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Description of a two level steady-state model.

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Abstract.

This paper is a technical description of a linear two-level steady-state primitive equation model. The model has been designed for the computation of stationary planetary waves arising from thermal forcing at the earth's surface. In particular we are interested in the explanation and prediction of the anomalies in monthly mean circulation patterns (Opsteegh and Van den Dool, 1980).

In the first part of the paper the model equations are derived by eliminating temperature, the zonal and the vertical velocities from the original set of equations. Next it is described how the matrix of the resulting linear set of equations is build up. Finally the computer program is given.

DESCRIPTION OF A TWO LAYER STEADY-STATE MODEL

1. Basic equations and discretization

The model equations can be derived by first applying a time average to the basic laws governing the atmosphere. Apart from new unknowns that appear in the momentum and thermodynamic equations as additional forcing and heating, the resulting equations look very much the same as the original ones (Opsteegh and Van den Dool, 1979). In order to construct model equations, we follow roughly the approach by Egger (1976).

The equations for the monthly mean flow are linearized around a basic state, which is the normal (or long-term) monthly mean circulation. For example

$$\overline{U} = U_n + \hat{u} . \qquad (1)$$

Here, $\overline{U}$, U_n and $\hat{u}$ are the monthly mean zonal wind for a particular month, the normal monthly mean zonal wind and the anomalous component. All three quantities depend on latitude, longitude and height.

We now substitute (1) in the equations for the time-averaged quantities and neglect the tendency term and terms nonlinear in $\hat{u}$, $\hat{v}$ etc. Further we subtract from the equations the terms describing the normal monthly mean atmospheric state. We then obtain a set of linear stationary equations in the deviations ($\hat{u}$, $\hat{v}$, etc.).

In order to derive tractable model equations, the normal flow is divided further into a zonally symmetric and a zonally asummetric part. For example

$$U_n = U_{sn} + U_{an} . \qquad (2)$$

Here, U_{sn} and U_{an} are the symmetric and the asymmetric part of the normal zonal wind. After substitution of (2) we get the final anomaly (or perturbation) equations. Because we will not deal in this study with the terms describing the interaction of the perturbations with both the mean meridional flow and the normal standing eddies, we transfer these terms to the right-hand side of the perturbation equations. The equations are expressed in curvilinear coordinates with pressure as vertical coordinate. They read as follows:

Zonal momentum balance

$$U_{sn} \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial U_{sn}}{\partial y} + \hat{\omega} \frac{\partial U_{sn}}{\partial p} - f\hat{v} + \frac{\partial \hat{\Phi}}{\partial x} - U_{sn} \hat{v} \frac{\tan\phi}{a} - \hat{F}_{Wx} = \hat{F}_{Ex} - \widehat{MU} - \widehat{SU} \quad (3)$$

Meridional momentum balance

$$U_{sn} \frac{\partial \hat{v}}{\partial x} + f\hat{u} + \frac{\partial \hat{\Phi}}{\partial y} + 2U_{sn} \hat{u} \frac{\tan\phi}{a} - \hat{F}_{Wy} = \hat{F}_{Ey} - \widehat{MV} - \widehat{SV} \quad (4)$$

First law of thermodynamics

$$U_{sn} \frac{\partial \hat{T}}{\partial x} + \hat{v} \frac{\partial T_{sn}}{\partial y} - \sigma_{sn} \hat{\omega} = \frac{\hat{Q}}{c_p} + \hat{Q}_E - \widehat{MT} - \widehat{ST} \ . \quad (5)$$

Continuity equation

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v} \cos\phi}{\cos\phi \ \partial y} + \frac{\partial \hat{\omega}}{\partial p} = 0 \ . \quad (6)$$

Hydrostatic approximation

$$\frac{\partial \hat{\Phi}}{\partial p} = - \hat{\alpha} \ . \quad (7)$$

Equation of state

$$p\hat{\alpha} = R \hat{T} \ . \quad (8)$$

The symbols u, v, ω, x, y, p, f, Φ, T, α and R have their conventional meaning. $\hat{F}_{Wx}$ and $\hat{F}_{Wy}$ are the dissipation terms and will be specified below.

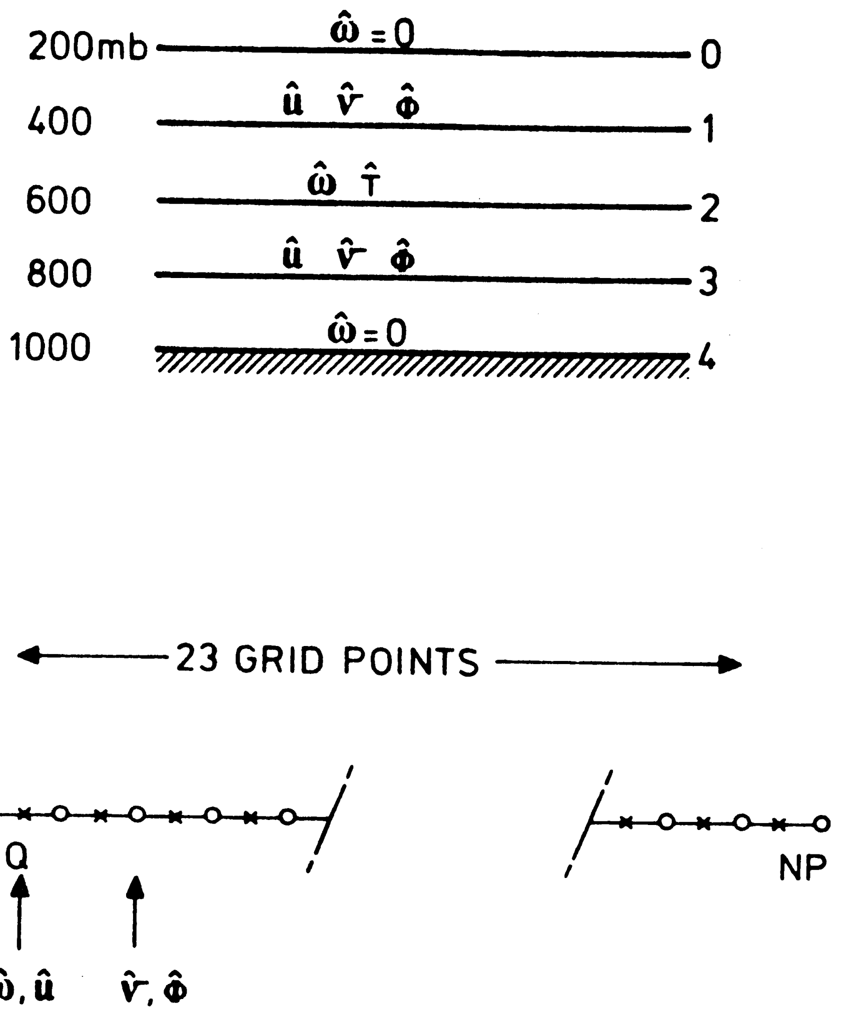

Fig. 1. Discretization of the two-level model in the vertical (upper part) and in the meridional direction (lower part).

$\hat{F}_{Ex}$, $\hat{F}_{Ey}$ and $\hat{Q}_E$ are the internal eddy sources of momentum and heat. $\widehat{MU}$, $\widehat{MV}$ and $\widehat{MT}$ describe the interaction of the perturbations with the mean meridional flow. $\widehat{SU}$, $\widehat{SV}$ and $\widehat{ST}$ describe the interaction with the normal standing eddies. Of all the terms at the right-hand side only the anomalous diabatic heating $\hat{Q}/c_p$ will be retained.

Vertical discretization

The vertical discretization of the model is shown in the upper part of Fig. 1. The momentum and continuity equations are applied at level 1 (400 mbar) and 3 (800 mbar), whereas the thermodynamic equation is applied at level 2 (600 mbar). In the last equation $\hat{T}_2$ is eliminated and expressed in $\hat{\Phi}_1$ and $\hat{\Phi}_3$, with the hydrostatic equation and the equation of state.

The boundary condition at levels 0 and 4 is:

$$\hat{\omega} = 0 \; . \tag{9}$$

We now have a set of seven equations in the seven variables $\hat{u}_1$, $\hat{u}_3$, $\hat{v}_1$, $\hat{v}_3$, $\hat{\Phi}_1$, $\hat{\Phi}_3$, and $\hat{\omega}_2$. For the friction terms Egger's (1976) approach is followed. So at level 3 we have surface friction and vertical diffusion, while at level 1 there is only vertical diffusion. In the zonal momentum equations they have the following form:

$$\hat{F}_{Wx3} = K_D(\hat{u}_1 - \hat{u}_3) - K_W\hat{u}_3 \; , \tag{10}$$

$$\hat{F}_{Wx1} = -K_D(\hat{u}_1 - \hat{u}_3) \; . \tag{11}$$

K_D and K_W are vertical diffusion and surface friction coefficients respectively. They are constants in the model. The friction terms in the meridional momentum equations are obtained by replacing $\hat{u}_1$ and $\hat{u}_3$ by $\hat{v}_1$ and $\hat{v}_3$ in (10) and (11).

Horizontal discretization

The various terms in the seven model equations contain coefficients that do not depend on longitude. Therefore the perturbation quantities are expanded in Fourier series along latitude circles. In the meridional direction we have chosen a gridpoint representation, with 23 gridpoints between pole and equator. As an example of the Fourier expansion we write:

$$\hat{u}_1 = \sum_{m=1}^{N} \hat{u}_{1m}(\phi)\, e^{-im\lambda} , \tag{12}$$

where $\hat{u}_{1m}$ is a complex coefficient. By substitution of these relations into the model equations we get new equations for the complex Fourier coefficients. These equations are given in section 2. Finally, we have N sets of 14 linear first-order differential equations for the real and imaginary parts of the Fourier coefficients.
The equations are solved on the grid that is shown in the lower parts of Fig. 1. There are 23 gridpoints, indicated as crosses with a distance of slightly less than 4^0. $\hat{v}$ and $\hat{\Phi}$ are formulated at intermediate points, indicated with open circles. The boundary conditions are:

$$\begin{aligned} &\hat{v}\cos(\phi) = 0 \quad \text{at} \quad \phi = \Pi/2 \\ &\frac{\partial\hat{\Phi}}{\partial\phi} = 0 \quad\quad \text{at} \quad \phi = 0 \end{aligned} \tag{13}$$

Here, $\hat{v}\cos\phi$ is a variable which is used in the model, instead of $\hat{v}$; it is zero at the pole. At the equator we use a symmetry condition.

2. The model equations

Zonal momentum balance at level 1:

$$- \frac{im\, U_{sn1}}{a \cos\phi} \hat{u}_{1m} + \frac{\partial U_{sn1}}{a \partial\phi} \hat{v}_{1m} + \left(\frac{\partial U_{sn}}{\partial p}\right)_1 \frac{\hat{\omega}_{2m}}{2} - f\, \hat{v}_{1m}$$

$$- \frac{im\, \Phi_{1m}}{a \cos\phi} - U_{sn1} \frac{\tan\phi}{a} \hat{v}_{1m} + K_D (\hat{u}_{1m} - \hat{u}_{3m}) = (\hat{F}_{Ex} - \widehat{MU} - \widehat{SU})_{1m} \qquad (14)$$

Zonal momentum balance at level 3:

$$- \frac{im\, U_{sn3}}{a \cos\phi} \hat{u}_{3m} + \frac{\partial U_{sn3}}{a \partial\phi} \hat{v}_{3m} + \left(\frac{\partial U_{sn}}{\partial p}\right)_3 \left(\frac{\hat{\omega}_{2m}}{2} + \frac{\hat{\omega}_{4m}}{2}\right) - f\hat{v}_{3m}$$

$$- \frac{im\, \hat{\Phi}_{3m}}{a \cos\phi} - U_{sn3} \frac{\tan\phi}{a} \hat{v}_{3m} - K_D (\hat{u}_{1m} - \hat{u}_{3m}) + K_w\, \hat{u}_{3m} = (\hat{F}_{Ex} - \widehat{MU} - \widehat{SU})_{3m} \qquad (15)$$

Meridional momentum balance at level 1:

$$- \frac{im\, U_{sn1}}{a \cos\phi} \hat{v}_{1m} + f\, \hat{u}_{1m}$$

$$+ \frac{\partial \hat{\Phi}_{1m}}{a \partial\phi} + 2\, U_{sn1} \frac{\tan\phi}{a} \hat{u}_{1m} + K_D (\hat{v}_{1m} - \hat{v}_{3m}) = (\hat{F}_{Ey} - \widehat{MV} - \widehat{SV})_{1m} \qquad (16)$$

Meridional momentum balance at level 3:

$$- \frac{im\, U_{sn3}}{a \cos\phi} \hat{v}_{3m} + f\, \hat{u}_{3m}$$

$$+ \frac{\partial\, \hat{\Phi}_{3m}}{a\, \partial\phi} + 2\, U_{sn3} \frac{\tan\phi}{a} \hat{u}_{3m} - K_D (\hat{v}_{1m} - \hat{v}_{3m}) + K_w\, \hat{v}_{3m} = (\hat{F}_{Ey} - \widehat{MV} - \widehat{SV})_{3m} \qquad (17)$$

Thermodynamic balance at level 2:

$$\frac{im\ p_2\ U_{sn2}}{2\ R\ \Delta\ p\ a\ \cos\ \phi}\ (\hat{\Phi}_{3m}-\hat{\Phi}_{1m}) + \frac{\partial\ T_{sn2}}{a\ \partial\phi}\left(\frac{\hat{v}_{1m}+\hat{v}_{3m}}{2}\right)$$

$$-\sigma_{sn2}\ \hat{\omega}_{2m} - \hat{Q}_{2m}/c_p = (\hat{Q} - \widehat{MT} - \widehat{ST})_{2m} \tag{18}$$

where p_2 and Δp are 600 mbar and 200 mbar respectively.

Continuity equation at level 1:

$$-\frac{im\ \hat{u}_{1m}}{a\ \cos\phi} + \frac{\partial(\hat{v}_{1m}\ \cos\phi)}{a\ \cos\ \partial\phi} + \frac{\hat{\omega}_{2m}}{2\Delta p} = 0\ . \tag{19}$$

Continuity equation at level 3:

$$-\frac{im\ \hat{u}_{3m}}{a\ \cos\phi} + \frac{\partial(\hat{v}_{3m}\ \cos\phi)}{a\ \cos\ \partial\phi} - \frac{(\hat{\omega}_{2m}-\hat{\omega}_{4m})}{2\Delta p} = 0\ . \tag{20}$$

Where $\hat{\omega}_{4m}$ is prescribed and usually is taken to be zero.
These seven equations have been reduced to four by eliminating $\hat{u}$ and $\hat{\omega}$ by substitution of (18) to (20) in (14) to (17). Expressed in the real and imaginary parts of the m'th Fourier coefficients we finally have N sets of eight equations. The equations can be solved by discretization of the differential equations and inverting the matrix of the resulting system of linear equations.

3. Method of solution

As mentioned in section 2, $\hat{u}$ and $\hat{\omega}$ have been eliminated by substitution of (18) to (20) in (14) to (17). For convenience we will drop the subscript m. From (18) it follows:

$$\hat{\omega}_2 = \frac{1}{a\,\sigma_{sn2}\cos\phi} \cdot \frac{\partial}{\partial\phi}\left(\frac{T_{sn1}+T_{sn3}}{2}\right) \cdot \left(\frac{\hat{v}_1^{*}+\hat{v}_3^{*}}{2}\right) +$$

$$i\frac{m}{a\,\sigma_{sn2}\cos\phi} \cdot \left(\frac{U_{sn1}+U_{sn3}}{2}\right) \cdot \frac{p_2}{2\,R\,\Delta\,p} \cdot (\hat{\Phi}_3-\hat{\Phi}_1) - \frac{1}{\sigma_{sn2}} \cdot (\hat{q}_2+\hat{x}_{32})$$

with:

$$\hat{v}^{*} = \hat{v}\cos\phi$$

$$\hat{x}_{32} = (\hat{Q}_E - MT - ST)_2 \; .$$

If we define:

$$k_1 = \frac{1}{4\,\sigma_{sn2}} \cdot \frac{\partial}{\partial\phi}\,(T_{sn1}+T_{sn3})$$

$$k_2 = \frac{p_2}{4\,\sigma_{sn2}\,\Delta\,p\,R} \cdot (U_{sn1}+U_{sn3})$$

$$\hat{\Phi}_1^{*} = i\,m\,\hat{\Phi}_1$$

$$\hat{\Phi}_3^{*} = i\,m\,\hat{\Phi}_3$$

we can write:

$$\hat{\omega}_2 = \frac{k_1}{a\cos\phi} \cdot (\hat{v}_1^{*}+\hat{v}_3^{*}) + \frac{k_2}{a\cos\phi} \cdot (\hat{\Phi}_3^{*}-\hat{\Phi}_1^{*})$$

$$- \frac{1}{\sigma_{sn2}} \cdot (\hat{q}_2+\hat{x}_{32}) \qquad (21)$$

From (19) it follows:

$$i\ m\ \hat{u}_1 = \frac{\partial}{\partial\phi}\hat{v}_1{}^* + \frac{a\ \cos\phi}{2\Delta p} \cdot \hat{\omega}_2 \ .$$

Substitution of (21) for $\hat{\omega}_2$ gives:

$$\hat{u}_1{}^* = \frac{\partial}{\partial\phi}\hat{v}_1{}^* + \frac{k_1}{2\Delta p} \cdot (\hat{v}_1{}^* + \hat{v}_3{}^*) + \frac{k_2}{2\Delta p} \cdot (\hat{\Phi}_3{}^* - \hat{\Phi}_1{}^*)$$

$$- \frac{a\ \cos\phi}{2\ \Delta\ p\ \sigma_{sn2}} \cdot (\hat{q}_2 + \hat{x}_{32}) \qquad (22)$$

where

$$\hat{u}_1{}^* = i\ m\ \hat{u}_1 \ .$$

Elimination of $\hat{u}_3$ from (20) gives:

$$i\ m\ \hat{u}_3 = \frac{\partial}{\partial\phi}\hat{v}_3{}^* - \frac{a\ \cos\phi}{2\Delta p} \cdot (\hat{\omega}_2 - \hat{\omega}_4) \ .$$

Substituting (21) for $\hat{\omega}_2$ we get:

$$\hat{u}_3{}^* = \frac{\partial}{\partial\phi}\hat{v}_3{}^* - \frac{k_1}{2\Delta p} \cdot (\hat{v}_1{}^* + \hat{v}_3{}^*) - \frac{k_2}{2\Delta p} \cdot (\hat{\Phi}_3{}^* - \hat{\Phi}_1{}^*)$$

$$+ \frac{a\ \cos\phi}{2\ \sigma_{sn2}\ \Delta\ p} \cdot (\hat{q}_2 + \hat{x}_{32}) + \frac{a\ \cos\phi}{2\Delta p} \cdot \hat{\omega}_4 \qquad (23)$$

The equations (21), (22) and (23) are substituted in (14) to (17), while the friction and vertical diffusion terms are neglected for the moment. This leads to the following set of equations:

1. Zonal momentum balance at level 1

$$- U_{sn1} \cdot \frac{\partial}{\partial\phi}\hat{v}_1{}^* + (k_1 k_4 + \frac{\partial}{\partial\phi} U_{sn1} - af - U_{sn1}\ \tan\phi) \cdot \hat{v}_1{}^*$$

$$+ (k_1 k_4) \cdot \hat{v}_3{}^* + (-1 - k_2 k_4) \cdot \hat{\Phi}_1{}^* + (k_2 k_4) \cdot \hat{\Phi}_3{}^* =$$

$$a \cos\phi \cdot (\hat{x}_{11} + \frac{k_4}{\sigma_{sn2}} \cdot (\hat{q}_2 + \hat{x}_{32})) \cdot \qquad (24)$$

2. Meridional momentum balance at level 1

$$2 \Delta p k_3 \cdot \frac{\partial}{\partial\phi} \hat{v}_1^* + (\frac{m^2 U_{sn1}}{\cos^2\phi} + k_1 k_3) \hat{v}_1^* + (k_1 k_3) \cdot \hat{v}_3^*$$

$$+ \frac{\partial}{\partial\phi} \hat{\Phi}_1^* + (-k_2 k_3) \cdot \hat{\Phi}_1^* + (k_2 k_3) \cdot \hat{\Phi}_3^* =$$

$$a \cos\phi \cdot (\frac{i\, m\, \hat{x}_{21}}{\cos\phi} + \frac{k_3}{\sigma_{sn2}} \cdot (\hat{q}_2 + \hat{x}_{32})) \qquad (25)$$

3. Zonal momentum balance at level 3

$$(k_1 k_5) \cdot \hat{v}_1^* - U_{sn3} \cdot \frac{\partial}{\partial\phi} \hat{v}_3^* + (k_1 k_5 + \frac{\partial}{\partial\phi} U_{sn3} - af - U_{sn3} \tan\phi) \cdot \hat{v}_3^*$$

$$+ (-k_2 k_5) \cdot \hat{\Phi}_1^* + (k_2 k_5 - 1) \cdot \hat{\Phi}_3^* =$$

$$a \cos\phi\, (\hat{x}_{13} + \frac{k_5}{\sigma_{sn2}} \cdot (\hat{q}_2 + \hat{x}_{32}) - k_{55} \cdot \hat{\omega}_4) \qquad (26)$$

4. Meridional momentum balance at level 3

$$(-k_1 k_6) \hat{v}_1^* + 2 \Delta p k_6 \cdot \frac{\partial}{\partial\phi} \hat{v}_3^* + (-k_1 k_6 + \frac{m^2 U_{sn3}}{\cos^2\phi}) \cdot \hat{v}_3^*$$

$$+ (k_2 k_6) \cdot \hat{\Phi}_1^* + \frac{\partial}{\partial\phi} \hat{\Phi}_3^* + (-k_2 k_6) \cdot \hat{\Phi}_3^* =$$

$$a \cos\phi\, (\frac{im}{\cos\phi} \cdot \hat{x}_{23} - \frac{k_6}{\sigma_{sn2}} \cdot (\hat{q}_2 + \hat{x}_{32}) - k_6\, \hat{\omega}_4) \qquad (27)$$

where

$$k_3 = \frac{a\ f + 2\ U_{sn1}\ \tan\phi}{2\Delta p}$$

$$k_4 = \frac{1}{2}\left(\frac{\partial}{\partial p} U_{sn}\right)_1 - \frac{U_{sn1}}{2\Delta p}$$

$$k_5 = \frac{1}{2}\left(\frac{\partial}{\partial p} U_{sn}\right)_3 + \frac{U_{sn3}}{2\Delta p}$$

$$k_{55} = \frac{1}{2}\left(\frac{\partial}{\partial p} U_{sn}\right)_3 - \frac{U_{sn3}}{2\Delta p}$$

$$k_6 = \frac{a\ f + 2\ U_{sn3}\ \tan\phi}{2\Delta p}$$

$$\hat{x}_{11} = (\hat{F}_{Ex} - \widehat{MU} - \widehat{SU})_1$$

$$\hat{x}_{21} = (\hat{F}_{Ey} - \widehat{MV} - \widehat{SV})_1$$

$$\hat{x}_{13} = (\hat{F}_{Ex} - \widehat{MU} - \widehat{SU})_3$$

$$\hat{x}_{23} = (\hat{F}_{Ey} - \widehat{MV} - \widehat{SV})_3$$

The equations for the complex fourier coefficients (24) to (27) are split in their real and imaginary components. This system of eight equations is solved simultaneously.

4. Derivation of the matrix of the linear system

The equations (24) to (27) are applied at each grid point. This leads to a system of linear algebraic equations:

$$
\begin{aligned}
[A_{11}]\,\hat{\underline{v}}_{1r}^{*} + [A_{12}]\,\hat{\underline{v}}_{3r}^{*} + [A_{13}]\,\hat{\underline{\Phi}}_{1r}^{*} + [A_{14}]\,\hat{\underline{\Phi}}_{3r}^{*} &= \underline{f}_{1r} \\
[A_{21}]\,\hat{\underline{v}}_{1r}^{*} + [A_{22}]\,\hat{\underline{v}}_{3r}^{*} + [A_{23}]\,\hat{\underline{\Phi}}_{1r}^{*} + [A_{24}]\,\hat{\underline{\Phi}}_{3r}^{*} &= \underline{f}_{2r} \\
[A_{31}]\,\hat{\underline{v}}_{1r}^{*} + [A_{32}]\,\hat{\underline{v}}_{3r}^{*} + [A_{33}]\,\hat{\underline{\Phi}}_{1r}^{*} + [A_{34}]\,\hat{\underline{\Phi}}_{3r}^{*} &= \underline{f}_{3r} \\
[A_{41}]\,\hat{\underline{v}}_{1r}^{*} + [A_{42}]\,\hat{\underline{v}}_{3r}^{*} + [A_{43}]\,\hat{\underline{\Phi}}_{1r}^{*} + [A_{44}]\,\hat{\underline{\Phi}}_{3r}^{*} &= \underline{f}_{4r}
\end{aligned}
\qquad (28)
$$

The 4 matrix equations (28) are the discretized form of equations (24) to (27) for the real components, respectively. The imaginary components obey a completely similar set of linear equations and are decoupled from the real components. The two sets of equations can therefore be solved independently. However, when friction is included, the variables are coupled. In that case the eight equations have to be solved simultaneously. The solution matrix is called RMAT. It has for the case of zero friction the following form:

$$
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & & \bigcirc & & \\
A_{31} & A_{32} & A_{33} & A_{34} & & & & \\
A_{41} & A_{42} & A_{43} & A_{44} & & & & \\
 & & & & A_{11} & A_{12} & A_{13} & A_{14} \\
 & \bigcirc & & & A_{21} & A_{22} & A_{23} & A_{24} \\
 & & & & A_{31} & A_{32} & A_{33} & A_{34} \\
 & & & & A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
\hat{v}_{1r}^{*} \\ \hat{v}_{3r}^{*} \\ \hat{\Phi}_{1r}^{*} \\ \hat{\Phi}_{3r}^{*} \\ \hat{v}_{1i}^{*} \\ \hat{v}_{3i}^{*} \\ \hat{\Phi}_{1i}^{*} \\ \hat{\Phi}_{3i}^{*}
\end{bmatrix}
=
\begin{bmatrix}
f_{1r} \\ f_{2r} \\ f_{3r} \\ f_{4r} \\ f_{1i} \\ f_{2i} \\ f_{3i} \\ f_{4i}
\end{bmatrix}
$$

5. The boundary conditions

The boundary conditions (13) are formulated as separate equations in the system of linear algebraic equations. For this purpose the 24'th row of the matrices A is used. So every 24'th equation is a formulation of a boundary condition.

24: $\hat{v}_{1r}^{*}\ (24) = 0$

48: $\hat{v}_{3r}^{*}\ (24) = 0$

where point 24 is situated at the north pole.

72: $\hat{\phi}_{1r}^{*}\ (1) - \hat{\phi}_{1r}^{*}\ (2) = 0$

96: $\hat{\phi}_{3r}^{*}\ (1) - \hat{\phi}_{3r}^{*}\ (2) = 0$

where point 1 is situated at the equator.
The equations 120, 144, 168 and 192 are similar compared to 24, 48, 72 and 96, but for the imaginary components.

6. The staggered gridsystem

As indicated in Fig. 1, the variables $\hat{V}^*$ and $\hat{\Phi}^*$ are formulated at intermediate points.
However, the equations are applied at the grid points. We therefore substitute for the variables at grid point i:

$$\alpha_i = (\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}})/2 \ ,$$

and for their gradients:

$$\left(\frac{\partial\alpha}{\partial\phi}\right)_i = (\alpha_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}})/\Delta\phi \ ,$$

where α is one of the variables, $i+\frac{1}{2}$ and $i-\frac{1}{2}$ are intermediate points surrounding i.

7. The incorporation of friction and vertical diffusion

When friction is included, additional terms appear in the model equations (24) to (27). In equation (26) the following expression appears in the left hand side of the equation:

$$a \cos\phi \, (-k_d \cdot (\hat{u}_1 - \hat{u}_3) + k_w \cdot \hat{u}_3) \, ,$$

where

$$k_w = \frac{C_D \, g |\mathbf{v}_4|}{R\bar{T}_4} \, .$$

Elimination of $\hat{u}_1$ and $\hat{u}_3$ with use of (22) and (23) gives:

$$\begin{aligned} i \cdot \frac{a \cos\phi}{m} \cdot \{ & k_d \frac{\partial}{\partial\phi} \hat{v}_1^{*} + \frac{k_1 \cdot (2k_d + k_w)}{2\Delta p} \cdot \hat{v}_1^{*} \\ & - (k_d + k_w) \cdot \frac{\partial}{\partial\phi} \hat{v}_3^{*} + \frac{k_1 \cdot (2k_d + k_w)}{2\Delta p} \cdot \hat{v}_3^{*} \\ & - \frac{k_2(2k_d + k_w)}{2\Delta p} \cdot \hat{\Phi}_1^{*} + \frac{k_2(2k_d + k_w)}{2\Delta p} \cdot \hat{\Phi}_3^{*} \\ & - a \cos\phi \, \frac{(2k_d + k_w)}{2 \, \sigma_{sn2} \, \Delta p} \cdot (\hat{q}_2 + \hat{x}_{32}) \} \, . \end{aligned} \tag{29}$$

The last term in this expression is an additional forcing term, which must be transported to the right hand side of the equation. In the real component of equation (26) frictional terms appear with the complex part of the variables and vice versa.

In the same way we can derive additional terms for the other three equations. Equation (24) gives:

$$\begin{aligned} - \frac{i \, a \cos}{m} \{ & k_d \cdot \frac{\partial}{\partial\phi} \hat{v}_1^{*} + \frac{2k_1 k_d}{2\Delta p} \cdot \hat{v}_1^{*} \\ & - \frac{2k_2 k_d}{2\Delta p} \cdot \hat{\Phi}_1^{*} + 2k_2 k_d \cdot \hat{\Phi}_3^{*} \\ & - \frac{2k_d}{2 \, \sigma_{sn2} \, \Delta p} \cdot a \cos\phi \cdot (\hat{q}_2 + \hat{x}_{32}) \} \, . \end{aligned} \tag{30}$$

Equation (25) gives:

$$\frac{i\ m\ a}{\cos\phi} \cdot (k_d \cdot \hat{v}_1^{*} - k_d \cdot \hat{v}_3^{*}) \ . \qquad (31)$$

Equation (27) gives:

$$\frac{i\ m\ a}{\cos\phi} \cdot (k_d\ \hat{v}_1^{*} - (k_d + k_w)\ \hat{v}_3^{*}) \ . \qquad (32)$$

The matrix RMAT now has the following form:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & B_{11} & B_{12} & B_{13} & B_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} & B_{21} & B_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & B_{31} & B_{32} & B_{33} & B_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} & B_{41} & B_{42} & 0 & 0 \\ -B_{11} & -B_{12} & -B_{13} & -B_{14} & A_{11} & A_{12} & A_{12} & A_{14} \\ -B_{21} & -B_{22} & 0 & 0 & A_{21} & A_{22} & A_{23} & A_{24} \\ -B_{31} & -B_{32} & -B_{33} & -B_{34} & A_{31} & A_{32} & A_{33} & A_{34} \\ -B_{41} & -B_{42} & 0 & 0 & A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \hat{v}^{*} \\ \hat{v}_{3r}^{*} \\ \hat{\Phi}_{1r}^{*} \\ \hat{\Phi}_{3r}^{*} \\ \hat{v}_{1i}^{*} \\ \hat{v}_{3i}^{*} \\ \hat{\Phi}_{1i}^{*} \\ \hat{\Phi}_{3i}^{*} \end{bmatrix} = \begin{bmatrix} f_{1r} + F_{1i} \\ f_{2r} \\ f_{3r} - F_{3i} \\ f_{4r} \\ f_{1i} - F_{1r} \\ f_{2i} \\ f_{3i} + F_{3r} \\ f_{4i} \end{bmatrix}$$

8. Inclusion of the heating

The heating scheme is described in Vernekar and Chang (1978). It computes the heating at the mid-level of the model atmosphere (600 mb). The scheme is interactive, which means that the heating depends on the surface conditions as well as on the atmospheric disturbances which are generated by these surface conditions.

The m'th component of the heating in equations (24) to (27) and in the expressions (29) and (30) can be formulated as follows:

$$\hat{q}_2 = A_{(\phi)}\ \hat{T}_2 + \hat{q}_2^{*} \qquad (33)$$

where $\hat{q}_2$, $\hat{T}_2$ and $\hat{q}_2^{*}$ are complex fourier coefficients for the m'th component as usual. A and $\hat{q}_2^{*}$ are known functions of ϕ.

Substitution of (33) in (24) to (27), adding the $\hat{q}_2$ terms in the expressions (29) and (30) and replacing $\hat{T}_2$ by $(\hat{\Phi}_3^{*} - \hat{\Phi}_1^{*})$ with use of the hydrostatic equation gives the following additional terms:

Equation (24):

$$- \frac{i\ k_4\ A^{*}}{m} \cdot a \cos\phi \cdot (\hat{\Phi}_3^{*} - \hat{\Phi}_1^{*})$$
$$- \frac{k_d\ A^{*}}{\Delta p m^2} \cdot a^2 \cos^2\phi \cdot (\hat{\Phi}_3^{*} - \hat{\Phi}_1^{*}) \ . \qquad (34)$$

Equation (25):

$$- \frac{i\ k_3\ A^{*}}{m} \cdot a \cos\phi \cdot (\hat{\Phi}_3^{*} - \hat{\Phi}_1^{*}) \ . \qquad (35)$$

Equation (26):

$$- \frac{i\ k_5\ A^{*}}{m} \cdot a \cos\phi \cdot (\hat{\Phi}_3^{*} - \hat{\Phi}_1^{*})$$
$$+ \frac{(2\ k_d + k_w) . A^{*}}{2 \Delta p m^2} \cdot a^2 \cos^2\phi \cdot (\hat{\Phi}_3^{*} - \hat{\Phi}_1^{*}) \ . \qquad (36)$$

Equation (27):

$$\frac{i\ k_6\ A^*}{m} \cdot a\ \cos\phi\ (\hat{\Phi}_3^* - \hat{\Phi}_1^*)\ , \qquad (37)$$

where

$$A^* = \frac{p_2\ A}{2\ R\ \Delta\ p\ \sigma_{sn2}}\ .$$

When (34) to (37) are added to the model equations (24) to (27), $\hat{q}_2$ in these equations and in the expressions (29) and (30) must be replaced by $\hat{q}_2^*$.

9. A note on the convection parameterization in the heating scheme

The original parameterization of the convection is (Saltzman, 1967):

$$H_4^{(3)} = -(\alpha(\theta - \overline{T}_7) + c) \ .$$

Where θ is the temperature at the surface and $\overline{T}_7$ is the temperature at 700 mb.

In our model we have a level at 600 mb, and therefore we apply a correction:

$$H_4^{(3)} = -(\alpha(\theta - \overline{T}_6) + c - \alpha(\overline{T}_7 - \overline{T}_6)) \ .$$

When climatological mean values for $\overline{T}_7$ and $\overline{T}_6$ at 45N for January conditions are used, the value of this correction is -63.
If we add this value to c, we get -135 instead of -72.

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