4.3.26}$$

$$\eta = -\frac{\beta\xi}{g}\frac{\hat{v}_0}{k}\left(\frac{c}{\beta}\right)^{1/2} 2^{-n/2}e^{-\xi^2/2}H_n(\xi)\sin(kx-\omega t). \tag{4.3.27}$$

One can check that both equations for v are now identical [see (4.2.27) and (4.3.26)].

By rearranging the terms in (4.3.22), the dispersion relation for the slow westward waves becomes

$$\boxed{-k^2 - \frac{\beta k}{\omega} = (2n+1)\frac{\beta}{c}}, \tag{4.3.28}$$

or

$$\boxed{\omega = \frac{-\beta k}{k^2+(2n+1)\frac{\beta}{c}}}. \tag{4.3.29}$$

We see that the dispersion relation, with approximated nondivergent velocity field, is exactly the same as neglecting term ω^2/c^2 in the original dispersion relation (4.2.37). Gill (1982) already suggests such approximation can be made to describe the equatorial Rossby waves. However, we now know the actual physical foundation that justifies this assumption. Our next goal is to investigate how good the slow westward equatorial waves can be described using this approximated solution.

4.4 Results

In the previous section we acquired a new approximated differential equation (4.3.19) written in terms of ψ, using the notion that perturbation of horizontal divergence is very small. By comparing the results with the original differential equation (4.2.21) we want to know how good the slow branch of the westward waves can be described. In the first subsection we compare the corresponding dispersion curves for two types of westward waves: the slow equatorial Rossby wave and the slow branch of the mixed Rossby-gravity wave. In the second subsection the two differential equations are compared for a Rossby wave by analyzing contour plots of height, velocity, divergence, and vorticity. Finally, we compare between different mode numbers the meridional cross-sections of u, v, and η.

4.4.1 Dispersion Diagrams

We analyze the original dispersion equation with the approximated equation by plotting the corresponding curves in a dispersion diagram. We look in particular to two types of waves that are close to a geostrophic balance: the Rossby waves and the slow branch of the mixed Rossby-gravity wave.

In Figure 4.2, curves from Rossby waves with modes $n = 1, 2, 10$ are visible. The red curves are determined with the original dispersion equation (4.2.37) and the blue curves are approximated using zero divergence (4.3.28). One can observe that there are no significant differences between these two solutions. The biggest error occurs for $n = 1$, where a small discrepancy is visible for the highest frequencies. The absence of term ω^2/c^2 is most prominent here. For $n = 10$, no error is visible, the blue curve is on top of the red one. Overall we can say that the approximated dispersion relation is describing the Rossby waves extraordinary well. Its maximum error is less than 2% (Gill, 1982).

Besides Rossby waves, we are also interested in the slow branch of the mixed Rossby-gravity wave. We like to know for which region of zonal wave numbers motions stay close to geostrophic balance. In Figure 4.3, curves for mixed Rossby-gravity wave are plotted. Again, the red curve is determined with the original dispersion equation (4.2.37) and the blue curve is approximated using zero divergence (4.3.28). As one can observe, the approximated equation only holds if the zonal wave number is larger than -2.5. That means our approximated equation can only describe the westward mixed Rossby-gravity wave for wave lengths smaller than roughly 9200 km (assuming shallow-water layer represents the troposphere).

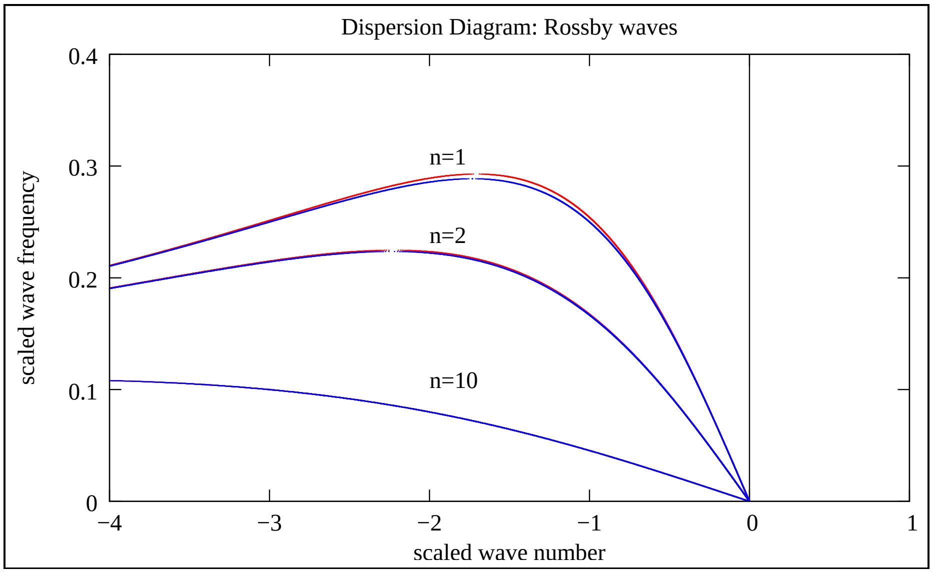


Figure 4.2: *Dispersion equations compared for Rossby waves with n=1,2,10. The Rossby wave determined with the original dispersion equation is red and the approximated Rossby wave is blue. Because the approximated Rossby wave lacks a term with ω, the biggest difference occurs at the highest frequency for n=1. Horizontal axis represents the scaled wave number in units of $(\beta/c)^{1/2}$ and vertical axis represents the scaled wave frequency in units of $(\beta c)^{1/2}$.*

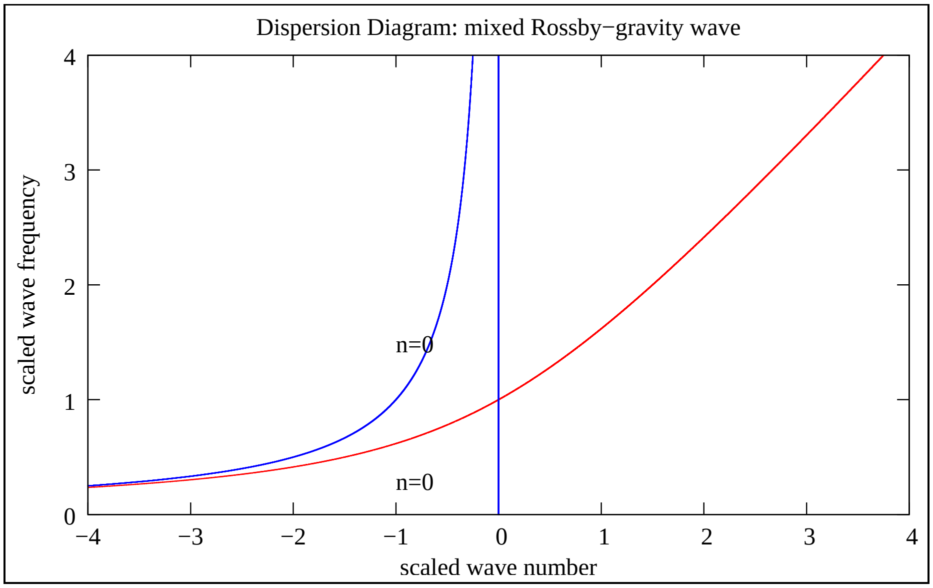


Figure 4.3: *Dispersion equations describing a mixed Rossby-gravity wave (n=0). The mixed Rossby-gravity wave determined with the original dispersion equation is red and the approximated mixed Rossby-gravity wave is blue. Horizontal axis represents the scaled wave number in units of $(\beta/c)^{1/2}$ and vertical axis represents the scaled wave frequency in units of $(\beta c)^{1/2}$.*

4.4.2 Horizontal Contour Plots

From the literature [e.g. Gill (1982)] it is known that the field in an equatorial Rossby wave is in quasi-geostrophic balance. To study this further contour plots are made of the original and approximated solutions. Figures 4.4 and 4.5 show height contour plots of a scaled Rossby wave ($n = 1$, $\tilde{k} = k(c/\beta)^{1/2} = -1$) determined respectively with the original solution (see section 4.2) and the approximated solution (see section 4.3). The contours represent the height anomalies and the arrows represent the horizontal motion. We can speak of geostrophic balance if the flow is going along the height contour lines. For both solutions we take $v_0 = 1$ ms^{-1}, but it can assume any value because the equations are linear. In both figures we can observe a certain degree of geostrophic balance, however, there are some differences. In the original solution, motions seem to be out of balance close to the equator, where height contour lines are crossed by arrows. Nonetheless, at higher latitudes, the motion is indeed close to a balanced state. In the approximated solution we observe geostrophic state at all latitudes, even around the equator. Our approximated equation for η (4.3.27) persists that no height anomaly can exist at the equator ($y = 0$) and thus the flow can remain in geostrophic balance.

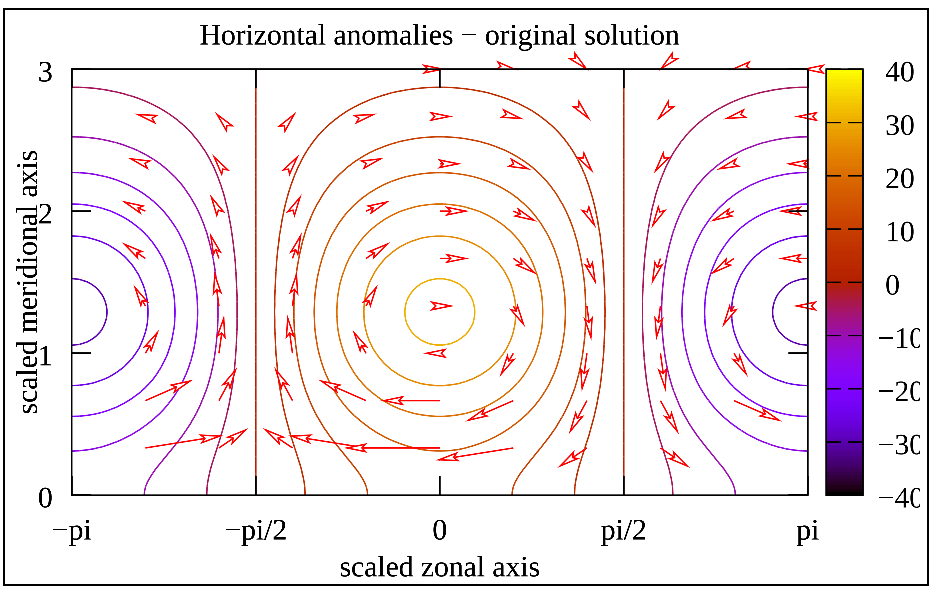


Figure 4.4: *Height anomalies (contour) and horizontal velocity (vectors) for the equatorial Rossby wave with n=1 and $\tilde{k}$=-1, using the original differential equation.*

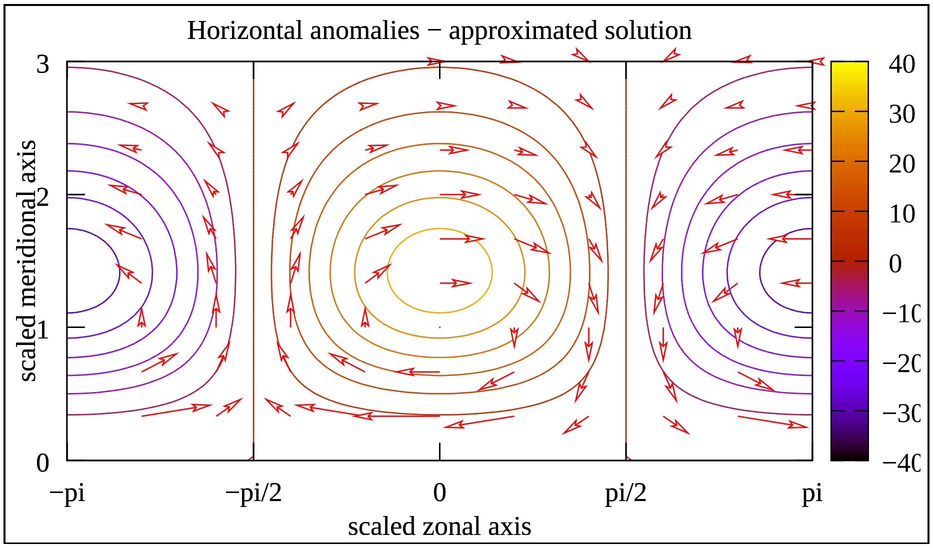


Figure 4.5: *Height anomalies (contour) and horizontal velocity (vectors) for the equatorial Rossby wave with n=1 and* $\tilde{k}$*=-1, using the approximated differential equation.*

In section 4.3 we have approximated the slow equatorial waves by assuming that the velocity field is nondivergent. To confirm if such strong constraint is really allowed one can choose to analyze the divergence and vorticity. Holton (1992) argues that synoptic scale motions must be quasi-nondivergent, where horizontal divergence must be small compared to the vorticity. Figures 4.6 and 4.7 illustrate the divergence and vorticity fields determined with the original solution. At most locations the magnitude of the divergence is order of ten smaller than the vorticity. It leads to the conclusion that the motions are indeed close to a state of nondivergence. However, we observe at the equator, close to locations $x = -\pi/2$ and $x = \pi/2$ that vorticity is actually smaller than divergence. It coincides with the out of balance motions observed in Figure 4.4.

The negative vorticity zones correspond with anticyclonic motions and the positive vorticity zones with cyclonic motions. Additionally, we observe in Figure 4.6 negative divergence (i.e. convergence) zones on westside of the height anomaly and positive divergence on the eastside. These divergence and convergence zones are associated with the westward traveling behavior of Rossby waves. On the westside more air is coming in than air is going out, i.e. a positive height anomaly must grow. However, on the eastside the opposite happens; a negative height anomaly must grow because more air is going out.

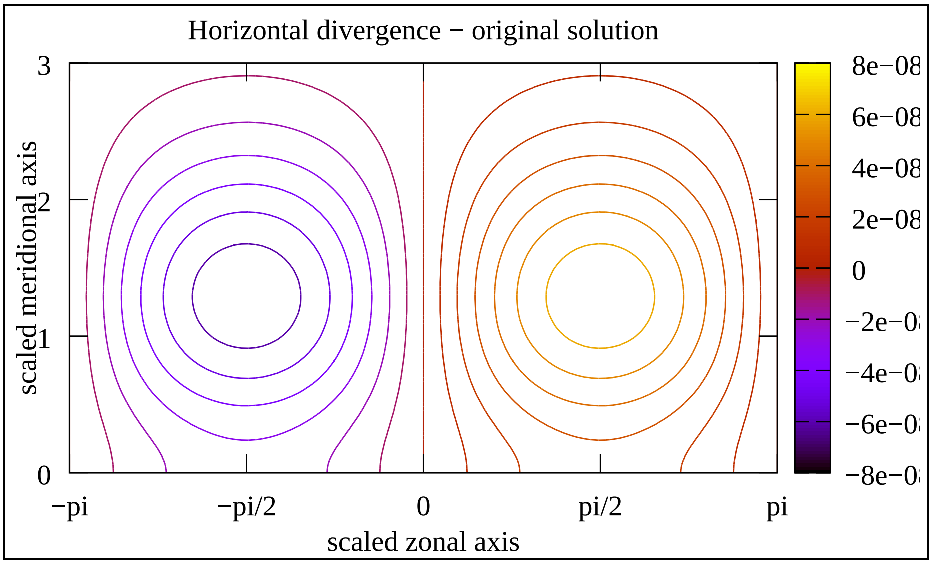


Figure 4.6: *Horizontal divergence for the equatorial Rossby wave with n=1 and $\tilde{k}$=-1, using the original differential equation.*

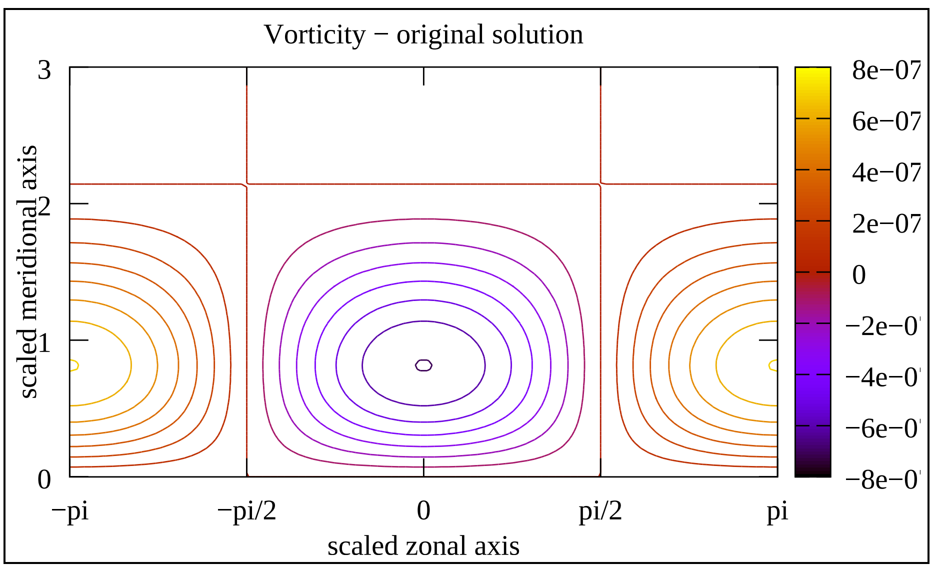
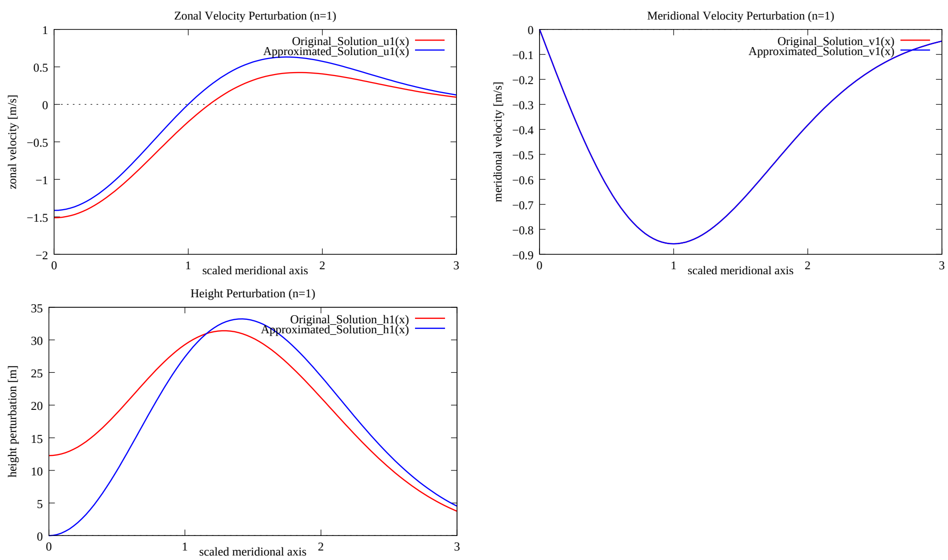


Figure 4.7: *Vorticity for the equatorial Rossby wave with n=1 and $\tilde{k}$=-1, using the original differential equation.*

4.4.3 Meridional Cross-sections

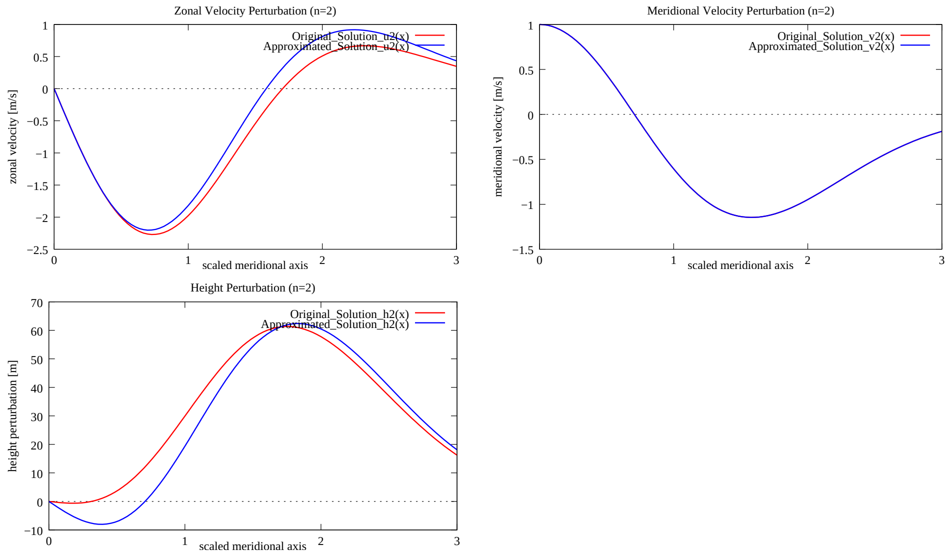
In this subsection we examine what effect the increasing mode number has on the horizontal velocity components (u and v) and height anomalies (η) of a scaled Rossby wave ($\tilde{k} = -1$). For three different mode numbers ($n = 1, 2, 4$) we determine, for the original and approximated solutions, meridional cross-sections for u, v, and η. In previous Figures 4.4 and 4.5 cross-sections are taken along location $x = 0$ for u and η. Because v remains zero along $x = 0$ we choose to take its cross-section along x=$\pi/2$ instead. These cross-sections are plotted in Figure 4.8. The red line illustrates the variable determined with the original solution and the blue line illustrates the variable determined with the approximated solution. We take for both solutions $v_0 = 1$ ms^{-1}. It is clear that for u and η the differences between the two solutions remain small, especially for higher mode numbers. For v the blue line lies on top of the red line, because the solutions are identical. The higher mode numbers provide Hermite polynomials of a higher degree and are associated with more complex meridional structures. One can observe that for higher modes a larger part of atmosphere (in poleward direction) is influenced by the Rossby wave. For example, for mode number 1 only one single height anomaly is visible and vanishes slowly around scaled latitude 3 ($\approx$ 11.000 km away from equator if we assume the shallow-water layer represents the troposphere). However, one can see that mode number 4 acquires a more complex height anomaly structure; including a large positive anomaly, which vanishes now far beyond scaled latitude 3.

A



[Figure 4.8 continues on the next page]

B



C

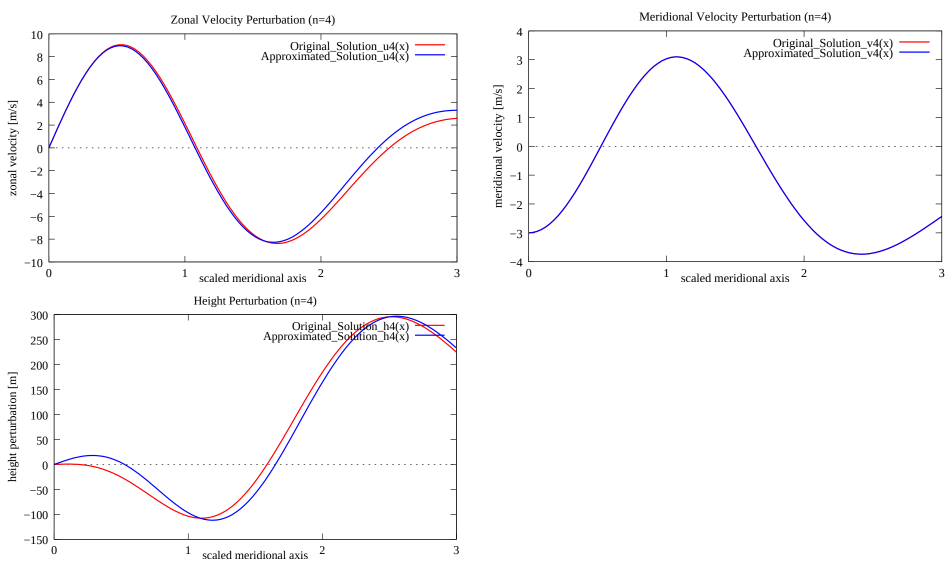


Figure 4.8: *Cross-sections to study the effect of increasing mode number on horizontal velocity components and height anomalies. The variables are determined with the original (red line) and approximated solution (blue line). The first 3 plots (A) are for n=1, (B) are for n=2, and (C) are for n=4. For u and η cross-sections are taken along x=0 of Figures 4.4 and 4.5. However, for v cross-sections are taken along $x=\pi/2$. The x-axis here shows scaled values of the latitude according to expression (4.2.22).*

5 Conclusions and Discussion

In this M.Sc.-thesis linear equatorial waves have been studied, with emphasis on describing their low-frequency branches. These slow waves are rather interesting because the motions are close to geostrophic balance. To approximate the slow waves around the equator a formulation of geostrophic balance is used based on zero divergence, instead of a balance between Coriolis force and pressure gradient force.

To solve this problem analytically, a simplified model is derived in Chapters 2 and 3. The main part of Chapter 2 is concerned with linearizing the three conservation laws of motion, mass, and energy using a background state that is motionless. Subsequently, in Chapter 3 we have proven how the linearized equations assume a barotropic form if the potential temperature of the background is taken to be constant with height. The barotropic form has the same appearance as the corresponding laws in the shallow-water system. In fact, these shallow-water equations, approximated on an equatorial plane, are used to describe linear waves around the equator.

In Chapter 4 we have derived a dispersion relation that describes all types of equatorial waves. The corresponding dispersion diagram reveals there are two types of slow waves that are interesting for our study; the Rossby wave and the slow branch of the mixed Rossby-gravity wave. Using the assumption of zero divergence we derived a form of the potential vorticity equation in terms of a single variable ψ, i.e. a streamfunction. By solving this equation, we have obtained an approximated solution for nondivergent motions.

We found that Rossby waves are well approximated by this solution, especially for higher mode numbers n. Its maximum error is less than 2% for $n = 1$ (Gill, 1982). Furthermore, meridional cross-sections of horizontal velocity components and height anomalies did not reveal significant differences between the approximated and original solutions. In fact, the solutions for v are the same. Unfortunately, for westward mixed Rossby-gravity wave we do observe large errors between the two solutions. The approximation is only valid if zonal wave numbers are large enough, i.e. for wave lengths smaller than 9200 km.

Height contour plots of the approximated Rossby wave have shown us that motions are in quasi-geostrophic balance for all local latitudes around the equator; even exactly at $y = 0$. However, the original solution shows near the equator locations where the motion is actually out of balance. This behavior is also confirmed after studying the divergence and vorticity contour plots. As a result, convergence and divergence zones that correspond with the westward movement of the wave are present on respectively west- and eastside of the height anomaly.

Because we have gained some insight in the equatorial wave theory, we are interested which of the different types of equatorial waves can be retrieved from observations. As part of my internship we will continue the study of equatorial waves. Timmermans (2005) studied, as part of her PhD-thesis, the detection of Kelvin waves using ozone measurements from the GOME-instrument. We want to perform the same kind of study, however we will use the more recent OMI-instrument, which has higher resolution in space and time than GOME.

Appendices

Appendix A: Alternative Form of the Thermodynamic Energy Equation

Derivation of an alternative form for the primitive thermodynamic energy equation is discussed here. The general entropy form of the first law of thermodynamics is given in section 2.2 (2.2.15), i.e.

$$\frac{c_p}{\theta}\frac{D\theta}{Dt} = \frac{c_p}{T}\frac{DT}{Dt} - \frac{R}{p}\frac{Dp}{Dt} = \frac{Q}{T} \equiv \frac{Ds}{Dt}, \tag{A-1}$$

where s is the entropy, c_p is the specific heat of dry air at constant pressure (1004 $\mathrm{JK^{-1}kg^{-1}}$), R is the gas constant for dry air (287 $\mathrm{JK^{-1}kg^{-1}}$), T is the absolute temperature, p is the air pressure, and Q is the rate of heat energy. The potential temperature θ is given as

$$\theta = T\left(\frac{p_r}{p}\right)^{R/c_p}. \tag{A-2}$$

With understanding of the ideal-gas law (2.2.8) it gives

$$\theta = \frac{p}{\rho R}\left(\frac{p_r}{p}\right)^{R/c_p}, \tag{A-3}$$

where p_r is the reference pressure and ρ is the air density. To derive the alternate thermodynamic form, take the natural logarithm of the potential temperature:

$$
\begin{aligned}
\ln(\theta) &= \ln\left(\frac{p}{\rho R}\right) + \frac{R}{c_p}\ln(p_r) - \frac{R}{c_p}\ln(p) = \\
& \ln\left(\frac{p}{R}\right) - \ln(\rho) + \frac{R}{c_p}\ln(p_r) - \frac{R}{c_p}\ln(p) = \\
& \ln(p) - \ln(R) - \ln(\rho) + \frac{R}{c_p}\ln(p_r) - \frac{R}{c_p}\ln(p) = \\
& \left(1 - \frac{R}{c_p}\right)\ln(p) - \ln(R) - \ln(\rho) + \frac{R}{c_p}\ln(p_r) = \\
& \left(1 - \frac{c_p - c_v}{c_p}\right)\ln(p) - \ln(R) - \ln(\rho) + \frac{R}{c_p}\ln(p_r) = \\
& \left(1 - 1 + \frac{c_v}{c_p}\right)\ln(p) - \ln(R) - \ln(\rho) + \frac{R}{c_p}\ln(p_r) = \\
& \frac{c_v}{c_p}\ln(p) - \ln(R) - \ln(\rho) + \frac{R}{c_p}\ln(p_r) = \\
& \frac{1}{\gamma}\ln(p) - \ln(R) - \ln(\rho) + \frac{R}{c_p}\ln(p_r),
\end{aligned}
$$

where c_v is the specific heat of dry air at constant volume (717 $\mathrm{JK^{-1}kg^{-1}}$) and $\gamma = \frac{c_p}{c_v}$. Differentiate $\ln(\theta)$ with respect to time to obtain an expression for the thermodynamic energy. Subsequently, constants $\ln(R)$ and $\frac{R}{c_p}\ln(p_r)$ disappear. After some manipulation the desired form for the thermodynamic equation is deduced:

$$
\begin{aligned}
\frac{D\ln(\theta)}{Dt} &= \frac{1}{\gamma}\frac{D\ln(p)}{Dt} - \frac{D\ln(\rho)}{Dt} = \frac{Q}{c_pT} \\
\frac{1}{\theta}\frac{D\theta}{Dt} &= \frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} = \frac{Q}{c_pT} \\
\frac{\rho}{\theta}\frac{D\theta}{Dt} &= \frac{\rho}{\gamma p}\frac{Dp}{Dt} - \frac{D\rho}{Dt} = \frac{\rho}{c_pT}Q \\
'' &= \frac{1}{\gamma RT}\frac{Dp}{Dt} - \frac{D\rho}{Dt} = \frac{\rho}{c_pT}Q \\
'' &= \frac{1}{c_s^2}\frac{Dp}{Dt} - \frac{D\rho}{Dt} = \frac{\rho}{c_pT}Q \\
\frac{c_s^2\rho}{\theta}\frac{D\theta}{Dt} &= \frac{Dp}{Dt} - c_s^2\frac{D\rho}{Dt} = \frac{c_s^2\rho}{c_pT}Q
\end{aligned}
$$

where $c_s^2 = \gamma RT = \frac{c_p}{c_v}RT$, i.e. the speed of sound squared. If we assume no heat exchange ($Q = 0$, dry-adiabatic), the alternative form of thermodynamic energy equation can be written as

$$
\boxed{\frac{c_s^2\rho}{\theta}\frac{D\theta}{Dt} = \frac{Dp}{Dt} - c_s^2\frac{D\rho}{Dt} = 0}. \tag{A-4}
$$

Appendix B: Alternative Form of the Brunt-Väisälä Frequency

To eliminate the density perturbation in the set of conservation equations it is useful to introduce the Brunt-Väisälä frequency N_0. In textbooks, like for instance Holton (1992), it is written as

$$N_0^2 = g\frac{d\ln\theta_0}{dz}. \tag{B-1}$$

Because the assumption is made that air parcels move dry-adiabatically, potential temperature takes, with the understanding of the ideal gas law (2.2.8), the following form

$$\theta_0 = \frac{p_0}{\rho_0 R}\left(\frac{p_r}{p_0}\right)^{R/c_p}. \tag{B-2}$$

Take the natural logarithm of the potential temperature equation likewise Appendix A. That yields,

$$\ln(\theta_0) = \frac{1}{\gamma}\ln(p_0) - \ln(R) - \ln(\rho_0) + \frac{R}{c_p}\ln(p_r), \tag{B-3}$$

where $\gamma = \frac{c_p}{c_v}$.
Now, take $g\partial/\partial z$ of (B-3). Subsequently, the constants $\ln(R)$ and $\frac{R}{c_p}\ln(p_r)$ disappear. Hence,

$$\begin{aligned} g\frac{d\ln(\theta_0)}{dz} &= \frac{g}{\gamma}\frac{d\ln(p_0)}{dz} - g\frac{d\ln(\rho_0)}{dz} = \\ &\frac{g}{\gamma p_0}\frac{\partial p_0}{\partial z} - \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}. \end{aligned}$$

Substitute the background hydrostatic approximation (2.3.4), thus

$$g\frac{d\ln(\theta_0)}{dz} = -\frac{g^2\rho_0}{\gamma p_0} - \frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}.$$

The ideal gas law (2.3.5) declares $\frac{\rho_0}{p_0} = \frac{1}{RT_0}$. The speed of sound is defined by $c_{0s}^2 = \gamma RT_0$. Therefore, the alternative expression for the Brunt-Väisälä frequency becomes

$$\boxed{N_0^2 = g\frac{d\ln\theta_0}{dz} = \frac{-g}{\rho_0}\frac{\partial\rho_0}{\partial z} - \frac{g^2}{c_{0s}^2}}. \tag{B-4}$$

Appendix C: General Characteristics Isentropic Layer

This appendix derives the dynamics within an isentropic layer, where potential temperature is uniform with height. This derivation is based on the article done by Verkley (2000). Eventually we acquire height-dependency expressions for absolute temperature T, pressure p, and density ρ.

Consider the diagnostic equations for potential temperature, ideal gas law, and the hydrostatic approximation:

$$\theta = T\left(\frac{p_r}{p}\right)^{\kappa}, \tag{C-1}$$

$$\rho = \frac{p}{RT}, \tag{C-2}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{C-3}$$

where $\kappa = R/c_p$. Assumption of potential temperature being constant implies that the absolute temperature decreases with height following the dry-adiabatic lapse rate. Rewrite equation (C-1) as

$$\frac{T}{\theta} = \left(\frac{p}{p_r}\right)^{\kappa}, \tag{C-4}$$

where temperature T acts as a function of pressure p. Differentiating (C-4) with respect to height z gives

$$\frac{\partial}{\partial z}\left(\frac{T}{\theta}\right) = \kappa\left(\frac{p}{p_r}\right)^{\kappa-1}\frac{1}{p_r}\frac{\partial p}{\partial z} = \kappa\left(\frac{p}{p_r}\right)^{\kappa}\frac{1}{p}\frac{\partial p}{\partial z}. \tag{C-5}$$

The term $\frac{1}{p}\frac{\partial p}{\partial z}$ can be rewritten with help of (C-1), (C-2), and (C-3). That gives accordingly

$$\frac{1}{p}\frac{\partial p}{\partial z} = \frac{-g}{RT},$$

$$\frac{1}{T} = \frac{1}{\theta}\left(\frac{p_r}{p}\right)^{\kappa},$$

and that leads to

$$\frac{1}{p}\frac{\partial p}{\partial z} = \frac{-g}{R\theta}\left(\frac{p_r}{p}\right)^{\kappa}. \tag{C-6}$$

By using a scale height $H \equiv \frac{c_p\theta}{g}$, (C-6) takes the following form

$$\frac{1}{p}\frac{\partial p}{\partial z} = \frac{-1}{\kappa H}\left(\frac{p_r}{p}\right)^{\kappa}. \tag{C-7}$$

Substitute (C-7) into (C-5) and find

$$\frac{\partial}{\partial z}\frac{T}{\theta} = \frac{-1}{H}. \tag{C-8}$$

Dependence of the temperature on height can be written as

$$\boxed{T(z) = \theta - \theta\frac{z}{H} = \theta - z\frac{g}{c_p}}. \tag{C-9}$$

Taking the z-derivative of T yields the the adiabatic lapse rate:

$$\frac{\partial T}{\partial z} = \frac{-g}{c_p}. \tag{C-10}$$

Now, rewrite (C-1) and obtain the dependence of pressure on height. It gives

$$\boxed{p(z) = p_r\left(\frac{T(z)}{\theta}\right)^{\frac{1}{\kappa}}}. \tag{C-11}$$

The density at reference pressure level can be found with the ideal gas law (C-2), i.e.

$$\rho_r = \frac{p_r}{R\theta},$$

and it follows that

$$\frac{\rho}{\rho_r} = \frac{p}{p_r}\frac{\theta}{T}. \tag{C-12}$$

Combining (C-11) and (C-12) gives us the dependence of density on height, i.e.

$$\boxed{\rho(z) = \rho_r\left(\frac{T(z)}{\theta}\right)^{\left(\frac{1}{\kappa}-1\right)}}. \tag{C-13}$$

Appendix D: Proving Meridional Velocity Relates to q and r

By examining the equatorial trapped waves in Chapter 4, we have obtained the differential equation for meridional velocity v (4.2.21). As discussed in section 4.2, by introducing a scaled meridional coordinate, a solution for v follows straight forward, i.e.

$$v = \hat{v}_n(\xi)e^{i(kx-\omega t)} = \hat{v}_0 2^{-n/2} e^{-\xi^2/2} H_n(\xi)\cos(kx-\omega t). \tag{D-1}$$

Solutions for zonal velocity u and height perturbation η follow less straight forward. To relate v to other variables, we follow the work done by Gill (1982). He introduced variables q and r:

$$q = \frac{g\eta}{c} + u, \tag{D-2}$$

$$r = \frac{g\eta}{c} - u. \tag{D-3}$$

These quantities emerge after taking the sum and difference between the x-momentum equation (4.2.3) and g/c times the continuity equation (4.2.5):

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + c\frac{\partial v}{\partial y} - \beta y v = 0, \tag{D-4}$$

$$\frac{\partial r}{\partial t} - c\frac{\partial r}{\partial x} + c\frac{\partial v}{\partial y} + \beta y v = 0. \tag{D-5}$$

If v takes the form of (D-1), the corresponding solutions for q and r are

$$q = \hat{v}_0 \frac{(2\beta c)^{1/2}}{(ck-\omega)} 2^{-(n+1)/2} e^{-\xi^2/2} H_{n+1}(\xi)\sin(kx-\omega t), \tag{D-6}$$

$$r = \hat{v}_0 \frac{(2\beta c)^{1/2}}{(ck+\omega)} 2^{-(n-1)/2} e^{-\xi^2/2} n H_{n-1}(\xi)\sin(kx-\omega t). \tag{D-7}$$

First this is proven for q by solving each term of (D-4). It gives

$$\frac{\partial q}{\partial t} = -\omega \frac{(2\beta c)^{1/2}}{(ck-\omega)} \hat{v}_0 2^{-(n+1)/2} e^{-\xi^2/2} H_{n+1}(\xi)\cos(kx-\omega t),$$

$$c\frac{\partial q}{\partial x} = ck \frac{(2\beta c)^{1/2}}{(ck-\omega)} \hat{v}_0 2^{-(n+1)/2} e^{-\xi^2/2} H_{n+1}(\xi)\cos(kx-\omega t),$$

$$c\frac{\partial v}{\partial y} = c\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} = c\left(\frac{\beta}{c}\right)^{1/2} \hat{v}_0 2^{-n/2} e^{-\xi^2/2}\left[-\xi H_n(\xi) + 2nH_{n-1}(\xi)\right]\cos(kx-\omega t),$$

$$\beta y v = c\left(\frac{\beta}{c}\right)^{1/2} \hat{v}_0 2^{-n/2} e^{-\xi^2/2} \xi H_n(\xi)\cos(kx-\omega t).$$

Subsequently, the sum of first two terms acquire the form

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} = \left[H_{n+1}(\xi)\right] 2^{-(n+1)/2} (2\beta c)^{1/2} \hat{v}_0 e^{-\xi^2/2}\cos(kx-\omega t). \tag{D-8}$$

The difference between the last two terms gives

$$c\frac{\partial v}{\partial y} - \beta y v = \left[-2\xi H_n(\xi) + 2nH_{n-1}(\xi)\right] 2^{-n/2} c \left(\frac{\beta}{c}\right)^{1/2} \hat{v}_0 e^{-\xi^2/2} \cos(kx - \omega t). \tag{D-9}$$

It is important to notice that

$$2^{-n/2} c \left(\frac{\beta}{c}\right)^{1/2} = 2^{-(n+1)/2} (2\beta c)^{1/2}.$$

One can check that the sum between (D-8) and (D-9) satisfies zero.

As for q it can also be proven for r. Each term of (D-5) gives

$$\frac{\partial r}{\partial t} = -\omega \frac{(2\beta c)^{1/2}}{(ck + \omega)} \hat{v}_0 2^{-(n-1)/2} e^{-\xi^2/2} n H_{n-1}(\xi) \cos(kx - \omega t),$$

$$c\frac{\partial r}{\partial x} = ck \frac{(2\beta c)^{1/2}}{(ck + \omega)} \hat{v}_0 2^{-(n-1)/2} e^{-\xi^2/2} n H_{n-1}(\xi) \cos(kx - \omega t),$$

$$c\frac{\partial v}{\partial y} = c\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} = c\left(\frac{\beta}{c}\right)^{1/2} \hat{v}_0 2^{-n/2} e^{-\xi^2/2} \left[-\xi H_n(\xi) + 2nH_{n-1}(\xi)\right] \cos(kx - \omega t),$$

$$\beta y v = c\left(\frac{\beta}{c}\right)^{1/2} \hat{v}_0 2^{-n/2} e^{-\xi^2/2} \xi H_n(\xi) \cos(kx - \omega t).$$

The difference between the first two terms gives

$$\frac{\partial r}{\partial t} - c\frac{\partial r}{\partial x} = \left[-nH_{n-1}(\xi)\right] 2^{-(n-1)/2} (2\beta c)^{1/2} \hat{v}_0 e^{-\xi^2/2} \cos(kx - \omega t). \tag{D-10}$$

The sum of the last two terms gives

$$c\frac{\partial v}{\partial y} + \beta y v = \left[nH_{n-1}(\xi)\right] 2^{-n/2} 2c \left(\frac{\beta}{c}\right)^{1/2} \hat{v}_0 e^{-\xi^2/2} \cos(kx - \omega t). \tag{D-11}$$

Notice that

$$2^{-n/2} 2c \left(\frac{\beta}{c}\right)^{1/2} = 2^{-(n-1)/2} (2\beta c)^{1/2},$$

so the sum of (D-10) and (D-11) satisfies zero again.

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