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Searching for stationary stable solutions of Euler's equation

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Abstract

In this paper an attempt of finding new stationary stable solutions of the Euler equation is presented. We will look into solutions of the *sinh-Poisson* equation, which lead to stationary stable solutions of the Euler equation. New solutions can be generated by applying a Bäcklund transformation method on known solutions of the *sine-Gordon* equation which is related to the *sinh-Poisson* equation. This method is applied to a circular symmetric solution. Unfortunately we're not able to present any new circular symmetric solutions.

1 Introduction

1.1 2-D Ideal fluid

Before starting to investigate the *sinh-Poisson* equation, it is first pointed out in what way we arrive at this equation from the Euler equation. The latter equation describes the motion of a two-dimensional (2-D) ideal fluid:

$$\frac{\partial \omega}{\partial t} + \vec{u} \cdot \nabla \omega = 0 \tag{1}$$

The incompressibility of the fluid leads to the introduction of the stream-function ψ. The velocity field $\vec{u}$ can then be written as $\vec{u} = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x})$. It can easily be seen that the vorticity field ω, defined by $\omega = \nabla \times \vec{u}$, is connected to ψ by $\omega = \Delta \psi$.

With the introduction of the Jacobian (also known as canonical Poisson bracket),

$$\{a, b\} = a_x b_y - a_y b_x$$

where a_x denotes the partial derivative of a with respect to x, (1) can be written in the form

$$\frac{\partial \omega}{\partial t} + \{\psi, \omega\} = 0 \tag{2}$$

A very interesting property of a 2-D ideal fluid is that the fluid motion conserves infinitely many quantities. A quantity $F(x,t)$ is a conserved quantity if $\frac{dF(x,t)}{dt} = 0$ in the direction of the fluid motion. According to (2) we have:

$$\frac{d}{dt}\int \omega^n d\vec{x} = \int n\omega^{n-1}(\frac{\partial \omega}{\partial t} + \{\psi, \omega\})d\vec{x} = 0$$

which gives us infinitely many conserved quantities.

As mentioned before, we are looking for stationary solutions of (2). In other words, the problem is to solve

$$\{\psi, \Delta\psi\} = 0 \tag{3}$$

If ψ satisfies the relation $\Delta\psi = F(\psi)$, with F an arbitrary continuously differentiable functional, then (3) is satisfied and thereby ψ is a stationary solution of (2).

The question now is which functional F(ψ) will lead to physically interesting stationary solutions of (2). Here comes in some statistical mechanical theory.

1.2 Physical links with *sinh-Poisson* equation

In [1,2,3], the continuous Euler equations are approximated by a system consisting of a large finite number of identical point vortices. Classical mechanical techniques can be applied on this finite Hamiltonian system. The system has only a limited (finite) number of conserved quantities, so the fluid motion does not stay on the "isovortical sheet" prescribed by the initial conditions of the continuous system. The standard statistical analysis for finite systems led to a $\omega - \psi$ relation given by:

$$\omega = \sinh\psi \tag{4}$$

or, written entirely in terms of ψ:

$$\Delta\psi = \sinh\psi \tag{5}$$

As mentioned before, the system described by Euler equations has infinitely many conserved quantities. In [4,6] the statistical mechanics for this system is developed. The approach made here takes into account all the constants of motion. In [4], two levels to describe the system are distinguished. At a microscopic level all vorticity fields having the appropriate initial conditions are considered. For the statistical description of the system averages of those vorticity fields are considered. The small scales of the fields are averaged out. This level of the description is the macroscopic level. (We concentrate on the vorticity fields, but in general all physical quantities can be treated in the same way).

Associated with the two levels an entropy functional $S = k_B log W$ is defined, where S is the entropy of a macrostate, k_B is the Boltzmann constant and W is the volume in phase space of all microstates sharing the same macrostate.

It was shown by R. Robert in [5] that most of the (microscopic) vorticity fields having the same constants of motion as initial condition, are close to a certain macrostate. This is the macrostate that can be reached by maximizing the entropy. It means that the macrostate with maximum entropy has a large probability of being observed, i.e. the most probable state.

It is not obvious that the $\omega - \psi$ relation given by (4) is found for the continuous system. This relation namely is strongly dependent on the way in which the system is discretized and on the initial values of the equations. The macrostate for a certain initial condition corresponds to the $\omega - \psi$ relation as given above. For more information on the statistical aspects of the *sinh-Poisson* equation we refer to [4,6].

Other proofs of the importance of (4) have been found recently with the aid of large computers. An example is given below.

In [1,4,6] the statistical mechanics of a Hamiltonian (inviscid) system is considered. In [7] the viscous system described by the following equations is considered:

$$\frac{\partial\omega^{\pm}}{\partial t} + \vec{u}\cdot\nabla\omega^{\pm} = \nu\nabla^2\omega^{\pm} \tag{6}$$

where $\omega = \omega^+ - \omega^-$ is satisfying the relation $\omega = \nabla \times \vec{u}$, ω^+ and ω^- are the absolute values of the positive and negative vorticity respectively, ν is the viscosity and assuming periodic boundary conditions. Contrary to (1) which has infinitely many conserved quantities in terms of analytical functionals of the vorticity, (6) only possesses two really conserved quantities and one almost conserved quantity. Total positive vorticity and total negative vorticity are conserved, and the energy of the system decays slowly.

The evolution of the system is simulated in a computer experiment. The main result of this simulation is that an initially 'small scale' vortex distribution evolves by like-sign vortex capture to a state in which one posive and one negative vortex remain. When this final situation is reached (after a few hundred eddy turnover times) the $\omega - \psi$ relation is locally described to good approximation by:

$$c\omega = \sinh(|\beta|\psi) \tag{7}$$

where c and β in (7) are proper constants adapted to the initial conditions.

The big question is why the viscous fluid system relaxes to an $\omega - \psi$ relation that coincides with the one obtained from a simple, discretized, inviscid model. There is no obvious reason for this. It is also not clear if other initial values lead to the same $\omega - \psi$ correspondence.

Finally I would like to present an example that is extensively discussed in [8,9].

Euler's equation describing a 2-D ideal fluid flow can be modified in such a way that energy is no longer conserved but all the vorticity invariants are. This can be done by adapting the velocity field slightly, such that energy changes monotonically in the direction of the flow. All vorticity invariants are preserved because the change of the vorticity field is accomplished by advection only, we say then that the flow stays on the same "isovortical sheet".

In 2-D ideal fluid, for a given vorticity distribution there is an upper bound for the kinetic-energy. Because the energy changes monotonically the system will evolve towards a state in which the energy is maximal. If this maximal energy state is an isolated state, a stable state is reached. Arnol'd in [10,11] showed (under certain conditions) that if a stationary point has an extremum of energy, the flow is Lyapunov stable.

It is shown in [9] that the steady states of the Euler equation and of the modified dynamical system coincide but in the latter the stationary stable states have extremal energy (which states can easily be found).

In a numerical simulation this technique was applied to a random initial condition. A stationary state was reached and that showed a non-linear $\omega - \psi$ relation which looks quite like the relation described by (4).

In principle there are infinitely many $\omega - \psi$ relations that lead to stationary stable solutions of Euler's equation. The importance of a $\omega - \psi$ relation given by (4) lies in the fact that this relation may be relevant to viscous systems. Moreover the most probable state and extremal kinetic energy states lead us to solutions of the *sinh-Poisson* equation. Enough reasons to have a closer look at the *sinh-Poisson* equation.

2 The *sine-Gordon* equation

2.1 From *sinh-Poisson* equation to *sine-Gordon* equation

The equation we're interested in, the *sinh-Poisson* equation, is not an equation that is intensively studied. However, there is a way in which we can get information on the *sinh-Poisson* equation by studying another equation, the famous *sine-Gordon* equation. In this procedure a so called Bäcklund transformation is involved.
Roughly said, a Bäcklund transformation is a transformation BT that maps a solution $v(x,t)$ of equation $E_1(v,x,t) = 0$ into a solution $u(x',t')$ of equation $E_2(u,x',t') = 0$, where E_1 and E_2 are partial differential equations (p.d.e.). The two equations $E_1 = 0$ and $E_2 = 0$ don't have to be different equations. It is possible to define a Bäcklund from an equation to itself, in order to find new solutions of that equation. To make things somewhat clearer, I'll give an example below:

Example: let $v(x,t)$ be a solution of the equation $E_1 = 0$,
the *Modified-Korteweg-de-Vries* (*M.K.d.V.*) equation given by:

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

and let equation $E_2 = 0$ be the *Korteweg-de-Vries* (*K.d.V.*) equation given by:

$$u_t + 6uu_x + u_{xxx} = 0$$

Now consider the transformation between $u(x,t)$ and $v(x,t)$

$$u = -v^2 - v_x$$

The Bäcklund transformation between the two given equations can now be defined as follows:

$$(-\frac{\partial}{\partial x} + 2v)(v_t - 6v^2 v_x + v_{xxx}) = u_t + 6uu_x + u_{xxx}$$

This Bäcklund transformation transforms the solution $v(x,t)$ of the *M.K.d.V.* equation into a solution $u(x,t)$ of the *K.d.V.* equation. In this way the Bäcklund transformation gives the possibility of generating a new solution of an equation.

In an article by Leibbrandt [12], it is shown that there is a connection given by a Bäcklund transformation between the *sinh-Poisson* equation given in (5) and the *elliptic sine-Gordon* equation:

$$\Delta\psi = \sin\psi \tag{8}$$

where ψ is a function of (x,t).

Solutions of (8) are used to construct solutions of the *sinh-Poisson* equation by a Bäcklund transformation. Let $u(x,t)$ be a solution of (8). Then a solution $v(x,t)$ of (5) can be generated by the Bäcklund transformation:

$$(\partial_x + i\partial_t)(\frac{u - iv}{2}) = \sin(\frac{u + iv}{2})e^{i\chi} \tag{9}$$

Here χ is a transformation parameter which enables us to generate a family of new solutions.

So, instead of studying the *sinh-Poisson* equation we can concentrate on the *elliptic sine-Gordon* equation. The equation (8) can be transformed into the *sine-Gordon* soliton equation by the transformation:

$$t \longmapsto it$$

which gives (8) the form:

$$(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})\psi = \sin\psi \tag{10}$$

This is the most familiar form of the *sine-Gordon* equation. But having in mind what we want to do with the equation later on we once more bring (10) into another form. By setting:

$$\xi = \tfrac{1}{2}(x + it)$$
$$\eta = \tfrac{1}{2}(x - it)$$

we arrive at the *sine-Gordon* equation in the form:

$$\psi_{\xi\eta} = \sin\psi$$

It is this form of the equation that we are going to study further on.

2.2 The Painlevé property

In the following part we want to explain what it means for an ordinary differential equation (o.d.e.) and a partial differential equation (p.d.e.) to possess the Painlevé property. This property is defined for both types of equations in a slightly different way. If an equation possesses the Painlevé property, this tells us something about the singularities of its solutions. In this way information on the integrability of the equation under consideration is gathered. Second order o.d.e. that possess the Painlevé property for example, are all integrable.

The notion of the Painlevé property for p.d.e. is less clear. The general thought is that in case one is dealing with a p.d.e. , the concepts of Painlevé property and integrability are closely related. The requirement that a p.d.e. have the Painlevé property has been proposed as a necessary condition for an equation to be integrable. If this condition is also sufficient is not known. For more information one is refered to [19]. Furthermore there is a close relationship between the set of p.d.e. on which Bäcklund transformations can be applied and those p.d.e. that are integrable. A precise description is given in [18,19,21]. This gives us confidence that it is possible to find a Bäcklund transformation from an equation to itself whenever that p.d.e. has the Painlevé property. Before defining the Painlevé property for a (p.d.e.), we will first define this property for an o.d.e. where it all started from. A detailed mathematical description of the Painlevé property for an o.d.e. and general information on o.d.e. can be found in [17], a more global description of the Painlevé property for o.d.e. is given in [18,19]. We follow the latter two.

Let us consider the following linear o.d.e. :

$$\frac{d^n w}{dz^n} + P_1(z)\frac{d^{n-1}w}{dz^{n-1}} + \ldots + P_n(z)w = 0 \tag{11}$$

Because the equation is linear in w, all the singularities of a certain solution of (11) must be singularities of the coefficient functions $P_i(z)$. These singularities are fixed singularities because they are independent of the initial values of the problem. In the case of a nonlinear equation not all the singularities need to be fixed. Singularities of a solution dependent on the initial values of the equation are called movable singularities. A movable singularity of a solution of an o.d.e. that is not a pole of the equation is an essential singularity or a branch point.

At the beginning of this century, Painlevé and coworkers classified all second order o.d.e. of the following type:

$$\frac{d^2 w}{dz^2} = F(z, w, \frac{dw}{dz})$$

where F is a function rational in $\frac{dw}{dz}$, algebraic in w and locally analytical in z. They showed that of all possible equations of this form, there are only fifty equations with the property of having no movable essential singularities and no movable branch points. What it meant for an o.d.e. not to have any movable singularities was first pointed out by S.Kowalevski. In two articles [22,23] she showed that the absence of movable singularities accounted for the integrability of an o.d.e.

The property having no movable singularities is called the Painlevé property. Of the fifty equations mentioned above, forty-four were already known at that time. The remaining six could not be solved in terms of known functions and therefore define new functions, the so called Painlevé transcendents.

Thusfar we have been talking about the Painlevé property for o.d.e. , but now we have arrived at the point where we can introduce the Painlevé property for p.d.e as defined by Weiss et al. In a sequence of articles [14,15, 16], J.Weiss defines the Painlevé property for partial differential equations. He checks in those articles for various p.d.e. whether they have this property or not. The method suggested in [14] by Weiss et al. , enables us to check integrability and determine Bäcklund transformations of the given equation and can in this way generate new solutions of the equation.

The Painlevé property for p.d.e. and the property for o.d.e. have a large amount of similarities. In the definitions of both properties the singular points play an important role. Theorems that are valid for o.d.e. with respect to the Painlevé property can in general not be passed over directly to p.d.e., nevertheless they are often rather alike.

Consider an arbitrary p.d.e. and let $u = u(z_1, z_2, \ldots, z_n)$ be a solution of this equation. Lets call the manifold on which the singularities of the solution are situated the singular manifold. The singular manifold can be represented by the following expression:

$$\phi(z_1, z_2, \ldots, z_n) = 0$$

where ϕ is an analytical function of $(z_1, z_2, \ldots, z_n)$ in a neighborhood of the singular manifold. So the function ϕ determines the position of the singularities of a particular solution.
Weiss et al. define the Painlevé property for a p.d.e. as follows: a p.d.e. has the Painlevé property when the solutions of the equation are single valued near the movable singular manifolds.

In other words, the solution $u = u(z_1, z_2, \ldots, z_n)$ can be expanded in the series:

$$u(z_1, z_2, \ldots, z_n) = \phi^\alpha \sum_{j=1}^{\infty} u_j \phi^j \qquad (12)$$

where $u_j = u_j(z_1, z_2, \ldots, z_n)$ has to be an analytical function in the neighborhood of the singular manifold, the equations for the functions u_j must have self-consistent solutions and α has to be a negative integer. This expansion can be considered as a Laurent expansion around the singular manifold.

This definition deserves further explication. What does the expansion in terms of ϕ mean ?
If we would be dealing with some ordinary differential equation having a solution $q(x)$ with a singularity in $x = x_0$, then ϕ would simply have had the form $\phi = x - x_0$. So expanding the solution in terms of that ϕ would have given a Laurent expansion in $(x - x_0)$ in the neighbourhood of x_0:

$$q(x) = (x - x_0)^{\alpha} \sum_{j=0}^{\infty} q_j(x)(x - x_0)^j$$

Thus $\phi(z_1, z_2, \ldots, z_n)$ can simply be regarded as a new coordinate. The Laurent expansion around the singular point x_0 can be generalized to an expansion around the singular manifold given by $\phi(z_1, z_2, \ldots, z_n) = 0$. This gives us the expansion (12).

Let us continue with the case in which $(z_1, z_2, \ldots, z_n) = (z_1, z_2)$. Substitution of the expansion given in (12) gives us equations for the coefficient functions $u_j(z_1, z_2)$, and by comparing the leading orders in ϕ on both sides of the equation, the value of α can be determined. It will become clear in a few moments, that for certain values of j the equation for the corresponding $u_j(z_1, z_2)$ will be a so called "empty condition", a condition on the coefficient functions $u_j(z_1, z_2)$ that is satisfied identically and does not put any restrictions on the coefficients, which allows us to introduce an arbitrary function for $u_j(z_1, z_2)$. The corresponding values of j are called resonances. (The resonance at $j = -1$ corresponds to the arbitrariness of ϕ).

The empty conditions on the coefficient functions in many cases allow us to cut the series (12) at resonant values of j. In these cases we obtain a finite expansion of u in terms of ϕ:

$$u(z_1, z_2) = \phi(z_1, z_2)^{\alpha}(u_0(z_1, z_2) + u_1(z_1, z_2)\phi(z_1, z_2) + \ldots + u_k(z_1, z_2)\phi^k(z_1, z_2))$$

Before the expansion series in ϕ was truncated, the function ϕ was arbitrary (as long as it satisfied the conditions mentioned in the definition of the Painlevé property). After the truncation this is no longer the case. The function ϕ has to satisfy certain equations imposed by the restrictions on the coefficient functions $u_j(z_1, z_2)$.

The expansion of $u(z_1, z_2)$, together with the introduction of arbitrary coefficient functions at the resonant values will provide the ingredients necessary to define a Bäcklund transformation.

2.3 Applications to the *sine-Gordon* equation

Our next intention is to find some special solutions of the *sine-Gordon* equation. We want to expand these solutions into a Laurent series the way we have outlined in the previous section. The truncated series will provide a Bäcklund transformation from the *sine-Gordon* equation to itself, and possibly new solutions of this equation.

In the following part we consider the *sine-Gordon* equation in the form:

$$u_{\xi\eta} = \sin u \tag{13}$$

A trivial solution of this equation is given by:

$$u = k\pi \qquad k = 0, 1, 2, \ldots$$

Another, cylindrical symmetric, solution can be found in quite an easy way. By introducing the coordinate transform:

$$\begin{aligned} r &= 2\sqrt{\xi\eta} \\ \theta &= \arctan(-i(\tfrac{\xi-\eta}{\xi+\eta})) \end{aligned} \tag{14}$$

and concentrating for the moment on finding a θ-independent solution of (13) we write this equation in a cylindrical symmetric form:

$$\frac{1}{r}\frac{d}{dr}(ru_r) = \sin u$$

Substitution of the expression $u(r) = 2i \ln w(r)$ gives:

$$\frac{1}{r}\frac{dw}{dr} - \frac{1}{w}(\frac{dw}{dr})^2 + \frac{d^2w}{dr^2} = \frac{1}{4}(w^3 - w)$$

and this is a special form of the III^c-Painlevé equation., i.e. the solution is one of the so called Painlevé transcendents. (Note: in [13] the "Weiss techniques" are applied to the *sinh-Gordon* equation

$$u_{\xi\eta} = \sinh u$$

directly. Because far more is known about the *sine-Gordon* equation than about the *sinh-Gordon* equation we decided to continue with the former).

To be able to use the method suggested by Weiss et al. we have to write the *sine-Gordon* equation in a slightly modified form, in order to get rid of the (non-algebraic) nonlinearity in (13), we introduce:

$$V(\xi,\eta) = e^{iu}$$

to write (13) in the form:

$$VV_{\xi\eta} - V_\xi V_\eta = \frac{1}{2}(V^3 - V) \tag{15}$$

Analogous to the previous part, we write:

$$V(\xi,\eta) = \phi^\alpha(\xi,\eta)\sum_{j=0}^{\infty} V_j(\xi,\eta)\phi^j(\xi,\eta) \tag{16}$$

The values of α can be found by substituting the expression $V = V_0\phi^\alpha$ into (15). If α is an integer and (16) is a valid expansion around the singular manifold $\phi = 0$, then the solution $V(\xi,\eta)$ is single valued around $\phi = 0$.

Comparing the leading order terms on both sides of (15) gives us:

$$(V_0^2\alpha(\alpha-1)\phi_\xi\phi_\eta - V_0^2\alpha^2\phi_\xi\phi_\eta)\phi^{2\alpha-2} = \frac{1}{2}V_0^3\phi^{3\alpha}$$

which leads us to the conclusion that $\alpha = -2$. The expansion for a solution $V(\xi,\eta)$ can be rewritten as:

$$V(\xi,\eta) = \phi^{-2}\sum_{j=0}^{\infty} V_j(\xi,\eta)\phi^j(\xi,\eta) \tag{17}$$

From the recursion relations given in [14] or by direct substitution of (17) into (15) and comparing powers of of ϕ, we find:

$$j = 0 \qquad V_0 = 4\phi_\xi\phi_\eta \tag{18}$$

$$j = 1 \qquad V_1 = -4\phi_{\xi\eta} \tag{19}$$

$$j = 2 \qquad 8\phi_\xi^2\phi_\eta(V_1 + 4\phi_{\xi\eta})_\eta + 8\phi_\xi^2\phi_\eta(V_1 + 4\phi_{\xi\eta})_\xi +$$

$$(-8\phi_{\xi\xi}\phi_\eta^2 - 8\phi_{\eta\eta}\phi_\xi^2 - 10\phi_\xi\phi_\eta V_1)(V_1 + 4\phi_{\xi\eta}) = 0 \tag{20}$$

By (19) the compatibility condition (20) at $j = 2$ is satisfied identically. Therefore, an arbitrary function $V_2(\xi,\eta)$ can be introduced at the resonance $j = 2$.

In this way we have shown that (17) is a proper single valued expansion around the manifold $\phi = 0$, which gives (15) the Painlevé property.

We now require the system to satisfy:

$$V_3 = V_4 = V_5 = V_6 = 0 \tag{21}$$

By this requirement all $V_j = 0$ for $j \geq 3$, which can easily be deduced from the recursion relations in [14], giving (17) the form:

$$V = \phi^{-2} V_0 + \phi^{-1} V_1 + V_2 \tag{22}$$

or, using (18) and (19):

$$V = -4 \frac{\partial^2}{\partial \xi \partial \eta} \ln \phi + V_2 \tag{23}$$

The restrictions given by (21) on the expansion (17), lead us to a system of four equations in (V_2, ϕ) which can be found explicitly by using the recursion relations given in [14] or by direct substitution of (22) into (15):

$$\begin{aligned}
j = 3 \quad & (-4\phi_\eta^2 \phi_{\xi\xi} - 4\phi_\xi^2 \phi_{\eta\eta} + 8\phi_\xi \phi_\eta \phi_{\xi\eta}) V_2 + 4\phi_\eta^2 \phi_\xi V_{2\xi} + 4\phi_\xi^2 \phi_\eta V_{2\eta} = \\
& 8\phi_{\xi\xi}\phi_{\eta\eta}\phi_{\xi\eta} + 8\phi_{\xi\xi\eta\eta}\phi_\xi\phi_\eta - 8\phi_{\xi\eta\eta}\phi_{\xi\eta}\phi_\eta - 8\phi_{\xi\xi\eta}\phi_{\eta\eta}\phi_\xi \qquad (24) \\
j = 4 \quad & (-8\phi_{\xi\eta}^2 + 4\phi_\xi\phi_{\xi\eta\eta} + 4\phi_\eta\phi_{\xi\xi\eta} + 2\phi_{\xi\xi}\phi_{\eta\eta}) V_2 - 3\phi_\xi\phi_\eta V_2^2 + \\
& 2\phi_\xi\phi_\eta V_{2\xi\eta} - (4\phi_{\xi\eta}\phi_\eta + 2\phi_{\eta\eta}\phi_\xi) V_{2\xi} - (4\phi_{\xi\eta}\phi_\xi + 2\phi_{\xi\xi}\phi_\eta) V_{2\eta} = \\
& -8\phi_{\xi\xi\eta\eta}\phi_{\xi\eta} + 8\phi_{\xi\xi\eta}\phi_{\xi\eta\eta} - \phi_\xi\phi_\eta \qquad (25) \\
j = 5 \quad & -2\phi_{\xi\xi\eta\eta} V_2 + 3\phi_{\xi\eta} V_2^2 - 2\phi_{\xi\eta} V_{2\xi\eta} + 2\phi_{\xi\eta\eta} V_{2\xi} + 2\phi_{\xi\xi\eta} V_{2\eta} = \phi_{\xi\eta} \qquad (26) \\
j = 6 \quad & V_2 V_{2\xi\eta} - V_{2\xi} V_{2\eta} = \tfrac{1}{2}(V_2^3 - V_2) \qquad (27)
\end{aligned}$$

As can be seen from the equation (27), this leads to the conclusion that V_2 has to satisfy the *sine-Gordon* equation in the form (15). Hence, the transformation given by (23) defines a Bäcklund transformation. What remains is the overdetermined system of four equations (24)...(27) in (V_2, ϕ).

In order to reduce the overdetermined system (21), Weiss introduces in [15] the expressions:

$$V_2 - \frac{\phi_{\xi\eta}^2}{\phi_\xi \phi_\eta}, \quad \Omega_1\Omega_2 - \tfrac{1}{4}, \quad \phi_\xi \frac{\partial}{\partial \xi}\Omega_1 + \phi_\eta \frac{\partial}{\partial \eta}\Omega_2 \tag{28}$$

where Ω_1 and Ω_2 are given by:

$$\Omega_1 = \frac{\phi_{\xi\eta\eta}}{\phi_\xi} - \frac{\phi_{\eta\eta}\phi_{\xi\eta}}{\phi_\xi\phi_\eta} - \frac{1}{2}\frac{\phi_{\xi\eta}^2}{\phi_\xi^2} \tag{29}$$

$$\Omega_2 = \frac{\phi_{\xi\xi\eta}}{\phi_\eta} - \frac{\phi_{\xi\xi}\phi_{\xi\eta}}{\phi_\xi\phi_\eta} - \frac{1}{2}\frac{\phi_{\xi\eta}^2}{\phi_\eta^2} \tag{30}$$

and he manages to write the overdetermined system (24)...(27) as combinations of expressions given in (28).
Now, if the three expressions in (28) all equal zero, (24)...(27) are immediately satisfied, and vice versa. We have established an equivalence between the overdetermined system (24)...(27) and the equations:

$$V_2 - \frac{\phi_{\xi\eta}^2}{\phi_\xi\phi_\eta} = 0 \tag{31}$$

$$\Omega_1\Omega_2 - \frac{1}{4} = 0 \tag{32}$$

$$\phi_\xi \frac{\partial}{\partial \xi}\Omega_1 + \phi_\eta \frac{\partial}{\partial \eta}\Omega_2 = 0 \tag{33}$$

After some calculations starting from (29) and (30), it can be shown that Ω_1 and Ω_2 satisfy the relation:

$$\phi_\xi \frac{\partial}{\partial \xi}\Omega_1 = \phi_\eta \frac{\partial}{\partial \eta}\Omega_2 \tag{34}$$

Using this relation, equation (33) can be further simplified. Combining (33) and (34) we conclude that:

$$\phi_\xi \frac{\partial}{\partial \xi}\Omega_1 = \phi_\eta \frac{\partial}{\partial \eta}\Omega_2 = 0 \tag{35}$$

Using (32) this eventually leads to:

$$\Omega_1 = \alpha \tag{36}$$

$$\Omega_2 = \beta \tag{37}$$

where α and β are constants satisfying $\alpha\beta = \frac{1}{4}$.

We have now arrived at the point where we can use the Bäcklund transformation given by (23). We have to take the conditions on V_2 and ϕ imposed by (31)...(37) into account, to transform the known solutions of the *sine-Gordon* equation mentioned at the beginning of section 2.3. First we consider the trivial solution $u = k\pi$ of (13).

As easily can be seen, $V_2 = 1$ is a solution of (15), corresponding to this solution of (13). Using the solution $V_2 = 1$ in (31), we find the equation:

$$\frac{\phi_{\xi\eta}^2}{\phi_\xi \phi_\eta} = 1$$

Two solutions of this equation are:

$$e^{(\xi+\eta)}, \quad e^{-(\xi+\eta)}$$

but substitution of these solutions into (17) leads to $V = 1$, which is not a new solution of the *sine-Gordon* equation.

We still have another solution available, namely the III^c-Painlevé transcendent, which is a circular symmetric solution. Our hope is that we can find a new circular symmetric solution $V_2(r)$ using the III^c-Painlevé transcendent as the $V(r)$ in the Bäcklund transformation (23).

First we made use of a function $\phi = \phi(r)$ relating $V(r)$ to $V_2(r)$ in the Bäcklund transformation. We wrote (29) and (30) in cylindrical coordinates given by (14) and tried to solve the resulting system. The problems that occurred are outlined in the following calculation:

$$\Omega_1 = \frac{\xi}{\eta}\left(-\frac{1}{2r^2} + \frac{\phi_{rrr}}{\phi_r} - \frac{3}{2}\frac{\phi_{rr}^2}{\phi_r^2}\right) \equiv \frac{1}{4\eta^2} G(r) \tag{38}$$

$$\Omega_2 = \frac{\eta}{\xi}\left(-\frac{1}{2r^2} + \frac{\phi_{rrr}}{\phi_r} - \frac{3}{2}\frac{\phi_{rr}^2}{\phi_r^2}\right) \equiv \frac{1}{4\xi^2} G(r) \tag{39}$$

which leads with the use of (32) to:

$$G(r) = \frac{r^4}{4}$$

On the other hand we have, using (33) and (35):

$$\frac{\partial}{\partial \xi}\Omega_1 = \frac{\partial}{\partial \eta}\Omega_2 = 0$$

which gives:

$$\frac{dG(r)}{dr} = 0$$

leading to a contradiction on $G(r)$. This contradiction can have several causes. One could be that there is only one circular symmetric solution of the *sine-Gordon* equation, so the Bäcklund transformation simply cannot generate any others. However, this fact should not lead to a contradiction, but could at the most be the reason no *new* solutions can be found. Another thing that could have caused the contradiction is that the form of the function ϕ being $\phi(r)$ is too restrictive. We examined this argument by introducing a function $\phi = \phi(r,\theta)$ in the Bäcklund transformation.

If it would be possible to find a function $\phi(r,\theta)$ which does not lead to a θ-dependent expression for $V(r)$ when substituting (r,θ) coordinates into (17) then we would still be in business. We proceeded then to check this possibility as follows.

With the use of the coordinate transform given by (14), it can be shown after some calculation that (23) can be written with respect to these new coordinates in the form:

$$V(r) = 4\phi_r^2\frac{1}{\phi^2} - \frac{4}{r}\phi_r\frac{1}{\phi} - 4\phi_{rr}\frac{1}{\phi} + \frac{4}{r^2}\phi_\theta^2\frac{1}{\phi^2} - \frac{4}{r^2}\phi_{\theta\theta}\frac{1}{\phi} + V_2(r) \qquad (40)$$

The question now is, under what conditions can a θ-dependent function $\phi(r,\theta)$ exist that leads to a θ-independent equation (40).

To make life a little bit easier, we made the following restriction:

$$\phi(r,\theta) = f(r)g(\theta) \qquad (41)$$

and substituted this expression into (40). This led to:

$$V(r) = 4\frac{f_r^2}{f^2} - \frac{4}{r}\frac{f_r}{f} - 4\frac{f_{rr}}{f} + \frac{4}{r^2}\frac{g_\theta^2}{g^2} - \frac{4}{r^2}\frac{g_{\theta\theta}}{g} + V_2(r)$$

To get rid of the θ dependence,

$$\frac{g_\theta^2}{g^2} - \frac{g_{\theta\theta}}{g} = C_1$$

has to be satisfied, where C_1 is a constant.

The general solution of this second order ordinary differential equation is given by:

$$g(\theta) = e^{\frac{1}{2}C_1\theta^2+C_2\theta+C_3}$$

with C_1, C_2 and C_3 constants. However, this solution for $\phi(r,\theta)$ should be compatible with (31) being:

$$V_2 = \frac{\phi_{\xi\eta}^2}{\phi_\xi\phi_\eta}$$

After some calculation it can be shown that this is only the case whenever $C_1 = 0$. This gives $g(\theta)$ the form:

$$g(\theta) = e^{C_2\theta+C_3} \tag{42}$$

Without loss of generality we can set $c_2 = im$ and $c_3 = 0$ in (42), which gives us for $\phi(r,\theta)$:

$$\phi(r,\theta) = f(r)e^{im\theta} \tag{43}$$

A function ϕ of the form (43) should enable us to transform a function $V(r)$ into $V_2(r)$. Now we repeat the method we used in case of the circular symmetric $\phi(r)$ (compare with (38) and (39)) and write the equations (29) and (30) for Ω_1 and Ω_2 in terms of the function $\phi(r,\theta)$ given by (43). This leads us to rather complicated equations for Ω_1 and Ω_2 which can however be represented in the compact form:

$$\Omega_1 = \frac{1}{8\eta^2}G_1(r)$$

$$\Omega_2 = \frac{1}{8\xi^2}G_2(r)$$

here $G_1(r)$ and $G_2(r)$ are expressions in $f(r)$ and derivatives of $f(r)$.

Using the transformation (14) to get expressions for ξ and η in terms of r and θ and using (36) and (37), the expressions for Ω_1 and Ω_2 can be written as:

$$\Omega_1 = \frac{1}{2r^2}e^{2i\theta}G_1(r) = \alpha \tag{44}$$

$$\Omega_2 = \frac{1}{2r^2}e^{-2i\theta}G_2(r) = \beta \tag{45}$$

The r-dependence of Ω_1 and Ω_2 does not necessarily give problems, but the θ-dependence is unavoidable, making it impossible for Ω_1 and Ω_2 to be constant. Again we are led to a contradiction.

Assuming that (41) is valid, it can be concluded that it is impossible to find a function $\phi(r,\theta)$ compatible with (32) and (33).

The only thing that remains to be done now is to use an arbitrary function $\phi(r,\theta)$ (not of the restricted form given in (41)) in the Bäcklund transformation (23). This would lead to a nonlinear differential equation in $\phi(r,\theta)$ with a structure as complicated as the *sine-Gordon* equation itself. Therefore we omit this general substitution.

3 Concluding remarks

We have seen that a circular symmetric solution $V(r)$ cannot generate a new circular symmetric solution $V_2(r)$ of the *sine-Gordon* equation if a $\phi = \phi(r)$ or a $\phi = \phi(r,\theta)$ of the form (41) is used in the Bäcklund transformation (23) relating the solution $V(r)$ to $V_2(r)$. The only circular symmetric solution of the *sine-Gordon* equation we were able to find is the III^c-Painlevé transcendent (see [24]). The only reason for this failure we are able to give is that the restriction imposed on the function $\phi(r,\theta)$ by (41) is too strong to be suitable in the Bäcklund transformation (23) and prevents the generation of a new circular symmetric solution. The method presented by Weiss et al. in [14] might still be usefull to generate new solutions of a partial differential equation, however, when applied to the *sine-Gordon* equation, it does not reduce the complexity of the problem.

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