Scientific report
WR 95-03

Ministerie van Verkeer en Waterstaat

Koninklijk Nederlands
Meteorologisch Instituut

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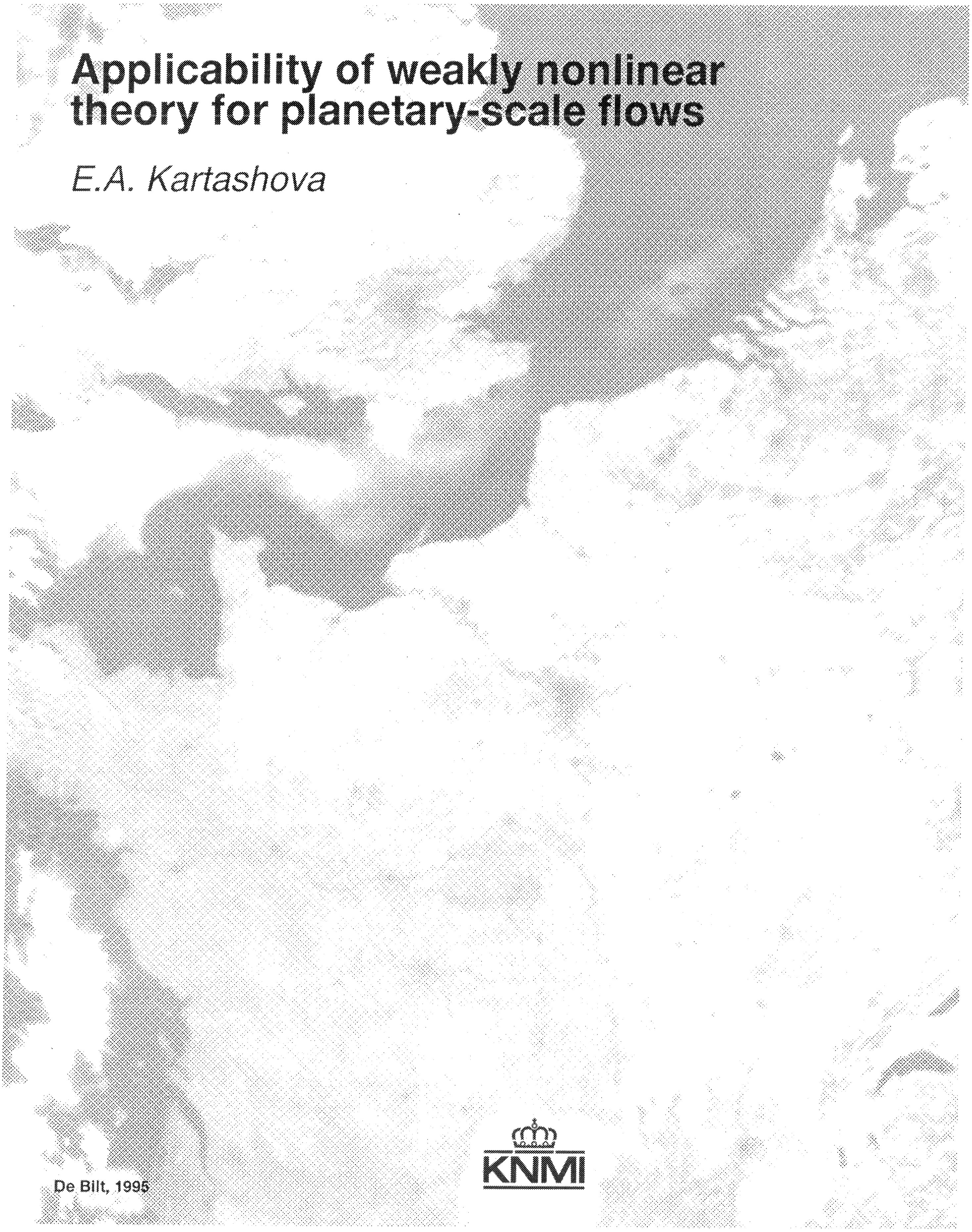

De Bilt, 1995

KNMI

Scientific report; WR 95-03

De Bilt 1995

Postbus 201
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Telefoon 030-206 911
Telefax 030-210 407

UDC: 532
551.511.33
551.513.1

ISSN: 0169-1651

ISBN: 90-369-2071-X

APPLICABILITY OF WEAKLY NONLINEAR THEORY FOR PLANETARY-SCALE FLOWS

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February 1995

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Abstract

The relevance of isolated resonant triad interactions for the dynamics of the large-scale atmospheric circulation is investigated. This is done within the context of the barotropic vorticity equation on the sphere. The equations governing the dynamics of a resonant triad on the sphere are of the same form as those on the beta-plane. They can be solved analytically in terms of elliptic integrals; giving a periodic vacillation of the amplitudes of the waves participating in the triad. The vacillation period depends on the total energy of the triad and on the initial energy distribution within the triad. This dependence is investigated numerically. It is investigated whether or not, for realistic energy distributions, there exists a time-scale for which the resonant interactions within the triad dominate over the interactions of the triad components with the rest of the spectrum. This is done within the context of a barotropic model truncated to T21. Several experiments are described. The overall results are that vacillation can be observed for a few cycles. The vacillation period lies between 30 and 60 days. After a few cycles the nonresonant interactions rapidly transfer the energy to the rest of the spectrum. However, after some time part of the energy lost is returned back to the resonant triad and vacillation again takes place. The results may shed some new light on the observed intraseasonal oscillations in the tropical and extratropical atmosphere.

1 Introduction

The first step in many investigations of geophysical flows is to linearize the equations of motion. The general solution is then relatively easy to obtain and consists of a superposition of noninteracting normal modes, or waves. When the spatial domain is of finite extent, then there is a countably infinite number of such waves, each of which can be characterized by a discrete label.

The fact that the waves behave independently of each other is the great advantage of the linear approximation. In the fully nonlinear case this is no longer true. It is still possible, however, to consider the flow as a superposition of waves. The difference is that now the waves interact among each other. If the nonlinear terms are quadratic then three waves are involved in each interaction. Whether, in the case of quadratic nonlinear terms, three waves interact or not, is determined by certain conditions on the labels of the waves. These conditions determine the structure of the set of waves participating in interactions. Usually this set is so complicated that handling of all the interactions can only be dealt with numerically.

If the nonlinear terms are small, then the situation can be simplified considerably. For small nonlinear terms the time scale of interaction between waves will be slow compared with the time scale of each wave's individual behaviour. This notion can be worked out systematically, using the method of multiple scales. It leads to the distinction of two time regimes: the (fast) wave propagation and the (slow) wave interaction regime. In the first regime the waves behave as if they were linear, i.e., independently of each other. In the second regime interactions do play a role, but only to a limited extent. In the interaction regime only interactions have to be taken into account which satisfy a resonance condition. These interactions are called resonant to distinguish them from the other, nonresonant, interactions. Because the set of resonant interactions is usually considerably smaller than the set of all interactions, limitation to the resonant interaction regime is an important simplification. Theories which handle nonlinearity in this particular way are called weakly nonlinear theories.

The aim of this paper is to obtain more insight into the question of how relevant resonant interactions are for the dynamics of large-scale atmospheric flow. A first-order description of this flow, neglecting vertical structure, horizontal divergence, forcing and dissipation, is the barotropic vorticity equation on a rotating sphere. The linear solutions (modes) of this equation are Rossby-Haurwitz waves, referred to as spherical planetary waves (SPW). They constitute a countably infinite set of solutions characterized by labels (m, n), where m and n are integers with $-n \leq m \leq n$ and $0 < n < \infty$. As the barotropic vorticity equation is a nonlinear equation the waves interact with each other. Three waves are involved in each interaction because the nonlinear (advective) term is quadratic.

The nonlinear interactions of planetary waves have been studied by many investigators. The method for solving the nonlinear barotropic vorticity equation by expanding the fields in terms of spherical planetary waves was given by Silberman (1954). Also, a few conditions for interaction between these waves were given. A systematic analytic investigation of the systems obtained in this way is given by Platzman (1962). It was mentioned also that the three component system, i.e. the system in which the fields are expanded in three spherical planetary waves, can be solved in terms of elliptic functions. The solution, and in particular the energy exchange among the components, is periodic in time. The solution, however, was not written out in explicit form. The investigation of the significance of Silberman's conditions for the case of an infinite beta-plane is given by Longuet-Higgins and Hill (1967). It was also pointed out that for application to the ocean and atmosphere it is generally desirable to consider resonantly interacting planetary waves in closed basins and on a sphere.

A recent exploration of the set of resonant interactions for the barotropic vorticity equation on a sphere was carried out in a number of papers by Kartashova (1990a, 1990b, 1991) and by Kartashova et al. (1990). In these studies it was proven that the set of all resonantly interacting planetary waves can be partitioned into disjoint subsystems. An important implication of these results is that, within the context of the weakly nonlinear theory, the different subsystems can be treated independently. The simplest isolated subsystem of resonantly interacting waves consists of three waves, called a resonant triad. The equations governing the dynamics of this system are of the same form as those for a resonant triad of Rossby waves on the beta-plane. These equations can be solved analytically in terms of elliptic integrals. The behaviour of one isolated triad can be described as follows. On the wave propagation time scale each of the three waves behaves as it was a linear free wave. On the interaction time scale the amplitudes of the waves change periodically, where the period depends on the initial amplitudes of the waves. This (slow) periodic exchange of energy among waves in a single isolated triad is the simplest manifestation of weak nonlinearity.

In this paper we wish to extend these earlier researches by studying the behaviour of resonant triads in a full spectral model of the barotropic vorticity equation. More specifically, it will be investigated whether the slow periodic exchange of energy among waves in an isolated triad is observed when all other modes (not only resonant ones) are present.

In section 2 we describe the main analytical results concerning the resonant interactions of spherical planetary waves (SPW) obtained by Kartashova (1990a, 1990b, 1991) and by Kartashova et al., (1990). Section 3 is devoted to the investigation of the formula for the energy exchange period among the modes of the resonant triads. In section 4 computer simulations with the numerical model of the barotropic potential vorticity equation with truncation T21 are described. Discussion of the obtained results can be found in section 5.

2 Resonance conditions for spherical planetary waves

To describe the nonlinear dynamics of large-scale atmospheric nondivergent flow we will use the barotropic vorticity equation on a rotating sphere. This equation describes the time evolution of a two-dimensional, incompressible, inviscid and unforced fluid:

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi + 2 \sin \varphi) = 0. \tag{1}$$

Here ψ is the streamfunction and ∇^2 and J are the Laplace and Jacobi operators for a spherical surface. The equation is written in nondimensional form using the radius of the Earth a (6.371×10^6 m) and the reciprocal of the Earth's angular velocity Ω ($7.292 \times 10^{-5} \mathrm{s}^{-1}$) as length and time scales, respectively. The coordinates specifying a point on the sphere are λ and φ, where λ is longitude and φ is latitude. It is assumed that these coordinates are expressed in radians.

The equations for the real-valued amplitudes of three resonantly interacting SPWs, denoted by $(m_1, n_1), (m_2, n_2), (m_3, n_3)$, have the form (Kartashova et al., 1990)

$$\begin{aligned} N_1 \dot{a_1} &= Z(N_2 - N_3) a_2 a_3, \\ N_2 \dot{a_2} &= Z(N_3 - N_1) a_1 a_3, \\ N_3 \dot{a_3} &= Z(N_1 - N_2) a_1 a_2, \end{aligned} \tag{2}$$

where $N_i = n_i(n_i + 1)$, $i = 1, 2, 3$, a_i is the amplitude of i-th wave, and the interaction coefficient Z is given by

$$Z = \frac{1}{2} \int_{-\pi/2}^{\pi/2} [m_1 P^{(1)} \frac{\partial P^{(2)}}{\partial \varphi} - m_2 P^{(2)} \frac{\partial P^{(1)}}{\partial \varphi}] P^{(3)} d\varphi, \tag{3}$$

where we use the notation $P^{(i)}$, $i = 1, 2, 3$, for the associated Legendre functions of the first kind $P_{n_i}^{m_i}$ of degree n_i and order m_i.

The conditions for resonant interactions between three spherical planetary wave solutions (SPWs) of system (1) are

$$\begin{aligned} \omega_1 + \omega_2 &= \omega_3, \\ m_1 + m_2 &= m_3, \\ m_i &\leq n_i, \forall i = 1, 2, 3, \\ |n_1 - n_2| &\leq n_3 \leq n_1 + n_2, \\ n_1 + n_2 + n_3 & \text{ is odd}, \\ n_1 \neq n_2 &\neq n_3, \end{aligned} \tag{4}$$

where $\omega = -2m/n(n+1)$ is the frequency of the SPW and m and n are the zonal and total wave numbers respectively. The first two equations of the system (4) are the constraints with respect to frequencies and wave numbers of resonantly interacting waves. All other conditions are necessary for nonzero interaction coefficients. The three last conditions are in accordance with the selection rules studied by Platzman (1962).

The investigation of the interaction coefficients as a function of $m_i, n_i, i = 1,2,3$ showed the existence of a hypothetical interaction latitude φ_0 defined by the following expression

$$\cos^2\varphi_0 = \frac{m_1^2(n_2^2+n_3^2-n_1^2)+m_2^2(n_1^2+n_3^2-n_2^2)+m_3^2(n_1^2+n_2^2-n_3^2)}{n_1^2n_2^2+n_1^2n_3^2+n_2^2n_3^2-0.25(n_1^4+n_2^4+n_3^4)}, \tag{5}$$

so that if $0 < \cos^2\varphi_0 < 1$ (We say in this case that the interaction latitude does exist.), then the interaction coefficient $Z \sim n^{3/2}$, where $n = \max n_i$. When the interaction latitude does not exist, $Z \sim n^{7/6}$, i.e. the interactions become less effective (Kartashova et al., 1990).

The system (4) has to be solved in integers and this is a nontrivial problem. In general there is no proof for the very existence of a solution. However, for the system (4) it turns out that there exists an infinite number of solutions; explicit formulae giving one- and two-parameter series of solutions have been found. The following question arises immediately: do all waves participate in resonant interactions or not? It appears that there are many waves that do not participate in resonant interactions. In fact, for a physically (meteorologically) relevant region of spectral space (we mean here $0 < m, n < 100$) the number of waves participating in resonant interactions does not exceed 32%; it decreases as r^{-1} where $r = (m^2+n^2)^{1/2}$, when r tends to infinity. It was shown that resonant interactions are local in the sense that for any fixed wave its interaction domain (i.e. the region in wavenumber space consisting of all waves that could interact with this one) is finite and is written out explicitly. For a wave with wave vector (m, n) the radius of this domain is of order n^2.

The set of all SPWs taking part in resonant interactions, can be partitioned into non-intersecting partial subsystems which do not cross-interact resonantly (Kartashova et al., 1990, Kartashova, 1990a) and this is a manifestation of a more general property of a wide class of wave systems with discrete spectra (Kartashova, 1990b). As a rule, such subsystems consist of 3-5 SPWs. There exists only one infinite subsystem (Kartashova, 1990a). It is very important to notice that there is no energy exchange in resonant interactions between these subsystems, so as a first approximation it is be possible to study each subsystem independently for some finite time. The spectral region T21 (waves with $m, n \leq 21$) will be further studied numerically. It contains 4 isolated triads, 3 groups of 2 connected triads and a group of 6 connected triads (see Table 1). As mentioned above, the case of isolated triads can be treated analytically. In this case, the equations for the slowly changing amplitudes of resonantly interacting SPW can be integrated exactly. The solution is expressed in Jacobian elliptic functions. The energy exchange between the modes of the triads is periodic and the period can be written out explicitly as a function of the initial energies of the modes, the wave numbers, the interaction coefficients etc. (see section 3). The groups of resonant SPW containing more than three waves can show a rather complex type of behaviour. In this case chaotic behaviour is possible. Indeed, starting with the case of 5 waves we are dealing with a physical system described by connected differential equations like those describing a system of coupled oscillators. This leads to complex behaviour (Rabinovitch and Trubetskov, 1989) and makes an analytical description of such systems a questionable matter. In the present paper we are not concerned with these groups. The subject of our study is the periodic energy exchange between the modes of one isolated resonant triad.

We would like to conclude this section by making the following remark. As the solutions found analytically do not exhaust all the solutions of (4), we have to investigate the finite region $n < 200$ numerically. This is a nontrivial problem, as any straightforward algorithm would take an extremely large amount of computer time. Indeed, combining the first two equations of (4), we immediately see the main computational difficulties. We have an equation in integers on five variables which, however, is not Diophantine (some of the variables occur in the denominators); to solve it numerically in integers, the common denominator has to be found. This procedure leads to a Diophantine equation on five variables of degree five, i.e. the calculation time grows with n^5 (a few weeks of calculation time for a PC AT even within the T21 region). Being interested not only in atmospheric, but also in large-scale oceanic waves, we have been studying a much larger region than T21 (see above). A specially developed fast algorithm has been used (Kartashov and Kartashova, 1991). It is based on divisibility properties of the triangle numbers $n(n+1)$. The required amount of computer time grows approximately as $\log n^{n^2}$.

3 Energy exchange between the modes of a triad

The solution of the system (2) has been obtained about one and a half century ago and may be written in terms of Jacobian elliptic functions (Whittaker, 1937). Here we are not going into the full detail of its derivation but just give its main idea. First of all, we note that the system has two conserved integrals. The first one is

$$N_1 a_1^2 + N_2 a_2^2 + N_3 a_3^2 = c_1. \tag{6}$$

This is the energy conservation law. It is obtained by multiplying the i-th equation by a_i, $i = 1, 2, 3$, and adding. We also have the enstrophy conservation law

$$N_1^2 a_1^2 + N_2^2 a_2^2 + N_3^2 a_3^2 = c_2. \tag{7}$$

It is obtained by multiplying the i-th equation by $N_i a_i$, $i = 1, 2, 3$, and adding. The constants c_1 and c_2 depend, of course, on the initial values of $a_i, i = 1, 2, 3$. From Equations (6) and (7) we obtain expressions for a_2 and a_3 in terms of a_1. Substitution of these expressions into the first equation of system (2) gives one differential equation of the first order which may be solved in terms of the elliptic functions cn, dn and sn. Let a_{i0} denote the initial value of the amplitude a_i, $i = 1, 2, 3$. Then the solution can be written as

$$\begin{aligned} a_1 &= b_1 \mathsf{cn}(t/t_0 - \lambda), \\ a_2 &= b_2 \mathsf{dn}(t/t_0 - \lambda), \\ a_3 &= b_3 \mathsf{sn}(t/t_0 - \lambda), \end{aligned} \tag{8}$$

where

$$\begin{aligned} b_1^2 &= a_{10}^2 + N_3(N_2 - N_3)/N_1(N_2 - N_1) a_{30}^2, \\ b_2^2 &= a_{20}^2 + N_3(N_3 - N_1)/N_2(N_2 - N_1) a_{30}^2, \\ b_3^2 &= a_{30}^2 + N_1(N_2 - N_1)/N_3(N_2 - N_3) a_{10}^2, \end{aligned} \tag{9}$$

and t_0 is given by the expression

$$t_0 = \frac{1}{Z}\left[\frac{N_2 - N_3}{N_1}\left(\frac{N_3 - N_1}{N_2} a_{30}^2 + \frac{N_2 - N_1}{N_3} a_{20}^2\right)\right]^{-1/2}. \tag{10}$$

The constant λ is obtained from the initial conditions. The Jacobian elliptic functions cn, sn, dn are defined in the following way (Abramowitz and Stegun, 1970). Let

$$u = \int_0^{\varphi} \frac{d\theta}{(1 - \mu \sin^2\theta)^{1/2}}. \tag{11}$$

Then we define

$$\mathsf{sn}u = \sin\varphi, \tag{12}$$

$$\mathsf{cn}u = \cos\varphi, \tag{13}$$

$$\mathsf{dn}u = (1-\mu\sin^2\varphi)^{1/2}. \tag{14}$$

These functions are periodic in u. The periods of $\mathsf{sn}u$ and $\mathsf{cn}u$ are equal to $4K$ and the period of $\mathsf{dn}u$ is equal to $2K$ where

$$K = \int_0^{\pi/2} \frac{d\theta}{(1-\mu\sin^2\theta)^{1/2}} \tag{15}$$

is called the complete elliptic integral and

$$\mu = \frac{a_{10}^2 \frac{N_1}{N_3-N_2} + a_{20}^2 \frac{N_2}{N_3-N_1}}{a_{30}^2 \frac{N_3}{N_2-N_1} + a_{20}^2 \frac{N_2}{N_3-N1}} \tag{16}$$

is the modulus of the elliptic functions. The modulus μ has to satisfy the following condition: $0<\mu<1$. It is easy to show that the expression (16) gives a positive value of μ for arbitrary a_{i0} and N_i, $i=1,2,3$. As to the other condition, we have to check it and if it is not satisfied, the solution may be obtained by interchanging the indices 1 and 2 in the formulae above. If $\mu=1$, then

$$K = \int_0^{\pi/2} \frac{d\theta}{\cos\theta} = \frac{1}{2}\ln\frac{1+\sin\varphi}{1-\sin\varphi}\,|_0^{\pi/2} \longrightarrow \infty \tag{17}$$

It means that the period of all these elliptic functions tends to infinity. To understand this let us look at how these elliptic functions degenerate at $\mu=1$. Since $u=\frac{1}{2}\ln\frac{1+\sin\varphi}{1-\sin\varphi}$ and $\mathsf{sn}u=\sin\varphi$ for $\mu=1$, we have that $\mathsf{sn}u$ turns into $1-\frac{2}{\exp 2u+1}$. For both $\mathsf{cn}u$ and $\mathsf{dn}u$ we obtain $\frac{2\exp u}{1+\exp 2u}$ and immediately notice that the degenerate $\mathsf{sn}u$ grows from 0 to 1 as u changes from 0 to ∞. At the same time the two other functions decrease from 1 to 0.

Now, keeping in mind (12)-(14), we can express the period τ of the energy exchange within the modes of the resonant triad as

$$\tau = 4pt_0K,\ \ p=1,2,3,\ldots \tag{18}$$

At first glance it is difficult to say anything definite about the period τ from this formula which contains 7 variables, most of which occur in a highly nontrivial manner. However, some qualitative observations can be made. To this end it is more convenient to speak in terms of energies rather than amplitudes. Let us denote the initial energies of each mode as E_{i0}, $i=1,2,3$, and define x_1, x_2 by

$$E_{10} = x_1\cdot E_{30}, \tag{19}$$

$$E_{20} = x_2\cdot E_{30}, \tag{20}$$

so that x_1 and x_2 show the fraction of the main mode's energy E_{30} that is initially contained in the first and second modes respectively. (Here and below the mode with largest frequency is called the main mode) Now we rewrite (10) and (18) in the new variables (using that $E_i = N_i a_i^2, i = 1,2,3$). The period becomes $\tau = p\tau_0,\ \ p = 1,2,3,\ldots$, where τ_0 is given by

$$\tau_0 = \frac{4K}{ZE_{30}^{1/2}}\Big[\frac{N_2 - N_3}{N_1 N_2 N_3}(N_3 - N_1 + x_2(N_2 - N_1))\Big]^{-1/2}, \tag{21}$$

and we can see immediately that τ is inversely proportional to the interaction coefficient Z of the triad and also to the square root of the initial energy of the main mode. For a fixed triad n_1, n_2, n_3 are, of course, fixed, so that τ_0 depends on x_2 as $1/(x_2 + const)^{1/2}$. Notice that there is no explicit dependence of τ_0 on x_1.

Now we rewrite μ in these variables as

$$\mu = \frac{x_1(N_2 - N_1) + (N_2 - N_3)}{x_2(N_2 - N_1) + (N_3 - N_1)} \cdot \frac{(N_3 - N_1)}{(N_2 - N_3)}. \tag{22}$$

The most interesting fact is that μ does not depend on the initial energies of the modes but only on the distribution of the initial energy among the modes of the triad. This, of course, applies also to $K(\mu)$. It is not difficult to show that $\mu = 1$ only if

$$\frac{x_1}{N_2 - N_3} = \frac{x_2}{N_3 - N_1}. \tag{23}$$

Thus, for any fixed N_i, $i = 1,2,3$ there exists a line in the plane (x_1, x_2) for which the integral K diverges; we will call it the degeneration line.

The previous discussion leads to the following qualitative description for the period τ of one fixed triad. For small values of x_1, x_2 and such that $x_1/x_2 \neq (N_2 - N_3)/(N_3 - N_1)$, the period $\tau \sim K(\mu)/ZE_{30}^{1/2}$. It is finite but can become very long when x_1, x_2 tend to zero. So the period becomes very long if almost all energy is initially contained in the main mode at the initial moment. For large values of x_1, x_2 such that $x_1/x_2 \neq (N_2 - N_3)/(N_3 - N_1)$, the period $\tau \sim K(\mu)/ZE_{30}^{1/2}x_2^{1/2}$. It is finite but can become very long when the ratio x_1/x_2 tends to $(N_2 - N_3)/(N_3 - N_1)$. When $x_1/x_2 = (N_2 - N_3)/(N_3 - N_2)$ the main mode continually gets energy from the two other modes and never loses it. It is very important to notice that theoretically for a fixed triad and arbitrary value of the initial energy we can always get the initial energy distribution in such a way that this non-periodic regime will appear.

For a fixed triad and fixed energy distribution between the modes, the period τ decreases when the energy grows. Using formula (21) describing the dependence of τ_0 on the energy distribution between the modes in general is not easy. So it was done numerically and the results are given in section 4. There we shall also make a comparative analysis of different triads to obtain some qualitative characteristics of their periods and their dependence on the initial energy distribution and on the individual parameters of each triad.

4 Numerical simulations

Our numerical simulations have two different objectives. First of all, we wish to verify the analytical expressions for the case that the spectral energy in the resonant triads is distributed according to observations. To characterize our second objective we have to remember that all results concerning the resonant triads have been obtained under the constraints of the weakly nonlinear theory of wave interactions. This means that these results are valid if there exists a time scale at which resonant interactions are dominant. To investigate whether or not such a time scale exists was our second goal.

We have used a numerical model of the barotropic vorticity equation on a sphere. The amplitudes of the streamfunction were chosen according to the analysis of observed data. Data of the atmospheric flow at the 500 hPa pressure level for each day of December 1989 at 0 and 12.00 GMT have been processed. These data consist of the complex expansion coefficients (in terms of spherical harmonics, i.e., spherical planetary waves) of the streamfunction fields. The root mean square of the (nondimensionalized) amplitudes of the coefficients was computed of the 62 fields of this period. The values are displayed in Table 1.

First of all, remembering the existence of the degeneration line, described by (23) in the parameter space (x_1, x_2), we want to find out the relative size of the line's neighbourhood in which the growth of the period becomes critical. The growth of K is critical when deviations of μ caused by variations in x_1, x_2 of the size that would have been caused by small errors in the initial data leads to a change of the order of K. In this area the period formula does not give reliable results. This only happens in a narrow region of μ values (close to 1) because if μ is considerably smaller than 1, $K(\mu)$ changes slowly enough (from $\pi/2$ to 3.69 while μ grows from 0 to 0.99, see Abramowitz and Stegun, 1970).

To estimate the size of this neighbourhood consider the representation of K as a power series:

$$K(\mu) = \frac{1}{2}\pi \sum_{n=0}^{n=\infty} p_n \mu^n = \frac{1}{2}\pi[1 + (\frac{1}{2})^2\mu + (\frac{1}{2}\frac{3}{4})^2\mu^2 + (\frac{1}{2}\frac{3}{4}\frac{5}{6})^2\mu^3 + \ldots] \tag{24}$$

Its coefficients p_n can be expressed in terms of Γ-functions and the following upper bound can be obtained:

$$p_n = [\frac{\Gamma(2n+1)}{\Gamma(n+1)^2 2^{2n}}]^2 < \frac{1}{n} \tag{25}$$

(The estimate of p_n can be obtained using the Stirling formula). This means that the sensitive area for $K(\mu)$ is very narrow. Indeed, for μ differing from 1 by no more than $\varepsilon = 5 \cdot 10^{-5}$ the asymptotic formula $\frac{1}{2}\ln\frac{16}{(1-\mu^2)}$ can be used for $K(\mu)$ (Zimring, 1988) and a simple calculation shows that the sensitive area has width $\Delta \leq \varepsilon + 10^{-50}$, i.e. the difference between ε and Δ is negligibly small. We have to evaluate the precision with which x_1 and x_2 should be given to yield μ with precision up to ε. Using the representation (22) it can be shown that the former should be equivalent to $\varepsilon \max[1+c, 1/c]$, where $c = (N_2 - N_3)/(N_3 - N_1)$. This means that if we take a point (x_1, x_2) from the sensitive area and change either x_1 or x_2 in the third digit then we get out of this area. Thus the area which has to be ignored due to uncontrolled growth of $K(\mu)$ within measurement errors of μ is so small that it has no practical importance. For every triad and observed distribution of the energy we have checked whether or not the growth of K is critical. It turned out that in all cases we are far away from the sensitive area. Therefore the period formula should give reliable results. To test this we made the first series of numerical simulations. Fig. 1 displays the results for the triad (4,12); (5,14); (9,13) (triad A_1 from Table 1). The horizontal axes depict the initial energy total energy $E = E_1 + E_2 + E_3$ of the triad and the energy in the two smallest modes as a fraction x of the energy in the main mode (it means that $x_1 = x_2 = x$). The vertical axis depicts the period $T = \tau_0$ in days. The values on the axis E are expressed in dimensionless units and were chosen according to observed data (see above). The figure gives the following qualitative picture: the period T grows when the initial energy decreases and T grows steeply if x and E are small simultaneously. Notice that for this case there is practically no dependence of T on x. Pictures for other triads are almost identical up to a rescaling of the axis.

The triads A_1 and A_3, for instance, have values of periods very close to those of triad A_1 for the same values of E and x, but the periods of the triad B_3 are substantially shorter. This can be explained by looking at the values of the interaction coefficients of all these triads $Z_{A_1}, Z_{A_3}, Z_{B_3}$ (see Table 1). While Z_{A_1} and Z_{A_3} differ from each other less than a factor of two, the coefficient Z_{B_3} is almost 10 times larger than Z_{A_1} and 5 times larger than Z_{A_3}. In all cases the values of the periods are between 30 and 60 days. The values of n_i for all these triads are so close that their influence can not be singled out.

In Figs. 2 and 3 the vertical axis is the same as before while the two horizontal axes depict x_1 and x_2; the energy E for which the figure was made, is printed at the top of the figure. Fig. 2 displays the results for triad (4,12); (5,14); (9,13) (triad A_1) and in Fig. 3 the same is shown for the triad (6,18); (7,20); (13,19) (triad A_3). In both figures we can see how the period tends to infinity near the degeneration line; the part of the picture that is symmetric with respect to the degeneration line has been omitted to make the picture more informative. For the same reason the triangle in the foreground has been omitted also. We can see that the period changes slowly if the ratio x_1/x_2 is far from the degeneration line and grows fast near this line. The same qualitative picture emerges for all other triads and for other values of the total energies in each triad.

After this detailed study of individual triads we may now raise a question with respect to the existence of weakly nonlinear regimes in a numerical model containing all modes in the region T21, not only resonant modes. A simple way to characterize the nonlinearity of a wave system is to check whether the ratio of the particle velocity $|\partial\psi/\partial\varphi|$ to the phase velocity $|2\pi\omega/(n-m)|$ is small enough (for instance, of order 10^{-2}). Often this ratio is chosen as a parameter of the systems' nonlinearity (Kamenkovich and Monin, 1978). Thus we have to estimate the amplitudes' values for which this ratio will be small enough to display weakly nonlinear behaviour. Let us denote

$$\varepsilon = |\frac{\partial\psi/\partial\varphi}{2\pi\omega/(n-m)}|, \tag{26}$$

and notice that

$$\partial\psi/\partial\varphi = a(m,n)\sin(m\lambda - \omega t)\frac{\partial}{\partial\varphi}P_{n_i}^{m_i}(\sin\varphi). \tag{27}$$

It means that

$$\varepsilon \leq a(m,n)\frac{n(n+1)(n-m)}{4\pi m} \mid \cos\varphi \cdot \frac{\partial}{\partial\varphi}P_{n_i}^{m_i}(\sin\varphi) \mid . \tag{28}$$

To estimate

$$\mid \cos\varphi \cdot \frac{\partial}{\partial\varphi}P_{n_i}^{m_i}(\sin\varphi) \mid \tag{29}$$

we use the formula which allows varying degree of Legendre functions to remove the derivative of the Legendre function (Abramowitz and Stegun, 1970). Then Rodrigues' formula allows us to obtain the expression for Legendre functions in terms of finite sums of cosines of even degree. As a result we obtain

$$\mid \cos\varphi \cdot \frac{\partial}{\partial\varphi}P_{n_i}^{m_i}(\sin\varphi) \mid < const \cdot n^n. \tag{30}$$

Now fixing some small ε we get the estimation of wave amplitude

$$a(m,n) < const \cdot n^{-n}. \tag{31}$$

(the constant now depends on ϵ) If this condition is valid for all three amplitudes of the triad's modes, the non-resonant interactions can be neglected and the behaviour of an isolated triad practically does not differ from the behaviour of the triad in the full wave field; the triad simply "does not notice" the presence of other modes of the spectral domain considered. But it is easy to see that for wave numbers 10 and larger this condition gives very small values of the amplitudes (order 10^{-10} in dimensionless units). So theoretically we cannot expect triads to behave as if they were isolated. To study the energy behaviour within the triad when the amplitudes are much larger, i.e. of the realistic order, was our next step.

In the first series of numerical simulations with the barotropic model we put at the initial time all the energy of the wave field into one isolated triad, for different energy distributions among the modes of the triad. It turned out that the energy oscillates between the modes of the triad for a few periods losing not more than $2-5\%$ of the initial energy and after that the energy goes out of the triad very fast. During a time comparable with the period of vacillation more then 90% of the energy is spreaded over the spectrum. Thus we have found two different phases in the evolution of the wave field: the vacillation phase and the spreading phase. The results for the triad A_1 (4,12); (5,14); (9,13) are shown in Figs. 4 to 6. The vertical axis depicts the initial energy of the triad which is the total initial energy of the wave field. The horizontal axis depicts time in hours. When the initial energy is very high (about two orders larger than observed values), energy begins to spread over the spectrum almost immediately, after two or three vacillation cycles with a period of about 3 days, and after that the triad loses more than 90% of its energy in 12 days (Fig. 4). Decreasing the initial energy to $3.3 \cdot 10^{-6}$, we see that the energy remains in the triad for about 120 days and this corresponds to 4 energy vacillation cycles with a period of 30 days (Fig. 5). After that the energy spreads over the spectrum and in about 250 days the triad loses more than 70% of its energy. In Fig. 6 when the initial energy is $E = 1.82 \cdot 10^{-6}$ the period is about 42 days and the time that the triad behaves as an isolated triad is 220 days. The experiment has also been done with the initial energy $E = 10^{-7}$. It turned out that in this case the energy does not spread over the spectrum during more than 2000 days and oscillates between the modes of the triad with a period of about 190 days. All the results shown in Figs. 4 to 6 have been obtained for the same distribution of the initial energy between the modes of the triad ($E_1 = E_2 = 0.25E_3$). Fig. 7 and Fig. 8 display the results for a realistic initial energy distribution (taken from the Table 1) between the modes of triad. Concerning Fig. 7, when the initial energy $E = 7.5 \cdot 10^{-7}$ the period is about 43 days and energy remains in the triad for about 470 days (this corresponds to 10 energy vascillation cycles). Increasing the initial energy to $E = 1.5 \cdot 10^{-6}$ we see that energy remains in the triad more then 240 days while the energy vascillation cycle is equal to 31 days.

These experiments have been repeated for the triad (4,12); (5,14); (9,13) with different initial distributions of the energy and for all other isolated triads in the spectral region T21. They all show the same qualitative behaviour.

From the figures we can see another interesting feature. It appears that part of the energy returns to the triad after spreading over the spectrum. This process of spreading and returning is repeated a few times. It happens for all triads that were studied.

5 Discussion

In the study of the periodic energy exchange in resonant triads in the spectral domain T21, some interesting results were obtained. First of all, it turned out that there exists a time scale dominated by the weakly nonlinear regimes for which vacillations occurs. There are 16 resonant triads within T21 whose vacillation periods are determined by the initial distribution of the energy. The values of these periods lie between 30 and 60 days for realistic values of the initial energy.

We have also found that an energy exchange cycle exists between the energy in the triad and that in the rest of the spectrum. This means that energy leaks out of the triad via a few modes. For instance, in the case of Fig. 9 energy leaks out of the triad mainly through two modes (this is immediately shown by applying the inverse Fourier transform to that part of the time evolution of inverse energy flow). The phenomenon of energy return deserves a separate study. Here we have only noticed its existence.

Thus we have found that there exists a time scale for which resonant triads behave as predicted by weakly nonlinear theory. Therefore it is not unlikely that some quasi-periodic low-frequency phenomena in the atmosphere or ocean can be understood and described within the context of weakly nonlinear theory. A large amount of evidence has accumulated that large-scale travelling planetary waves exist and that these waves propagate as expected from the linear theory. The periods of these waves depend on the total wavenumber n and range from 5 to 20 days. The fact that these waves can be detected in the observations is important from the viewpoint of predictability. They form the more predictable aspects of atmospheric flow. The observed large-scale waves have a nearly equivalent barotropic vertical structure, i.e., there is almost no shift of phase with height (Madden, 1979). Therefore these waves can be studied to some extent under the constraints of the barotropic vorticity equation. The results obtained may be used as a basis for studying resonant triads in more complicated models (with forcing, dissipation, more realistic initial conditions, etc.). If the periodic energy exchange among the waves in resonant triads is indeed observed in more realistic models, this may shed some more light on the intraseasonal oscillations in the extratropical atmosphere (Ghil, 1991, Jin and Ghil, 1990). These oscillations have periods of about 40-50 days. These periods are too long to be explained in terms of linear planetary waves but they may occur as a result of the slow energy exchange among waves in a resonant triad.

Here we would like to make a remark on the robustness of the obtained results, i.e. on their lack of sensivity to small deviations in the initial energy distributions and in the total initial energy. We see immediately from Figs. 1 to 3 that small variations in the energy distribution (i.e. in x_1 and x_2 when x_1/x_2 is far from the degeneration line) leads to small changes in the values of the periods. The qualitative picture does not change. It was shown analytically that the sensitive area close to the degeneration line is negligibly small and has no practical importance. Small variations in the total initial energy also do not lead to significant changes in vacillation behaviour, because the period is a smooth continuous function of the initial energy.

A few numerical simulations were performed in which initially 95% of the field energy is concentrated in the triad while the rest is randomly distributed over the field. Preliminary results show the same character of the wave field evolution though the vacillation time becomes shorter.

Finally we would like to note that the results obtained by studying the period formula (section 3) are of their own interest. These results are due to the general form of the system (2) and remain valid for all systems of differential equations of this form. Investigators have had to spend a lot of computer time to find out in numerical simulations at least a few general tendencies of the wave energy behaviour (Loesch and Deininger, 1979) that now are described and explained analytically.

Acknowledgements

Many thanks are due to the members of the predictability group at the Royal Netherlands Meteorological Institute (KNMI) for stimulating discussions during the preparation of this paper. The author wishes to thank Frank Selten for helping with processing the observed data and also with preparing the figures for publication. This investigation was supported by the Netherlands Organization for the Advancement of Research (NWO) by grant 07-30-015 of the NWO-program for aid to the former Soviet Union. Computing facilities were generously provided by the Royal Netherlands Meteorological Institute.

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Table 1. All resonant triads in spectral domain T21.

Triad	Wave (m,n)	Participating of the wave in the other triads	Interaction coefficient Z of the triad	Undimensional observed values of streamfunction amplitude	Undimensional observed energy of the triad
1	2	3	4	5	6
Isolated triads					
A1	(4,12) (5,14) (9,13)	- - -	7.82	1.35E-5 9.37E-6 1.18E-5	1.45E-7
A2	(1,20) (3,14) (4,15)	- - -	37.46	2.99E-6 7.43E-6 6.98E-6	5.4E-8
A3	(6,18) (7,20) (13,19)	- - -	13.66	4.38E-6 3.47E-6 3.41E-6	3.22E-8
A4	(1,14) (11,21) (12,20)	- - -	47.67	9.97E-6 2.87E-6 3.23E-6	5.84E-8
Groups of two connected triads					
B1	(2,6) (3,8) (5,7)	B2 - -	3.14	4.25E-5 3.57E-5 3.93E-5	5.09E-7
B2	(2,6) (4,14) (6,9)	B1 - -	14.63	4.25E-5 9.27E-6 3.39E-5	3.59E-7
B3	(2,20) (6,14) (8,15)	- B4 -	69.25	3.52E-6 8.31E-6 6.70E-6	6.1E-8
B4	(3,6) (6,14) (9,9)	- B3 -	11.31	6.19E-5 8.31E-6 6.70E-6	3.6E-7
B5	(3,10) (5,21) (8,14)	- B6 -	61.99	1.97E-5 2.68E-6 9.88E-6	1.32E-7

Table 1 (Continued)

1	2	3	4	5	6
B6	(8,11) (5,21) (13,13)	- B5 -	8.71	1.62E-5 2.68E-6 2.59E-6	6.9E-8

Group of six connected triads

C1	(1,6) (2,14) (3,9)	C3 C6 C5	28.98	5.37E-5 8.35E-6 3.52E-5	4.95E-7
C2	(2,7) (11,20) (13,14)	- C3 -	2.77	3.85E-5 2.95E-6 3.27E-6	1.78E-7
C3	(1,6) (11,20) (12,15)	C1 C2 C4	15.08	5.37E-5 2.95E-6 5.04E-6	2.62E-7
C4	(9,14) (3,20) (12,15)	- - C3	74.93	7.99E-6 3.41E-6 5.04E-6	4.87E-8
C5	(3,9) (8,20) (11,14)	C1 - -	32.12	3.52E-5 3.46E-6 6.52E-6	2.68E-8
C6	(2,14) (17,20) (19,19)	C1 - -	11.05	8.35E-6 1.80E-6 1.31E-6	3.34E-8

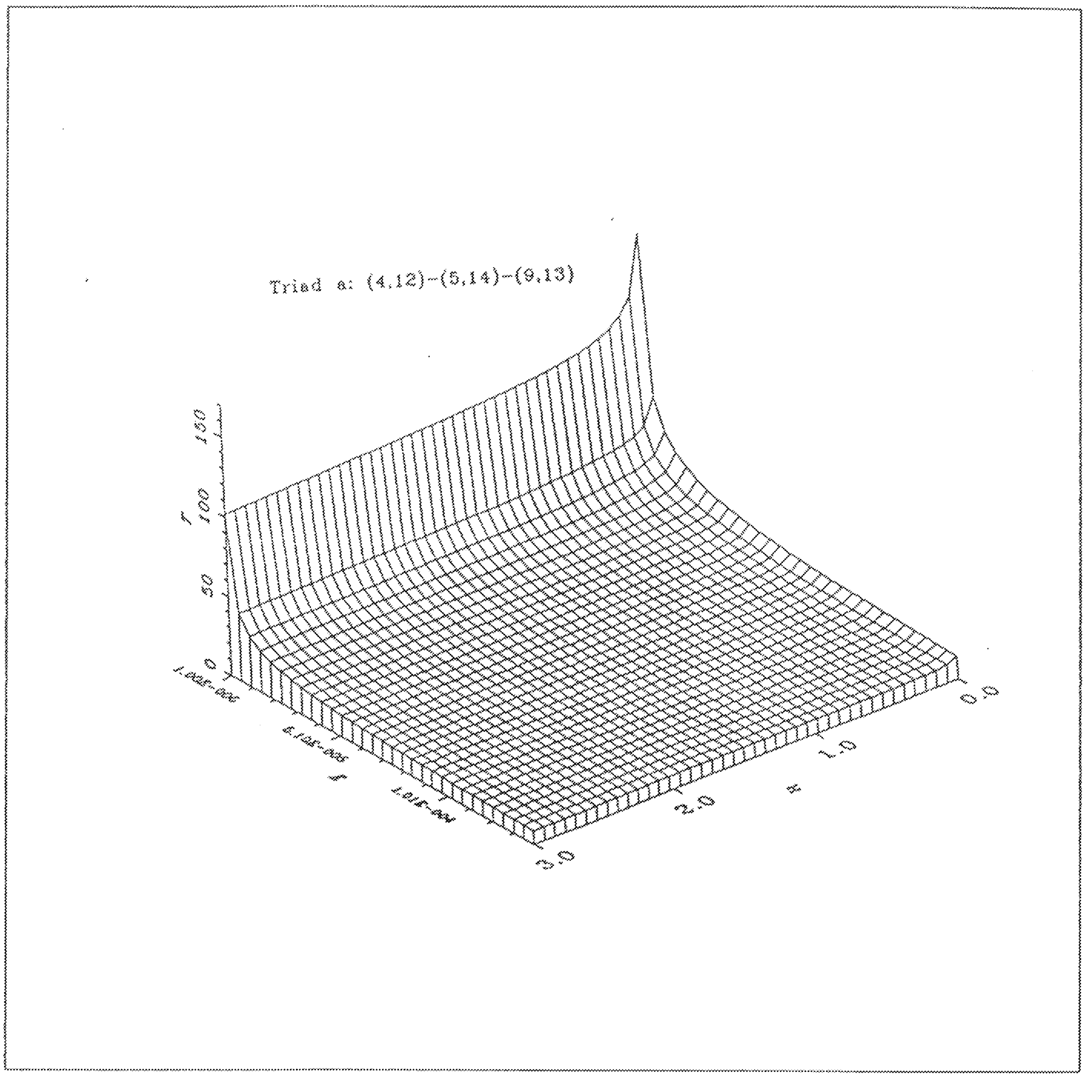

Figure 1: The value of the energy exchange period $T = \tau_0$ (in days) for the triad A_1 on the initial energy $E = E_1 + E_2 + E_3$ (nondimensional) and the initial energy distribution between the modes of the triad taken in the form $E_1 = E_2 = x \cdot E_3$, where E_1 and E_2 are the energies of the modes with the smallest frequencies (4, 12) and (5, 14).

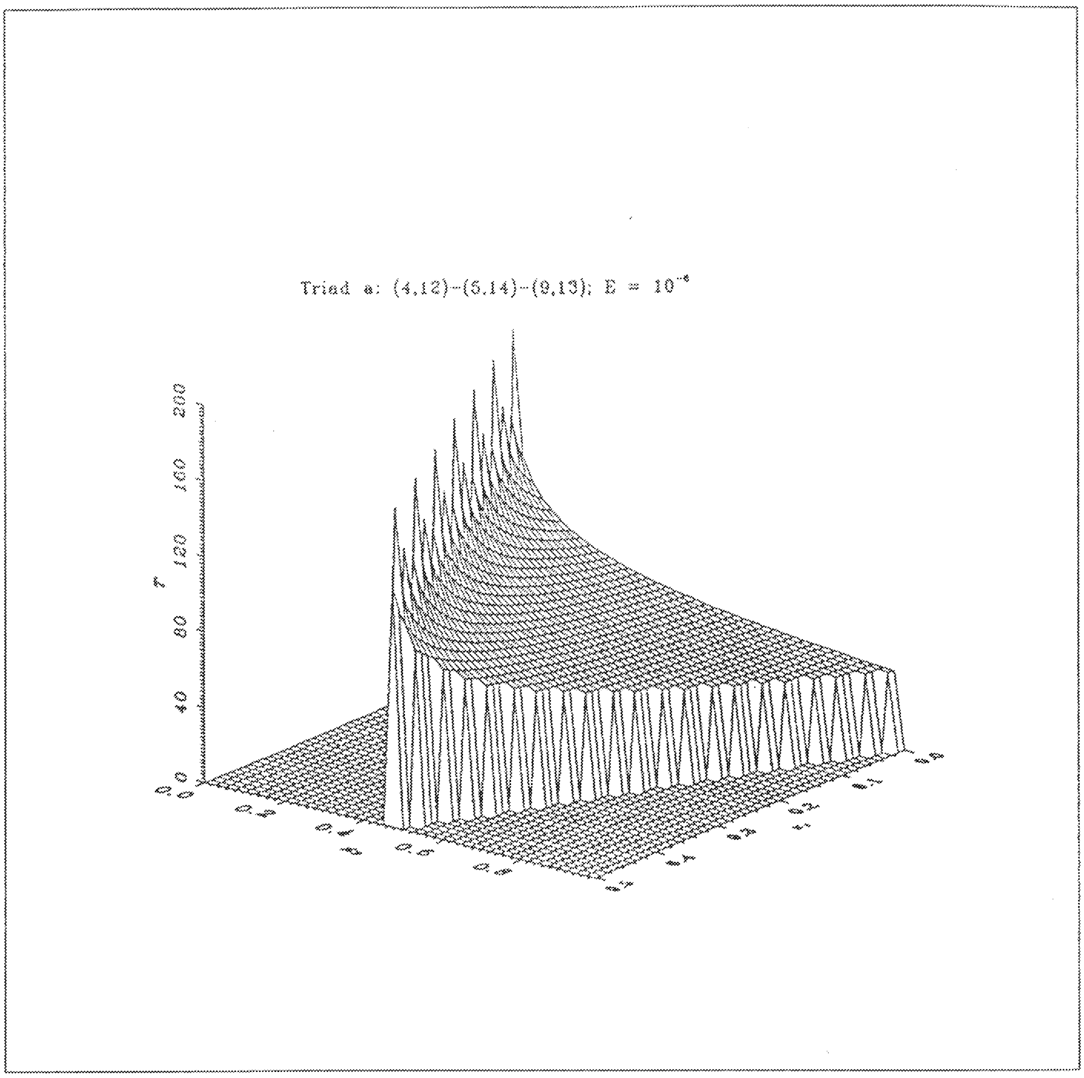

Figure 2: The value of the energy exchange period T (in days) for the triad A_1 with the total initial energy $E = 10^{-5}$ and the initial energy distribution between the modes of the triad taken in the form $E_1 = x_1 \cdot E_3$, $E_2 = x_2 \cdot E_3$ where E_1 and E_2 are the energies of the modes with the smallest frequencies (4, 12) and (5, 14).

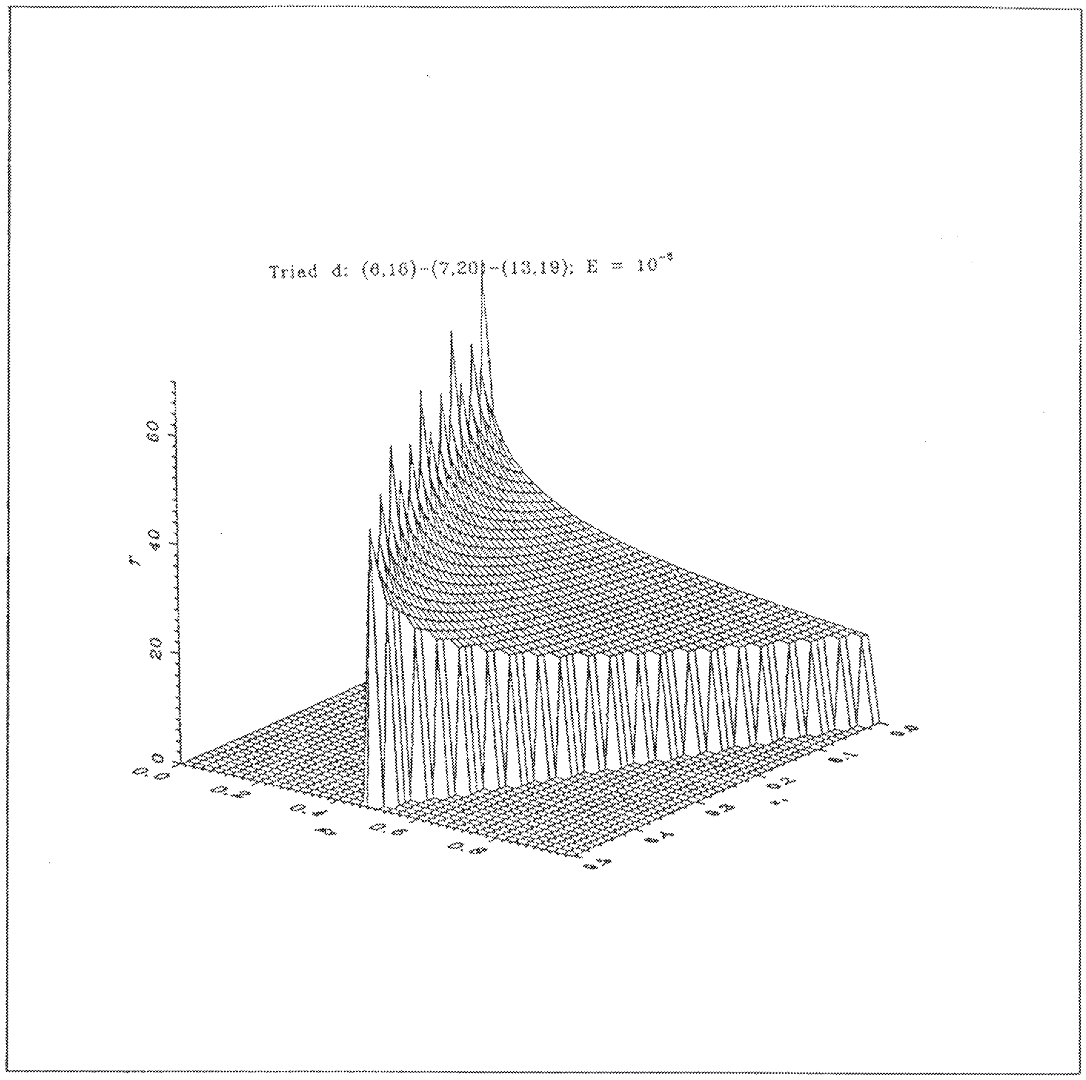

Figure 3: Same as Fig. 2 but for the triad A_3; the modes with the smallest frequencies are (6, 18) and (7, 20).

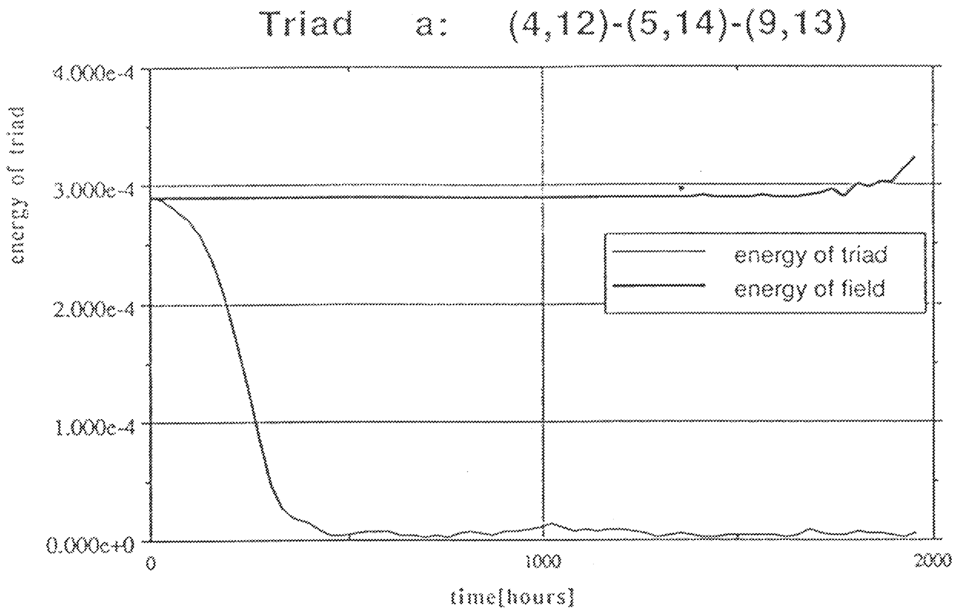

Figure 4: The behaviour of the total energy of the triad A_1 with the initial energy $E = 2.9 \cdot 10^{-4}$ in the complete wavefield (barotropic model truncated to T21). At the initial time the total energy of the wavefield contains in the triad. Initial energy distribution between the modes of the triad is taken in the form $E_1 = E_2 = 0.25 \cdot E_3$.

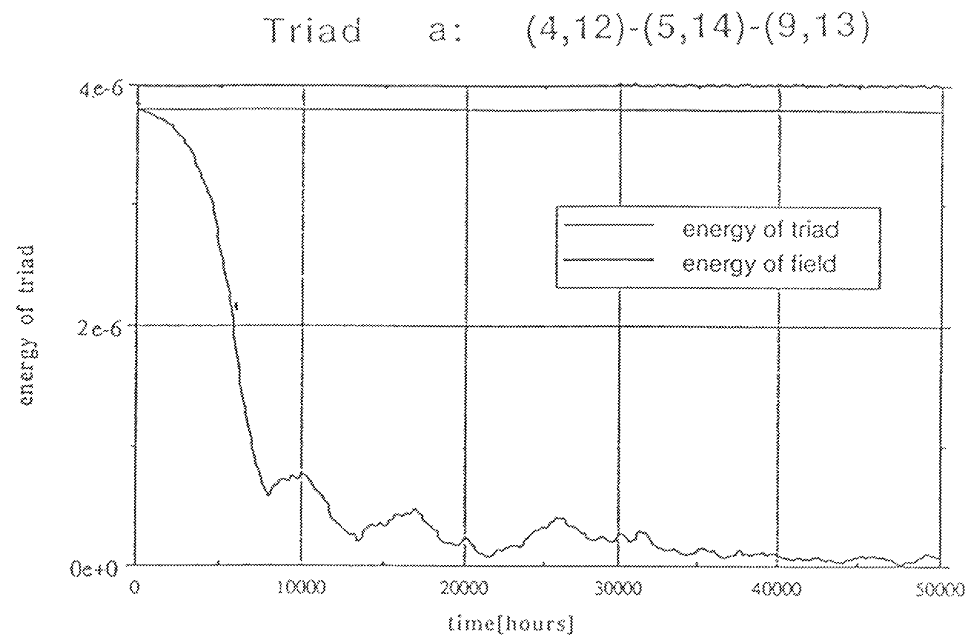

Figure 5: Same as Fig. 4 but for the initial energy $E = 3.9 \cdot 10^{-6}$.

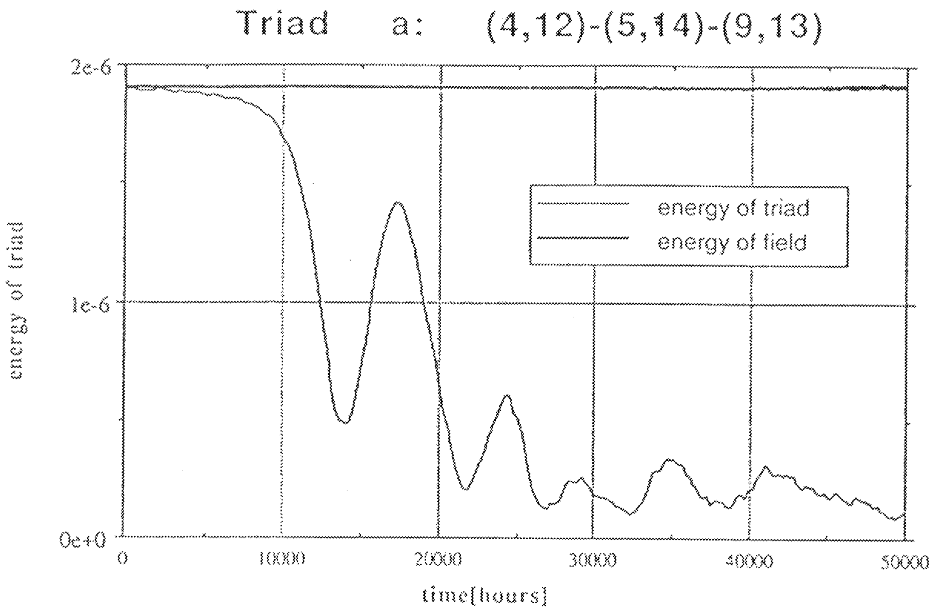

Figure 6: Same as Fig. 4 but for the initial energy $E = 1.9 \cdot 10^{-6}$.

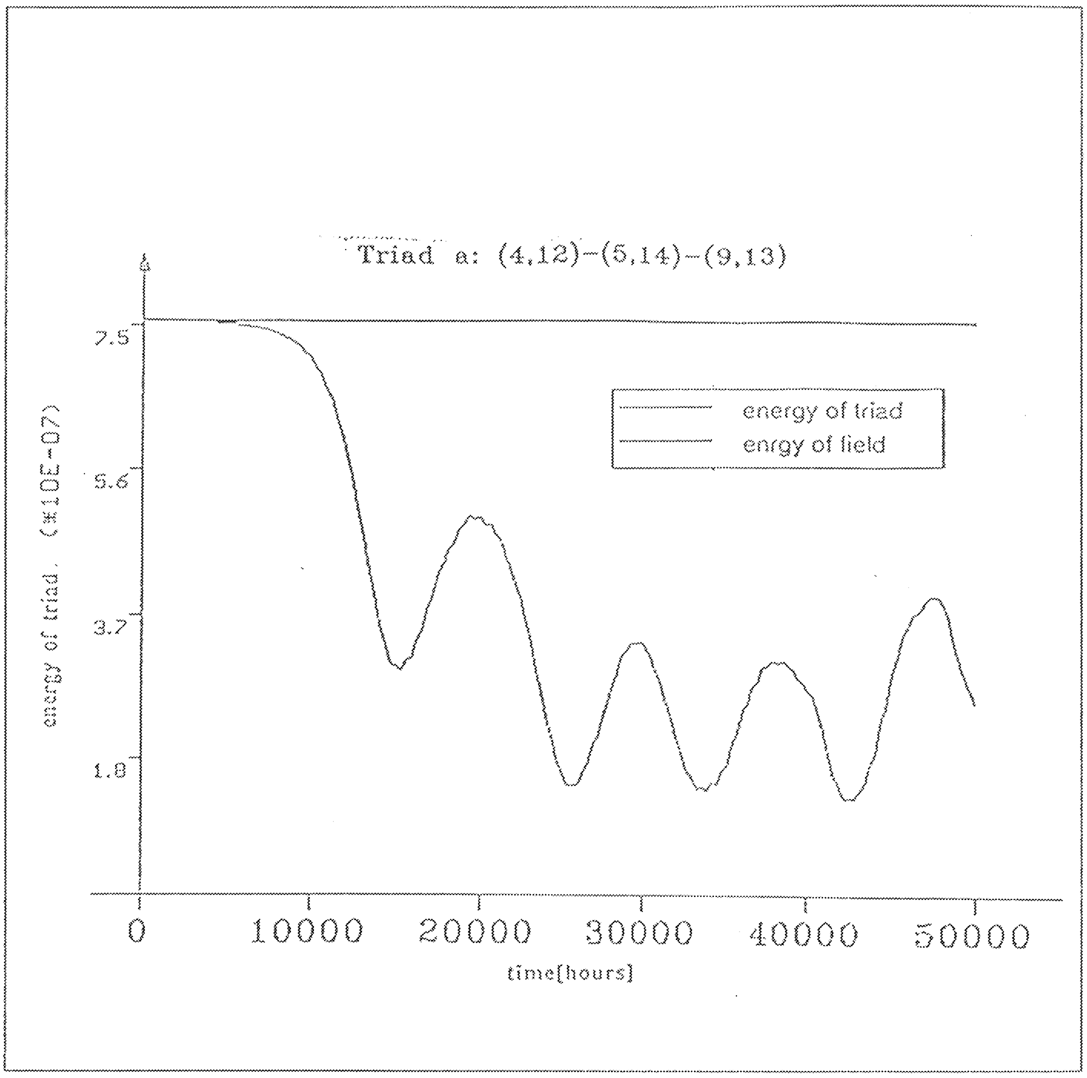

Figure 7: Same as Fig. 4 but for the initial energy $E = 7.5 \cdot 10^{-7}$ and realistic initial energy distribution taken from the Table 1 (the mode (4, 12) has the maximal initial energy).

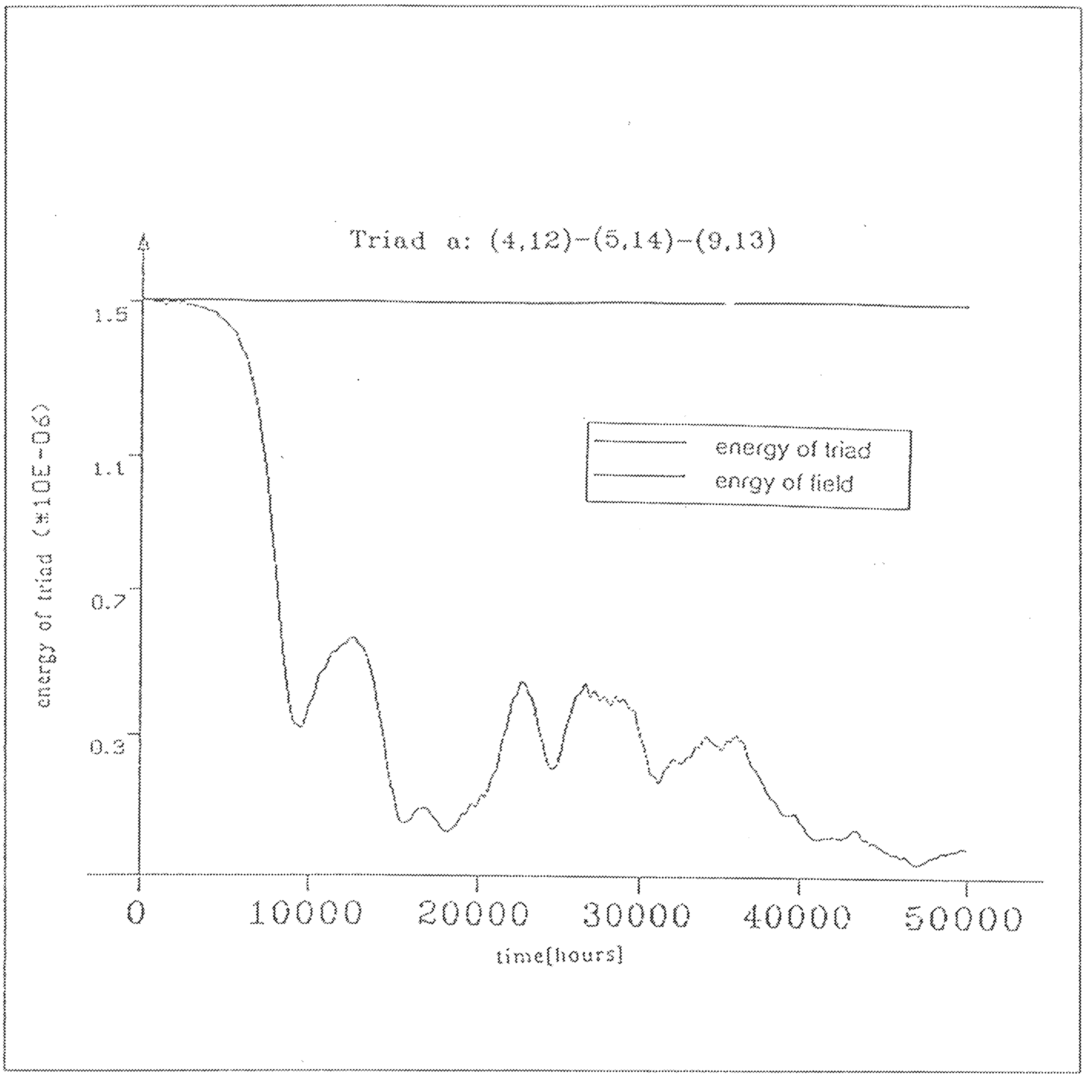

Figure 8: Same as Fig. 7 but for the initial energy $E = 1.5 \cdot 10^{-6}$.

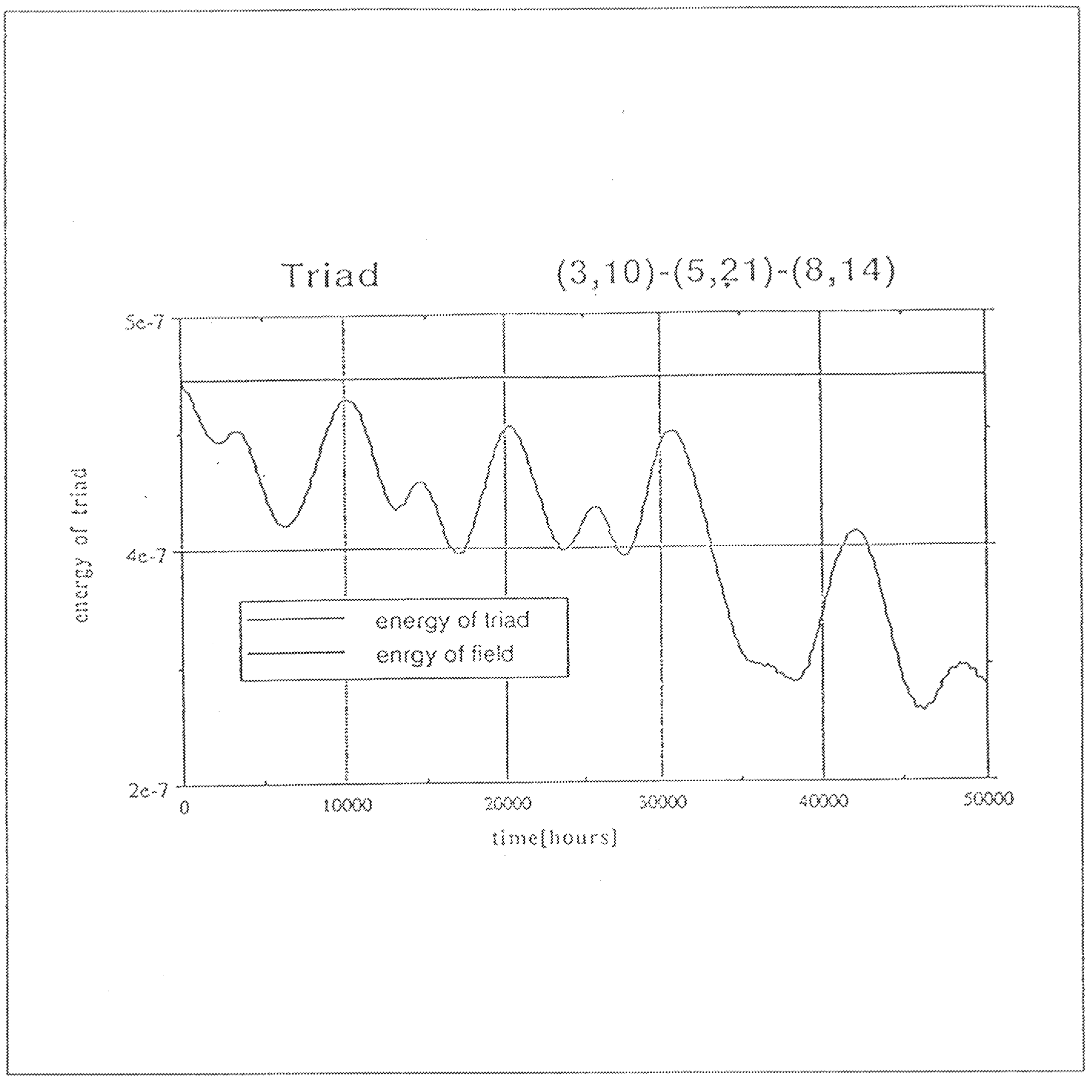

Figure 9: Same as Fig. 4 but for the triad B_5 and the initial energy $E = 4.75 \cdot 10^{-7}$.

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