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J. G. J. SCHOLTE

ON SEISMIC WAVES IN A SPHERICAL EARTH

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STAATSDRUKKERIJ- EN UITGEVERIJBEDRIJF / 'S-GRAVENHAGE

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FOREWORD

This paper deals with the theory of seismic waves in the mantle and the core of the earth; it differs from many other publications on this subject by taking the curvature of the boundaries into account. In this way Dr J. G. J. Scholte undertook to investigate several phenomena due to the sphericity of the earth, e.g. the diffraction of the *P* waves by the core and the behaviour of the waves in the vicinity of a caustic.

Dr J. Veldkamp, director of the geophysical division of this Institute gave valuable contribution by reading the manuscript and discussing it with the author. Mr J. A. As, assistant, carried out the numerical computations.

Director in Chief RNMI,
Ir C. J. Warners

INTRODUCTION

A theory concerning the propagation of seismic waves has to start with some suppositions defining the origin of these waves and the transmitting medium. In this paper it has been assumed that the medium consists of a solid spherical shell surrounding a liquid sphere and that both materials are perfectly elastic, homogeneous and isotropic. The primary disturbance is supposed to be confined to a space the dimensions of which are negligible in comparison to all other occurring linear dimensions ("point-source").

These assumptions have been made in order to simplify the mathematical treatment of the propagation as much as possible without losing every resemblance to actual conditions; a further simplification would be effected by neglecting the sphericity of the Earth but then some essential features of the seismic waves as e.g. the existence of a caustic, the diffraction or screening by the core, would be neglected as well. Moreover it is felt that it is worth while to ascertain the circumstances in which this simplification would be justified.

It is evident that the propagation in this elastic system will be described by means of spherical coordinates with their origin in the centre of the system. Writing the equations of motion in these coordinates we derive (in § 1) three general kinds of motion into which any movement can be decomposed: a movement without rotation and two movements without divergence, namely one where the radial component is identical to zero and the other where the rotation in the radial direction is zero. Each of these waves appears to be derivable from a function satisfying the simple wave equation.

Apart from the equations of motion the elastic displacements and tensions have to satisfy some conditions at the two spherical surfaces of discontinuity existing in the assumed model of the Earth. These conditions can only be treated mathematically if the radial coordinate appears separatedly in the formulae; this requirement leads directly (§ 2) to a special kind of movement, the spherical wave, which is harmonic (or exponential) in the time t and proportional to the product of a spherical surface harmonic and a cylindrical function. In these expressions two parameters appear: the coefficient ω of t and the order n of the spherical harmonic. For each value of n and ω a spherical wave system exists that satisfies the boundary conditions.

Several seismic waves are connected with only one of the surfaces of discontinuity; *PKP* is independent of the existence of the outer surface and *PP* has no connection with the surface of the core. These waves may be derived from a system that satisfies the boundary equations at both surfaces (as will be seen in § 11), but this method is rather cumbersome. A more expedient way is to derive waves connected with only one surface of discontinuity by means of a model with only one boundary; therefore we consider firstly (in § 3) the wave systems that are possible in an infinite solid surrounding a liquid sphere, and secondly (§ 9) the systems existing in a solid sphere without a core. Finally in § 11 the system that is possible in a solid spherical layer bounded by the vacuum and by a liquid will be calculated.

The further treatment of the rather complicated expressions which represent these

§ 1. GENERAL SOLUTIONS

We start the investigation with a survey of the different kinds of waves that are possible in a homogeneous elastic medium. The equations of motion expressed in spherical coordinates (r, ϑ, φ) are given by LOVE (1944):

$$\sigma \frac{\partial^2}{\partial t^2} \operatorname{div} l = (\lambda + 2\mu) \triangle \operatorname{div} l$$

$$\sigma \frac{\partial^2}{\partial t^2} (r \operatorname{curl}_r l) = \mu \triangle (r \operatorname{curl}_r l)$$

$$\sigma \frac{\partial^2}{\partial t^2} (r l_r) = \mu \triangle (r l_r) + (\lambda + \mu) r \frac{\partial}{\partial r} (\operatorname{div} l) - 2\mu \operatorname{div} l,$$

where l = the displacement, λ and μ = the constants of LAMÉ, and σ = the density.

The right-hand side of the last equation may be written in a more symmetrical form; by using the identity, valid for any vector B:

$$r \operatorname{curl}_r \operatorname{curl} B = \frac{1}{r} \frac{\partial}{\partial r} (r^2 \operatorname{div} B) - \triangle (r B_r)$$

that part of the third equation becomes $(\lambda + 2\mu) r\, \partial \operatorname{div} l / \partial r - \mu r \operatorname{curl}_r \operatorname{curl} l$. Introducing the velocities $\alpha = \left(\frac{\lambda + 2\mu}{\sigma}\right)^{1/2}$ and $\beta = \left(\frac{\mu}{\sigma}\right)^{1/2}$ the three equations are:

$$\left(\frac{\partial^2}{\partial t^2} - \alpha^2 \triangle\right) \operatorname{div} l = 0, \qquad \left(\frac{\partial^2}{\partial t^2} - \beta^2 \triangle\right) (r \operatorname{curl}_r l) = 0$$

$$\frac{\partial^2 l_r}{\partial t^2} = \alpha^2 \frac{\partial}{\partial r} \operatorname{div} l - \beta^2 \operatorname{curl}_r \operatorname{curl} l.$$

The displacement is derivable from a scalar potential — Φ and a vector potential A:

$$l = \nabla \Phi + \operatorname{curl} A;$$

as the divergence of A is of no importance whatever we assume $\operatorname{div} A = 0$.

Using again the identity above mentioned the equations are easily reduced to

$$(\partial^2/\partial t^2 - \alpha^2 \triangle)\, \Phi = 0$$

$$(\partial^2/\partial t^2 - \beta^2 \triangle)\, (r A_r) = 0$$

$$(\partial^2/\partial t^2 - \beta^2 \triangle)\, (r \operatorname{curl}_r A) = 0.$$

Consequently any motion can be decomposed into three kinds of motion, namely in those movements for which two of the three functions Φ, A_r and $\operatorname{curl}_r A$ vanish identically.

I. The movement of the first kind is defined by $A_r = 0$ and $\operatorname{curl}_r A = 0$; it follows that $A = 0$, so that $l = \nabla \Phi$, Φ satisfying the α-wave equation. The seismic P waves form an example of this kind of movement.

II. Movements of the second kind are characterised by $\Phi = 0$ and $A_r = 0$.

§ 2. SPHERICAL WAVES

In the problem we are concerned with in this paper we suppose that a spherical body is embedded in an infinite and homogeneous medium with different elastic properties; the movements possible in this system have to satisfy some boundary conditions at the surface of the sphere. In order to deal with these conditions it is necessary to use expressions for the displacement and the stress in which the coordinate r appears separatedly from the other variables. These waves will be called "spherical waves" (with centre at $r = 0$).

Consequently we put $\Phi = f(r) . F(t, \vartheta, \varphi)$, which gives by substitution in the α-wave equation:

$$\frac{r}{f}\frac{d^2(rf)}{dr^2} - \frac{r^2}{\alpha^2}\cdot\frac{1}{F}\frac{\partial^2 F}{\partial t^2} + \frac{\Delta' F}{F} = 0$$

It follows that $\frac{\Delta' F}{F}$ = a constant, which we denote by $-n(n+1)$, where n is a arbitrary number; then the equation becomes

$$\frac{1}{fr}\frac{d^2(fr)}{dr^2} - \frac{n(n+1)}{r^2} - \frac{1}{\alpha^2}\frac{1}{F}\frac{\partial^2 F}{\partial t^2} = 0;$$

it is evident that $\frac{1}{F}\frac{\partial^2 F}{\partial t^2}$ must be a constant too, which will be denoted by $-\omega^2$, where ω is an arbitrary quantity.

The equation is now

$$\frac{1}{r}\frac{d^2 fr}{dr^2} + \left\{\frac{\omega^2}{\alpha^2} - \frac{n(n+1)}{r^2}\right\} f = 0,$$

hence $F = e^{-i\omega t} S_n(\vartheta, \varphi)$ and $f = \left(\frac{\omega r}{\alpha}\right)^{-1/2} H_{n+1/2}\left(\frac{\omega r}{\alpha}\right)$; the function S_n is a spherical surface harmonic of the n^{th} order and $H_{n+1/2}$ is a Hankel function of the first or second kind, of order $n + \frac{1}{2}$. As we shall only investigate waves that are independent of φ the spherical harmonic is a Legendre function $P_n(\cos\vartheta)$.

Thus the conditions of the problem lead us directly to wave-potentials of the form

$$e^{-i\omega t} . \left(\frac{\omega r}{\alpha}\right)^{-1/2} H_{n+1/2}\left(\frac{\omega r}{\alpha}\right) . P_n(\cos\vartheta),$$

where ω and n are arbitrary constants (the expression M for transverse waves contains β instead of α).

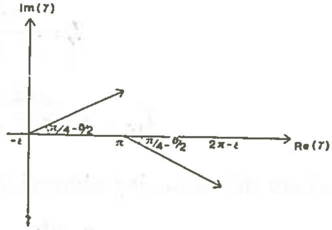
As these functions appear in most of the following considerations it is of some importance to ascertain the kind of waves they represent. This may be realized by using the saddle-point approximation, which has been developed by DEBIJE when

The saddlepoints situated on the path of integration are $\gamma = 0$ and $\gamma = \pi$, and the direction of the lines of steepest descent are as indicated in fig. 2. The two saddlepoints yield

$$\frac{e^{\pm i\left\{(n+{}^1\!/_2)\vartheta - \frac{\pi}{4}\right\}}}{\sqrt{2\pi\, n \sin\vartheta}}\,;$$

it follows that for not too small values of n

$$P_n(\cos\vartheta) \approx \sqrt{\frac{2}{\pi\, n \sin\vartheta}} \cdot \cos\left\{(n+\tfrac{1}{2})\,\vartheta - \pi/4\right\} \qquad (5)$$



Directions of steepest descent
FIG. 2

Combining the two results (2) and (5) we obtain two waves with the phase-factor e^{is} where $s = \sqrt{h^2 r^2 - (n+\frac{1}{2})^2} - (n+\frac{1}{2})$ arc cos $(n+\frac{1}{2})/hr \pm (n+\frac{1}{2})\vartheta$, and $h = \omega/\alpha$, which may be interpreted (fig. 3) as rays passing the centre O at a distance equal to $(n+\frac{1}{2})/h$. The angle of incidence $\varphi =$ arc sin $(n+\frac{1}{2})/hr$ satisfies the well-known ray-equation: $r \sin\varphi =$ a constant.

The Hankel function of the first kind thus appears to be connected with a wave progressing in the positive radial direction; the function $H^{(2)}(z) \, . \, e^{-i\omega t}$ describes an inward moving wave.

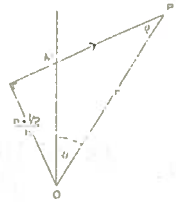


FIG. 3

Finally we calculate the quantities that will be needed in the next section to express the boundary conditions at the surface of the sphere $r = a$; these conditions involve the displacement and the stress components. The latter are given by:

$$T_{rr} = \lambda \operatorname{div} l + 2\mu \frac{\partial l_r}{\partial r}$$

$$T_{r\vartheta} = \mu \left(\frac{1}{r}\frac{\partial l_r}{\partial \vartheta} + \frac{\partial l_\vartheta}{\partial r} - \frac{l_\vartheta}{r} \right)$$

$$T_{r\varphi} = \mu \left(\frac{1}{r \sin\vartheta}\frac{\partial l_r}{\partial \varphi} + \frac{\partial l_\varphi}{\partial r} - \frac{l_\varphi}{r} \right).$$

Henceforward we shall use the $\zeta_n^{(1)}$ and $\zeta_n^{(2)}$ functions as defined by HEINE-DEBIJE (FRANK and v. MISES):

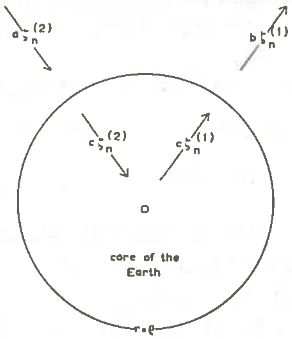
$$\zeta_n^{\substack{(2)\\(1)}}(z) = \sqrt{\frac{\pi}{2}} \cdot z^{-{}^1\!/_2}\, H_{n+\frac{1}{2}}^{\substack{(2)\\(1)}}(z).$$

The displacements and stresses are for

I. Longitudinal waves with scalar potential

$$\Phi = e^{-i\omega t}\, P_n(\cos\vartheta)\, \zeta_n^{\substack{(2)\\(1)}}(\omega r/\alpha)$$

§ 3. COEFFICIENTS OF SPHERICAL REFLECTION AND REFRACTION



Schematic representation of a set of elementary waves. FIG. 4

The wave systems possible in a body consisting of an elastic sphere surrounded by an infinite, elastic medium are composed of spherical waves as described in the previous paragraph. For each value of the wave parameters ω and n a set of spherical waves of one of the three kinds P, SV or SH exists. Such a set consists of a $\zeta_n^{(1)}$ and a $\zeta_n^{(2)}$ wave with amplitudes which may be different in the region outside the sphere $r = \varrho$ but which are equal inside that sphere as the movement has to remain finite at $r = 0$ (each function ζ_n becomes infinite at $r = 0$, but $\zeta_n^{(1)} + \zeta_n^{(2)}$ remains finite). Hence the possible sets are:

1. $\Phi = b\,\zeta_n^{(1)}(\omega r/\alpha) + a\,\zeta_n^{(2)}(\omega r/\alpha)$ for $r > \varrho$ and $\Phi = c\,\psi_n(\omega r/\alpha')$ for $r < \varrho$ where $\psi_n = \zeta_n^{(1)} + \zeta_n^{(2)}$, $\alpha' =$ longitudinal velocity inside the sphere and a, b, c are constants.

2. $A_\varphi = -\,\partial M/\partial\vartheta$, with $M = B\,\zeta_n^{(1)}(\omega r/\beta) + A\,\zeta_n^{(2)}(\omega r/\beta)$ for $r > \varrho$ and $M = C\psi_n(\omega r/\beta')$ ($\beta' =$ transverse velocity for $r < \varrho$).

3. $A_r = rM$ with M identical to M of the second kind. For brevity's sake we have omitted the factors containing t and ϑ.

We now consider whether these sets can satisfy the boundary conditions at the surface of the sphere; these conditions relate to the normal and tangential components of movement and stress. In the case of waves of the first and second set they are expressed by four equations concerning l_r, l_ϑ, T_{rr} and $T_{r\vartheta}$ respectively, while for waves of the third set only two equations appear as only l_φ and $T_{r\varphi}$ do not vanish identically. The equations being homogeneous in the three amplitudes a, b and c or A, B and C) it follows that only a set of waves of the last kind can fulfil the boundary conditions; waves of the first or second set are each separatedly incapable of satisfying the equations and consequently the solution will consist of linear combinations of those two sets.

In other words in this elastic body (sphere + surrounding infinite medium) two kinds of independent wave systems are possible, namely a set of SH waves and a combination of P and SV waves. (This conclusion applies also to waves in an inhomogeneous medium; here too SH waves can exist without other kinds of waves, but between SV and P waves a coupling occurs).

In connection with the application of this theory (§ 4) we shall confine ourselves to the case of P and SV waves; the calculation of the SH system, which is rather more simple, will therefore be omitted. Moreover, we restrict the investigation still more by assuming the rigidity of the sphere to be equal to zero; as seismic data appear to indicate that transverse waves are not able to travel through the core of the earth this assumption is not unwarranted.

incident wave $\zeta_n^{(2)}$ a longitudinal wave $\zeta_n^{(1)}(hr)$ and a transverse wave $\zeta_n^{(1)}(kr)$ reflected directly at the surface of the sphere; in addition inside the sphere a refracted wave $\zeta_n^{(2)}(h'r)$ will occur. The amplitudes of these reflected waves will depend on the values at $r=\varrho$ of the three ζ-functions involved and consequently be independent from the value of $\zeta_n^{(1)}(h'r)$ at $r=\varrho$. In the expressions for b and B this function $\zeta_n^{(1)}(h'\varrho)$ appears implicitely as the quantities x' and z' contain the function $\psi_n(h'\varrho)$ which is equal to $\zeta_n^{(1)}(h'\varrho)+\zeta_n^{(2)}(h'\varrho)$. It follows that in order to obtain the amplitudes of the directly reflected wave we have to split up these expressions into a part which is independent of $\zeta_n^{(1)}(h'\varrho)$ and a part wich contains this function.

This is realised by means of the identities

$$\psi_n(h'\varrho)=\zeta_n^{(1)}(h'\varrho)+\zeta_n^{(2)}(h'\varrho)$$

$$x'\,\psi_n(h'\varrho)=x_1'\,\zeta_n^{(1)}(h'\varrho)+x_2'\,\zeta_n^{(2)}(h'\varrho).$$

The expressions b, B and c become in the case of an incident longitudinal wave

$$b=-a\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(1)}(h\varrho)}\cdot\left[1-\frac{x_1-x_2}{D}\left((x_2'+1)\,y_1k^2\varrho^2+z_2'\{y_1+2n(n+1)\}\right)\right]+$$

$$+a\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)}\,\frac{(x'-x_2)\,y_1k^2\varrho^2}{D}\cdot\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(h\varrho)}\,\frac{\sigma'}{\sigma}\,\frac{(x_1'-x_2')\,y_1k^2\varrho^2}{D}\cdot\frac{1}{D'}$$

$$B=-a\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(1)}(k\varrho)}\,\frac{(x_1-x_2)\,.\,2z_2'}{D}+$$

$$+a\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)}\,\frac{(x_1-x_2)\,y_1k^2\varrho^2}{D}\cdot\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(k\varrho)}\,\frac{\sigma'}{\sigma}\,\frac{(x_1'-x_2')\,.\,2x_1k^2\varrho^2}{D}\cdot\frac{1}{D'}$$

$$c=+a\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)}\,\frac{(x_1-x_2)\,y_1k^2\varrho^2}{D}\cdot\frac{1}{D'}$$

and in the case of an incident SV wave

$$B=-A\,\frac{\zeta_n^{(2)}(k\varrho)}{\zeta_n^{(1)}(k\varrho)}\cdot\left[1-\frac{y_1-y_2}{D}\{(x_2'+1)\,x_1k^2\varrho^2+z_2'(x_1+1)\}\right]+$$

$$+A\,\frac{\zeta_n^{(2)}(k\varrho)}{\zeta_n^{(2)}(h'\varrho)}\,\frac{(y_1-y_2)\,n(n+1)\,x_1k^2\varrho^2}{D}\cdot\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(k\varrho)}\,\frac{\sigma'}{\sigma}\,\frac{(x_1'-x_2')\,2x_1k^2\varrho^2}{D}\cdot\frac{1}{D'}$$

$$b=-A\,\frac{\zeta_n^{(2)}(k\varrho)}{\zeta_n^{(1)}(h\varrho)}\,\frac{(y_1-y_2)\,n(n+1)\,z_2'}{D}+$$

$$+A\,\frac{\zeta_n^{(2)}(k\varrho)}{\zeta_n^{(1)}(h'\varrho)}\,\frac{(y_1-y_2)\,n(n+1)\,x_1k^2\varrho^2}{D}\cdot\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(h\varrho)}\,\frac{\sigma'}{\sigma}\,\frac{(x_1'-x_2')\,y_1k^2\varrho^2}{D}\cdot\frac{1}{D'}$$

$$c=+A\,\frac{\zeta_n^{(2)}(k\varrho)}{\zeta_n^{(2)}(h'\varrho)}\,\frac{(y_1-y_2)\,n(n+1)\,x_1k^2\varrho^2}{D}\cdot\frac{1}{D'}$$

In the case of a longitudinal incident wave the expression c becomes

$$c = a\,.\,(LL')\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)}\sum_{s=0}^{\infty}\left\{(L'L')\,\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(2)}(h'\varrho)}\right\}^s$$

representing the amplitude of waves which are refracted into the core and are s times internally reflected.

Turning now to the second term of b and B we note that the first factor in these terms is the coefficient of refraction (LL') or (TL') for waves entering the sphere; the second factor must be interpreted as coefficients of refraction for waves emerging from the sphere into the solid medium, namely:

longitudinal — longitudinal refraction:

$$\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(h\varrho)}\,(L'L),\ \text{with}\ (L'L) = -\frac{x_2'-x_1'}{D}\,\frac{\sigma'}{\sigma}\,y_1 k^2\varrho^2$$

longitudinal — transverse refraction:

$$\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(k\varrho)}\,(L'T),\ \text{with}\ (L'T) = -\frac{x_2'-x_1'}{D}\,\frac{\sigma'}{\sigma}\cdot 2x_1 k^2\varrho^2.$$

Developing again the remaining factor $1/D'$ into the series

$$\sum_{s=0}^{\infty}\left\{(L'L')\,\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(2)}(h'\varrho)}\right\}^s$$

we obtain the amplitudes of waves which have been s times internally reflected during their stay inside the core.

In this way the original expressions for b, B, . . . are changed into easily interpretable series; in recapitulation we write down the new form of the amplitude b of the total outwards running longitudinal wave connected with an incident longitudinal wave:

$$b = a(LL)\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(1)}(h\varrho)} + a(LL')\,\frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)}\cdot\sum_{s=0}^{\infty}\left\{(L'L')\,\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(2)}(h'\varrho)}\right\}^s\cdot(L'L)\,\frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(h\varrho)}.$$

Apparently the first term represents the wave reflected directly at the surface of the core (PcP) and each term of the second part expresses the amplitude of a wave which has entered the core and emerges again after suffering s internal reflections ($PK^{s+1}P$ wave). Analogous expressions for the other waves are obtained without difficulty.

It follows that the secondary movement connected with this incident wave is equal to the primary one, but with an opposite sign; apparently the continuous part of the primary movement disappears in every point in- and outside the sphere.

The total movement outside the sphere is therefore equal to the discontinuous part of the primary motion together with its appropriate secondary wave system, consisting of

$$PcP \text{ wave: } \Phi = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, \zeta_n^{(1)}(h\varrho_1) \,.\, (LL) \frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(1)}(h\varrho)} \cdot \zeta_n^{(1)}(hr)\, P_n(\cos\vartheta)\, e^{-i\omega t}$$

$$PcS \text{ wave: } M = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, \zeta_n^{(1)}(h\varrho_1) \,.\, (LT) \frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(1)}(k\varrho)} \cdot \zeta_n^{(1)}(kr)\, P_n(\cos\vartheta)\, e^{-i\omega t}$$

$$PK^{s+1}P \text{ wave: } \Phi = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, \zeta_n^{(1)}(h\varrho_1) \,.\, (LL') \frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)} \cdot \left\{ (L'L') \frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(2)}(h'\varrho)} \right\}^s \cdot (L'L) \frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(k\varrho)} \cdot \zeta_n^{(1)}(kr)\, P_n(\cos\vartheta)\, e^{-i\omega t}$$

$$PK^{s+1}S \text{ wave: } M = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, \zeta_n^{(1)}(h\varrho_1) \,.\, (LL') \frac{\zeta_n^{(2)}(h\varrho)}{\zeta_n^{(2)}(h'\varrho)} \cdot \left\{ (L'L') \frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(2)}(h'\varrho)} \right\}^s \cdot (L'T) \frac{\zeta_n^{(1)}(h'\varrho)}{\zeta_n^{(1)}(k\varrho)} \cdot \zeta_n^{(1)}(kr)\, P_n(\cos\vartheta)\, e^{-i\omega t}$$

Infinite series of the form

$$F = \sum_{n=0}^{\infty} f(n) \,.\, P_n(\cos\vartheta)$$

are made more tractable by means of the transformation of WATSON (1919); in applying this method we change this series into the integral

$$F = \frac{1}{2i} \int_l \frac{f(\nu - \tfrac{1}{2})}{\cos\nu\pi} P_{\nu-1/2}\{\cos(\pi-\vartheta)\}\, d\nu$$

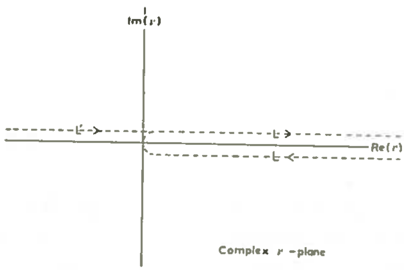


FIG. 6

taken along the path L indicated in fig. 6. This integral is obviously equal to the series F if no poles of $f(\nu)$ are situated on the real axis; as the elastic constants λ and μ appearing in the equations of motion are in fact not quite real (although the imaginary parts, causing an absorption of the waves, are very small) the poles of $f(\nu)$ will be situated at some distance of the real ν axis and it is therefore possible to choose L close enough to this axis to exclude every pole of $f(\nu)$. The next step in the Watson-transformation is the change of the part of L on the negative imaginary side of the real axis into the traject L', which is only valid if the integrand is an odd function of ν. It follows from the properties of the Hankel-functions (NIELSEN, 1903):

$$H^{(1)}_{-\nu}(z) = -e^{(\nu+1)\pi i} H^{(1)}_{\nu}(z) \text{ and } H^{(2)}_{-\nu}(z) = -e^{-(\nu+1)\pi i} H^{(2)}_{\nu}(z),$$

that $\zeta^{(1)}_{\nu-1/2}(x) \,.\, \zeta^{(2)}_{\nu-1/2}(y)$ and $\dfrac{\zeta^{(i)}_{\nu-1/2}(x)}{\zeta^{(i)}_{\nu-1/2}(y)}$ are even functions of ν.

§ 5. THE SEISMIC-RAY APPROXIMATION OF P AND PcP WAVE

In order to use the saddle-point method the integrals have to be written in the form

$$\int A(z)\, e^{i\psi(z)}\, dz$$

in such a way that along the chosen path of integration (which is the line of steepest descent) the exponent $\psi(z)$ changes much more rapidly than $A(z)$; then we may regard $A(z)$ to be a constant. The requirement leads immediately to the substitution of the P and ζ functions by their integral representations (1, 4) and moreover to a removing of exponential functions in the denominator of the integrand. Hence we develop in all expressions (7) the function $(\sin \nu\pi)^{-1}$ into the series $-2i\, e^{i\nu\pi} \sum_{m=0}^{\infty} e^{2im\nu\pi}$; again we multiply the function $\zeta_\nu^{(1)}(h\varrho)$ occurring in each expression (with the exception of that for the P wave) in the denominator by $\zeta_\nu^{(2)}(h\varrho)$ thus removing the exponential character of that function, and add the same factor to the nominator.

In this way we obtain multiple integrals which can be approximated by the saddle-point method in a manner analogous to the method for single integrals. First the saddle-points of the exponent of

$$\int \ldots \int A(x_1, x_2, \ldots x_n)\, e^{i\psi(x_1, x_2, \ldots x_n)}\, dx_1 \ldots dx_n$$

are determined by the equations $\partial\psi/\partial x_\varkappa = 0$ $(\varkappa = 1, 2, \ldots n)$; we denote the values of $x_\varkappa$ in the saddle-points by $\xi_\varkappa$. Then ψ is developped up to and including the second term of the Taylor series in the neighbourhood of $\xi_\varkappa$, which development consists of the value ψ_s of ψ in the saddle-point under consideration and a term which is quadratic in $(x_\varkappa - \xi_\varkappa)$. The coefficients of this quadratic function are the values of the second derivates $\dfrac{\partial^2\psi}{\partial x_\varkappa \varrho x_\lambda}$ in the saddle-point. This function can be reduced to a sum of squares of linear functions of $(x_x - \xi_x)$, and by introducing these functions $y_\varkappa$ as new variables the integral becomes

$$e^{i\psi_s} \int \ldots \int A(y_1, y_2, \ldots y_n)\, e^{i\sum_{\varkappa=1}^{n} \beta_\varkappa y_\varkappa^2} \frac{\partial(x_1 \ldots x_n)}{\partial(y_1 \ldots y_n)}\, dy_1 \ldots dy_n.$$

The functional determinant is equal to the square root of the discriminant of the quadratic function in $(x_\varkappa - y_\varkappa)$, or $\sqrt{H_s}$ where H_s is the determinant of Hesse. The contribution of one saddle-point is therefore

$$A_s \cdot e^{i\psi_s} \frac{(e^{1/4\pi i}\sqrt{2\pi})^n}{\sqrt{H_s}}. \tag{9}$$

Proceeding with the approximation of the potential Φ_P of the P wave we obtain for the term $m = 0$ and $r < \varrho_1$:

$$\Phi_P = \frac{1}{4\pi^2 h \sqrt{\varrho_1 r}} \int_{-\infty}^{+\infty}\!\!\int \int \int_{-\pi}^{+\pi} (\nu + \tfrac{1}{2})\, e^{i\psi}\, d\varphi\, d\eta_1\, d\eta_2\, d\nu,$$

The following results are readily found:

$$\psi_s = h\varrho_1 \cos\eta_\varrho + hr\cos\eta_r - 2h\varrho\cos a - \tfrac{1}{2}\vartheta, \text{ or } \psi_s = h(D_1 + D_2) - \tfrac{1}{2}\vartheta$$

$$H_s = -\nu\sin\vartheta\, e^{-i\vartheta}.\ h^3\varrho\cos a\,(r\,D_1\cos\eta_r + \varrho\,D_2\cos\eta_\varrho).$$

By substitution we obtain:

$$\Phi_r = K.(LL)_s.\frac{e^{-i\omega t + ih(D_1+D_2)}}{ih(D_1+D_2)}, \qquad (10)$$

with

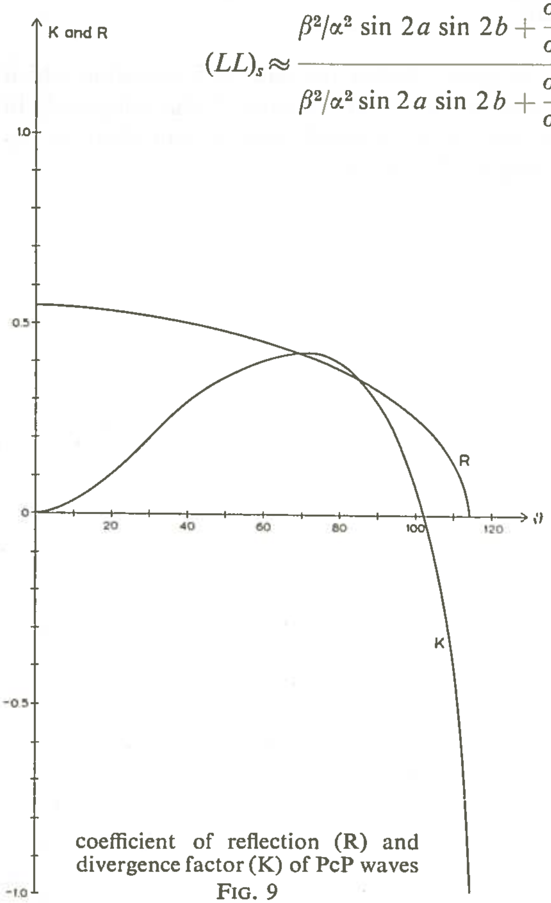
$$K = \frac{\varrho}{\varrho_1}\sqrt{\frac{\cos a}{\cos\eta_\varrho}}\;\frac{D_1 + D_2}{\sqrt{\frac{\varrho\sin\vartheta}{\sin\eta_r}}\left(D_1\frac{r\cos\eta_r}{\varrho\cos\eta_\varrho} + D_2\right)}.$$

The spherical reflection coefficient $(LL)_s$ is for large values of $h\varrho$ not very different from the coefficient of reflection against a flat surface

$$(LL)_s \approx \frac{\beta^2/\alpha^2\sin 2a\sin 2b + \dfrac{\sigma'\alpha'\cos a}{\sigma\alpha\cos a'} - \cos^2 2b}{\beta^2/\alpha^2\sin 2a\sin 2b + \dfrac{\sigma'\alpha'\cos a}{\sigma\alpha\cos a'} + \cos^2 2b}$$

In fig. 9 this value of $(LL)_s$ is shown as a function of the angle ϑ; the following values of the constants were used:
$\sigma = 5.7$, $\sigma' = 9.7$ gr/cm^3; $\alpha = 13.6$, $\beta = 7.3$ and $\alpha' = 8.1$ km/sec. (Bullen's model A).

Apart from the decrease of the amplitude indicated by the quantity (LL) the amplitude shows in comparison to the P wave a further decrease expressed by the divergence factor K. As this factor tends to the value 1 if ϱ becomes very large compared with D_1 and D_2 this decrease is entirely due to the curvature of the reflecting surface. In fig. 9 the value of K is drawn against the angle a, assuming $\varrho =$ the radius of the core to be 3500 km and $\varrho_1 = r = 6400$ km, the radius of the Earth. For $a = 0$ the value of $K = \varrho/\varrho_1$ and for $a = \pi/2$ its value is zero.



coefficient of reflection (R) and divergence factor (K) of PcP waves
FIG. 9

§ 6. THE DIFFRACTED *PcP* WAVE

As already mentioned in § 4 the last step in the method of WATSON consists in the closing of the path of integration by a semicircle with infinite radius in the positive imaginary part of the complex ν-plane. The integral Φ_r is then equal to the sum of the residues yielded by the poles of the integrand:

$$= -\pi e^{-i\omega t} \sum_{n=0}^{\infty} \frac{(\nu_n + \frac{1}{2}) P_{\nu_n}\{\cos(\pi - \vartheta)\}}{\sin \nu_n \pi} \left[(x_1 - x_2) \frac{(x_2' + 1) y_1 k^2 \varrho^2 + z_2' \{ y_1 + 2\nu(\nu+1) \}}{\partial D / \partial \nu} \right]_{\nu_n} \frac{\zeta_{\nu_n}^{(1)}(h\varrho) . \zeta_{\nu_n}^{(2)}(h\varrho) . \zeta_{\nu_n}^{(1)}(hr)}{\zeta_{\nu_n}^{(1)}(h\varrho)}$$

where ν_n are the roots with positive imaginary parts of the equation $D = 0$, or

$$(x_2' + 1) x_1 y_1 k^2 \varrho^2 + z_2' \{ x_1 y_1 + 2\nu(\nu+1) x_1 + y_1 \} = 0.$$

Using the fact that ν_n satisfies this equation and replacing $(x_1 - x_2)$ by its value (NIELSEN § 7)

$$x_1 - x_2 = \frac{2i}{h\varrho\, \zeta_\nu^{(1)}(h\varrho)\, \zeta_\nu^{(2)}(h\varrho)}$$

we obtain

$$\Phi_r = 2\pi i e^{-i\omega t} \sum_{n=0}^{\infty} \frac{(\nu_n + \frac{1}{2}) P_{\nu_n}\{\cos(\pi - \vartheta)\}}{h\varrho \sin \nu_n \pi} \left(\frac{y_1 z_2'}{x_1\, \partial D / \partial \nu} \right)_{\nu_n} \frac{\zeta_{\nu_n}^{(1)}(h\varrho_1)\, \zeta_{\nu_n}^{(1)}(hr)}{\zeta_{\nu_n}^{(1)}(h\varrho)\, \zeta_{\nu_n}^{(1)}(h\varrho)} \qquad (11)$$

A further simplification of this expression is effected by means of the approximative values of the P and ζ functions as given in § 2; however, in that paragraph these approximative formulas have been discussed for large values of the argument assuming that both the order and the argument were real, whereas in the above series the order ν_n will generally be a complex number. We therefore have to extend the discussion of the saddle-point approximation to complex values of the order; this theory has been developed by DEBIJE (1909).

The saddle-points of

$$H_n^{(1)}(z) = \frac{1}{\pi} \int_{(1)} e^{iz\cos\gamma + in(\gamma - \pi/2)}\, d\gamma$$

are determined by $\sin \gamma = n/z$. The path (1) of integration can be diverted for every value of n and z through one or at most two of these points; as shown in § 2 the contribution of a saddle-point to the value of $H_n^{(1)}(z)$ is

$$\sqrt{\frac{2}{\pi}}\, \frac{e^{i(z\cos\gamma_s - n \arccos n/z - \pi/4)}}{\sqrt{z \cos \gamma_s}} \qquad (12)$$

If the path of integration meets one saddle-point this expression represents the second order approximation of the Hankel-function; even if two saddle-points are met this expression suffices as in most cases the contributions of the two saddle-points are of a different order. The values of these contributions (and the fact whether one or two saddle-points will be met) depend on only one parameter, namely n/z; this

This equation is the Stoneley-wave equation for a flat interface between two semi-infinite media; it is well known — and indeed apparent from this form of that equation — that the root ν_n is real and greater than the arguments $h\varrho$, $k\varrho$ and $h'\varrho$ (the cosines being in these circumstances imaginary).

However, this root is of no importance to our problem; as the factor

$$\frac{\zeta^{(1)}_{\nu_n}(h\varrho_1)}{\zeta^{(1)}_{\nu_n}(h\varrho)} \cdot \frac{\zeta^{(1)}_{\nu_n}(hr)}{\zeta^{(1)}_{\nu_n}(h\varrho)}$$

occurring in Φ_r is in this approximation proportional to $e^{ih(\varrho_1 \cos\eta_\varrho + r\cos\eta_r - 2\varrho\cos}$ which is very small, cos a being positive imaginary, the contribution of this root to the value of Φ_r may be neglected.

It remains to consider the possibility of roots in the neighbourhood of the arguments; substituting the asymptotic formulae for ζ_n in $D = 0$ this equation becomes if $\nu/h\varrho$ is close to the transition curve in the ν/z plane:

$$\{\sqrt{h^2\varrho^2 - \nu^2} - \nu \arccos \nu/h\varrho + \pi/4\} = \frac{\sqrt{\nu^2 - h'^2\varrho^2}}{\sqrt{\nu^2 - h^2\varrho^2}} \frac{(2\nu^2 - k^2\varrho^2)^2}{-4\nu^2\sqrt{\nu^2 - h'^2\varrho^2)}(\nu^2 - k^2\varrho^2) + \frac{\sigma'}{\sigma}k^4\varrho^4}$$

For values of $\nu/h\varrho \approx 1$ we put $\nu/h\varrho = 1 + \varepsilon$ with $|\varepsilon| \ll 1$ and reduce the equation to

$$(2\varepsilon)^{1/2}\, tg\left\{\frac{\pi}{4} + \frac{i}{3}h\varrho(2\varepsilon)^{3/2}\right\} = -\frac{\sqrt{\frac{\alpha^2}{\alpha'^2} - 1}\,.\left(2 - \frac{\alpha^2}{\beta^2}\right)^2}{4\sqrt{\left(\frac{\alpha^2}{\alpha'^2} - 1\right)\left(\frac{\alpha^2}{\beta^2} - 1\right)} + \frac{\alpha^4\sigma'}{\beta^4\sigma}}.$$

It follows that the absolute value of $h\varrho\varepsilon^{3/2}$ is a finite number δ; this quantity is the root of

$$\sqrt{2\delta}\,.\, tg\left\{\frac{\pi}{4} + \frac{i}{3}(2\delta)^{3/2}\right\} = -\sqrt[3]{h\varrho}\,\frac{\left(\frac{\alpha^2}{\alpha'^2} - 1\right)^{1/2}\left(2 - \frac{\alpha^2}{\beta^2}\right)^2}{4\left(\frac{\alpha^2}{\alpha'^2} - 1\right)^{1/2}\left(\frac{\alpha^2}{\beta^2} - 1\right)^{1/2} + \frac{\sigma'\alpha^4}{\sigma\beta^4}}.$$

The right-hand side of this equation is equal to $gT^{-1/3}$, where g is a quantity determined by the elastic system and T is the period $2\pi/\omega$; using the values given by BULLEN we find $g = 1.182$. For each value of T we obtain an infinite series of roots δ; e.g. in the two limiting cases $T = 0$ and $T = \infty$ these roots are respectively

$$\delta = {}^1/_2\{3(q + {}^3/_4)\pi\}^{2/3} e^{1/3\pi i} \text{ and}$$

$$\delta = {}^1/_2\{3(q + {}^1/_2)\pi\}^{2/3} e^{1/3\pi i},\ q = 0, 1, \ldots.$$

In fig. 10 we have drawn the roots of the series $q = 0$ and $q = 1$ as a function of the period; it appears that with increasing q the imaginary part of δ increases rapidly.

§ 7. THE *PcS* WAVE

The vector potential of this transverse wave is directed in the φ direction and equal to

$$\Phi = -\tfrac{1}{2} i e^{-i\omega t} \int_{-\infty}^{+\infty} \frac{\nu+\frac{1}{2}}{\sin \nu\pi} \cdot \frac{1}{\nu} \zeta_\nu^{(1)}(h\varrho_1) . (LT) \frac{\zeta_\nu^{(2)}(h\varrho)}{\zeta_\nu^{(1)}(h\varrho)} \cdot \zeta_\nu^{(1)}(kr) \frac{\partial}{\partial\vartheta} P_\nu\{\cos(\pi-\vartheta)\}\, d\nu$$

As

$$\frac{\partial}{\partial\vartheta} P_\nu\{\cos(\pi-\vartheta)\} = \int_{-\pi}^{+\pi} i\nu \frac{\sin\vartheta - i\cos\varphi\cos\vartheta}{\cos\varphi\sin\vartheta - i\cos\vartheta}\{\cos(\pi-\vartheta) - i\cos\varphi\sin(\pi-\vartheta)\}\, d\varphi$$

this expression becomes when the ζ-functions are replaced by their integral-representations:

$$\Phi = \frac{-i}{(2\pi)^3 hk\sqrt{\varrho^2\varrho_1 r}} \int \cdots \int \frac{(\nu+\frac{1}{2})(LT)}{\zeta_\nu^{(1)}(h\varrho)\,\zeta_\nu^{(2)}(h\varrho)} \cdot \frac{\sin\vartheta - i\cos\varphi\cos\vartheta}{\cos\varphi\sin\vartheta - i\cos\vartheta} e^{i\psi}\, d\varphi\, d\eta_1 \ldots d\eta_4\, d\nu,$$

with

$$\psi = k\varrho_1\cos\eta_1 + kr\cos\eta_2 + h\varrho\cos\eta_3 + h\varrho\cos\eta_4 + (\nu+\tfrac{1}{2})\sum_1^4 \eta_i$$
$$-i\nu\log\{\cos(\pi-\vartheta) - i\cos\varphi\sin\vartheta\} - (\nu+1)\pi.$$

The calculation proceeds analogous to that of the *PcP* wave and results in:

$$\Phi = (LT)_s K \frac{e^{i(hD_1+kD_2)}}{i(hD_1+kD_2)}$$

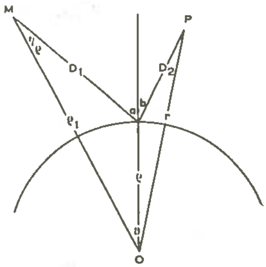
where $(LT)_s$ is equal to the coefficient of reflection for plane waves reflected by a plane surface:

$$(LT)_s = -\frac{2\frac{\beta^2}{\alpha^2}\sin 2a\cos 2b}{\frac{\beta^2}{\alpha^2}\sin 2a\sin 2b + \frac{\sigma'\alpha'\cos a}{\sigma\alpha\cos a'} + \cos^2 2b}$$

and

$$K = \frac{\varrho}{\varrho_1}\sqrt{\frac{\cos a}{\cos\eta_\varrho}} \cdot \frac{tg\, a}{tg\, b} \cdot \frac{D_1 + \frac{\alpha}{\beta}D_2}{\sqrt{\frac{\varrho\sin\vartheta}{\sin\eta_r}\left(D_1\frac{tg\, a}{tg\, b}\frac{r\cos\eta_r}{\varrho\cos\eta_\varrho} + D_2\right)}}$$

(see fig. 13).



PcS—ray

FIG. 12

In contrast with the divergence factor of the *PcP* wave this quantity K depends not only on the curvature of the reflecting surface, but also on the value of α/β. If the curvature tends to zero the factor K approaches the limiting value

$$K_0 = \frac{tg\, a}{tg\, b} \cdot \frac{D_1 + \frac{\alpha}{\beta}D_2}{\sqrt{\left(\frac{\alpha}{\beta}D_1 + D_2\right)\left(D_1\frac{\alpha\cos^2 b}{\beta\cos^2 a} + D_2\right)}}.$$

This quantity which is in fact a convergence factor, as $K_0 > 1$ if $\alpha > \beta$ is shown in fig. 14 against a; comparing K_0 with K, which is drawn in the same figure, we observe that the diverging effect of the curvature is of importance only for small values of the angle of incidence.

It is of some interest to derive this value of K_0 in an elementary way; to this end we consider a small pencil of rays, contained in the solid angle $M.BCDE$ (fig. 15). The rays situated in a meridional plane, as for instance MBE, will meet the reflecting surface in the points of LP; covering a distance D_2 after reflection these rays reach arc QS, which is equal to $(D'' + D_2)\, db$ (see fig. 16).

As $\dfrac{D''\, db}{\cos b} = LP = \dfrac{D_1\, da}{\cos a}$ we have $D'' = D_1 \dfrac{\cos b\, da}{\cos a\, db}$ and $D'' = D_1 \dfrac{\alpha \cos^2 b}{\beta \cos^2 a}$.

Again rays in the plane MBC are reflected in the points of arc FG (fig. 17); after traveling the distance D_2 they reach arc $HK = (D' + D_2) \sin b\, d\varphi$ where $D' = \alpha/\beta\, D_1$. Hence the area of the cross section of the reflected beam is

$$(\alpha/\beta\, D_1 + D_2)\left(\frac{\alpha \cos^2 b}{\beta \cos^2 a} D_1 + D_2\right) \sin b\, db\, d\varphi.$$

The time necessary to reach this cross-section is equal to $D_1/\alpha + D_2/\beta$; if the rays did not suffer a reflection they would have covered a distance $D_1 + \alpha/\beta\, D_2$ in that time. The area of the cross-section would then have been

$$(D_1 + \alpha/\beta\, D_2)^2 \sin a\, da\, d\varphi.$$

Supposing for a moment that the energy in both beams would be equal it follows that the amplitude at a distance D_2 of the transverse movement would be

$$\frac{\alpha \sqrt{\cos b}}{\beta \sqrt{\cos a}}\;\frac{D_1 + \dfrac{\alpha}{\beta} D_2}{\sqrt{\left(\dfrac{\alpha}{\beta} D_1 + D_2\right)\left(\dfrac{\alpha \cos^2 b}{\beta \cos^2 a} D_1 + D_2\right)}}$$

times the amplitude of the longitudinal rays, which have traveled during the same time.

The energy in the transverse beam being $(LT)_s^2 \,.\, \dfrac{\cos b}{\cos a}$ times that of the incident longitudinal beam, the amplitude has to be multiplied by $(LT)_s \sqrt{\dfrac{\cos b}{\cos a}}$; hence the amplitude is finally $(LT)_s \,.\, K_0$ times that of the unreflected longitudinal movement.

The above calculations are of course only valid in the lit area (extending to an epicentral distance of 72°); in the shadow area the amplitude decreases exponentially in the same way as the PcP wave.

the wave parameter ν; plotting the function $3\pi - \Sigma \eta_i$, which we denote by $\theta(\nu)$, against ν the remaining equation $\vartheta = \theta(\nu)$ is then to be solved graphically.

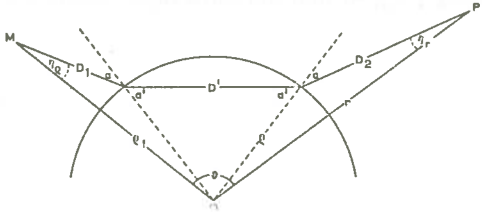


FIG. 18 PKP—ray

It now appears that in the case of the Earth the deviation $\theta(\nu)$ of the *PKP* ray emerging at the surface has a minimum value of 143°; this point is the intersection of the Earth's surface and the caustic, which curve is determined by $\vartheta = \theta(\nu)$ and $\partial \theta / \partial \nu = 0$. In the neighbourhood of this caustic the approximation we used above is not sufficient, as is apparent from the fact that the determinant of HESSE is then about equal to zero $\left(\frac{\partial \theta}{\partial \nu} = \frac{2}{h'\varrho \cos a'} - \frac{1}{h\varrho_1 \cos \eta_\varrho} - \frac{1}{hr \cos \eta_r} - \frac{2}{h'\varrho \cos a'}\right)$.

Therefore we have to complete the calculation of the *PKP* wave for points near to the caustic. In order to simplify the expression for Φ as much as possible we substitute in the original form (14) of Φ the asymptotic formulae of ζ_ν and P_ν, which procedure is allowable as we are only concerned with values of the order and argument quite distant from the transition curves. In this way we obtain instead of the multiple integral the comparatively simple expression:

$$\Phi = -\frac{1}{h^2 \varrho_1 r} \int_{-\infty}^{+\infty} (LL')(L'L) \sqrt{\frac{\nu}{2\pi \sin \vartheta \cos \eta_1 \cos \eta_2}}\, e^{i\psi + \frac{1}{4}\pi i}\, d\nu,$$

with $\quad \psi = h\varrho_1 \cos \eta_1 + hr \cos \eta_2 + 2h\varrho \cos \eta_3 + 2h'\varrho \cos \eta_4 + (\nu + \frac{1}{2})(\vartheta - \theta)$

$$\theta = 3\pi - \Sigma \eta_i \text{ and } \nu = h\varrho_1 \sin \eta_1 = \ldots . = h'\varrho \sin \eta_4.$$

The saddle-point of ψ is given by $\partial \psi / \partial \nu = 0$ or $\vartheta = \theta(\nu)$; if the point P is a point of the caustic we have also $\partial^2 \psi / \partial \nu^2 = -\partial \theta / \partial \nu = 0$ and then the development of ψ is

$$\psi = \psi_c - \left(\frac{\partial^2 \theta}{\partial \nu^2}\right)_c \frac{(\nu - \nu_c)^3}{3!} + \ldots$$

Introducing the variable $s = (\nu - \nu_c) \cdot \left(\frac{1}{6} \frac{\partial^2 \theta}{\partial \nu^2}\right)_c^{1/3}$ the expression for Φ is approximated by

$$\Phi = -\frac{e^{i\psi_c + \frac{1}{4}\pi i}}{h^2 \varrho_1 r \sqrt{2\pi \sin \vartheta}} \left\{\frac{(LL')(L'L)}{\sqrt{\cos \eta_1 \cos \eta_2}} \sqrt[3]{\frac{6}{\partial^2 \theta / \partial \nu^2}}\right\}_c \int_{-\infty}^{+\infty} e^{-is^3}\, ds$$

which can be reduced to

$$\frac{2}{3}\left\{\cos\left(-\frac{\pi}{6}\right)\Gamma(1/3) + \cos\left(\frac{\pi}{6}\right)\Gamma(2/3)\frac{\varkappa}{1!} + \cos\left(\frac{5\pi}{6}\right)\Gamma(4/3)\frac{\varkappa^3}{3!} + \ldots\ldots\right\}$$

Hence the potential of the *PKP* wave in points very close to the caustic is

$$\Phi_c\,(1 + 0.505\,\varkappa - 0{,}042\,\varkappa^3 - 0.007\,\varkappa^4 \ldots\ldots)$$

which result is valid at both sides ($\vartheta > \vartheta_c$ and $\vartheta < \vartheta_c$) of this curve. We note that the maximum intensity of the *PKP* ray is not on the caustic but at a point on its convex side.

For points P not very near to the caustic, but still in its vicinity we use the same development of ψ, changing however the path of integration to the saddle-point of $i\varkappa s - i s^3$. If $\vartheta > \vartheta_c$ we have two saddle-points $s = \pm\sqrt{\varkappa/3}$; the lines of steepest descent are drawn in fig. 19. With $s = -\sqrt{\varkappa/3} + t\,e^{1/4\,\pi i}$ and $s = +\sqrt{\varkappa/3} + t\,e^{3/4\,\pi i}$ on the two branches respectively the integral is approximated by:

$$e^{-2/3\,i\varkappa\sqrt{\varkappa/3} + 1/4\,\pi i}\int_{-\infty}^{+\infty} e^{-t^2\sqrt{3\varkappa} + t^3 e^{1/4\,\pi i}}\,dt + e^{2/3\,i\varkappa\sqrt{\varkappa/3} - 1/4\,\pi i}\int_{-\infty}^{+\infty} e^{-t^2\sqrt{\varkappa 3} + t^3 e^{-1/4\,\pi i}}\,dt$$

which is easily developed into a series of powers of $(3\varkappa)^{1/4}$. As the first term of this series is $2\sqrt{\pi}\,.\,(3\varkappa)^{-1/4}\cos\left(\frac{2}{3}\varkappa\sqrt{\frac{\varkappa}{3}} - \frac{\pi}{4}\right)$
the *PKP* potential is in first approximation

$$\Phi_c\cdot\frac{2\sqrt{3\pi}}{\Gamma(1/3)}\,\frac{\cos(2/3\,\varkappa\sqrt{\varkappa/3} - \pi/4)}{\sqrt[4]{3\varkappa}};$$

this expression obviously represents the result of the interference of two *PKP* waves.

If P is situated on the concave side of the caustic $\vartheta < \vartheta_c$ and $\varkappa < 0$; hence the two saddle-points are $s = \pm i\sqrt{-\varkappa/3}$. It appears that the path of integration has to be shifted to the point $s = -i\sqrt{-\varkappa/3}$ as shown in fig. 19; with $s = -i\sqrt{-\varkappa/3} + t$ the integral becomes

$$e^{2/3\,\varkappa\sqrt{-\varkappa/3}}\int_{-\infty}^{+\infty} e^{-t^2\sqrt{-3\varkappa} - it^3}\,dt$$

and the *PKP* potential is then in first approximation

$$\Phi_c\cdot\frac{\sqrt{3\pi}}{\Gamma(1/3)}\,\frac{e^{2/3\,\varkappa\sqrt{-\varkappa/3}}}{\sqrt[4]{-3\varkappa}}$$

The important quantity $\varkappa$ is for points on the surface of the Earth equal to

$$\frac{37.1\,(\vartheta - \vartheta_c)}{\sqrt[3]{\lambda^2\dfrac{\partial^2\theta}{\partial(\sin\eta_r)^2}}},$$

§ 9. REFLECTION AT THE EARTH'S SURFACE

In the foregoing we investigated the effect of the Earth's core on elastic waves emitted from a point outside this core; in this paragraph we shall consider the effect of the other boundary of the mantle namely the surface of the Earth. To this end we consider the motion inside a homogeneous sphere with radius ϱ, surrounded by vacuum; adding the results obtained here to those of previous paragraphs we arrive at a complete picture of the waves inside a homogeneous spherical shell bounded at the inside by a liquid and at the outside by the vacuum.

We assume the same primary movement as in § 4 with its centre inside the sphere; each of the partial waves, given by (6), may be interpreted as a wave emitted by the sphere $r = \varrho_1$ and traveling both away from and towards the centre $r = 0$. This wave is represented by the discontinuous potential

$$\Phi_p = \begin{cases} e^{-i\omega t}\, \zeta_n^{(2)}(h\varrho_1)\, \zeta_n^{(1)}(hr)\, P_n(\cos\vartheta) \text{ for } r > \varrho_1 \\ e^{-i\omega t}\, \zeta_n^{(1)}(h\varrho_1)\, \zeta_n^{(2)}(hr)\, P_n(\cos\vartheta) \text{ for } r < \varrho_1. \end{cases}$$

Moreover a continuous wave, proportional to $\zeta_n^{(1)}(h\varrho_1)\, \zeta_n^{(1)}(hr)$, appears at the reflection in the centre.

In order to satisfy the boundary condition at the free surface, which requires the vanishing of the tension at this surface, we have to add to the incident wave $a\, \zeta_n^{(1)}(hr)$ a reflected longitudinal wave $\varphi = b\, \psi_n(hr)$ and a transverse wave determined by the Hertzian function $M = B\, \psi_n(kr)$. This condition is expressed by two equations:

$$-(4x_1 - m^2)\, a\, \zeta_n^{(1)}(h\varrho_1) - (4x - m^2)\, b\, \psi_n(h\varrho) + n(n+1)(y + m^2)\, B\, \psi_n(k\varrho) = 0$$

$$-2x_1\, a\, \zeta_n^{(1)}(h\varrho) \qquad - 2x\, b\, \psi_n(h\varrho) \qquad + y\, B\, \psi_n(k\varrho) \qquad = 0$$

It follows that

$$= -a \frac{\zeta_n^{(1)}(h\varrho)}{\zeta_n^{(2)}(h\varrho)} \frac{\{(4x_1 - m^2)y_1 - 2n(n+1)x_1(y_1 + m^2)\}\, \zeta_n^{(1)}(k\varrho) + \{(4x_1 - m^2)y_2 - 2n(n+1)x_1(y_2 + m^2)\}\, \zeta_n^{(2)}(k\varrho)}{L \,.\, \zeta_n^{(2)}(k\varrho)}$$

and

$$B = \frac{a}{n} \cdot \frac{\zeta_n^{(1)}(h\varrho)}{\zeta_n^{(2)}(k\varrho)} \cdot \frac{2m^2(x_2 - x_1)}{L}$$

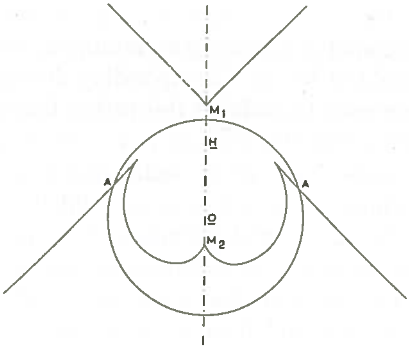
with

$$L = \{(4x_2 - m^2)\, y_2 - 2n(n+1)\, x_2(y_2 + m^2)\} +$$
$$+ \{(4x_1 - m^2)\, y_2 - 2n(n+1)\, x_1(y_2 + m^2)\} \frac{\zeta_n^{(1)}(h\varrho)}{\zeta_n^{(2)}(h\varrho)} +$$
$$+ \{(4x_2 - m^2)\, y_1 - 2n(n+1)\, x_2(y_1 + m^2)\} \frac{\zeta_n^{(1)}(k\varrho)}{\zeta_n^{(2)}(k\varrho)} +$$
$$+ \{(4x_1 - m^2)\, y_1 - 2n(n+1)\, x_1(y_1 + m^2)\} \frac{\zeta_n^{(1)}(h\varrho)}{\zeta_n^{(2)}(h\varrho)} \cdot \frac{\zeta_n^{(1)}(k\varrho)}{\zeta_n^{(2)}(k\varrho)}.$$

(for notation see § 2).

To this wave system we have to add the reflected waves due to the incident wave $e^{-i\omega t}\,\zeta_n^{(1)}(h\varrho_1)\,\zeta_n^{(1)}(hr)\,P_n(\cos\vartheta)$; for our purpose, however, this system may be ignored as this incident wave does not appear if the sphere contains a core. The presence of a core changes this system into the waves considered in § 4, 5 and 6 and the waves originating from these by reflection at the free surface.

ϱ_1/ϱ (down to a depth of the hypocentre of 800 km) the epicentral distance of this focus is in first approximation equal to $4\sqrt{1-\varrho_1/\varrho}$; in fig. 22 the caustic is shown for a depth of 600 km. Quite near to this point the amplitude of the movement resulting from the interference of the *pP* and *PP* phase is at a maximum; at smaller distances ϑ the amplitude rapidly decreases.



PP—caustic
FIG. 22

In analogy to formula (11) of § 5, expressing the Stoneley wave connected with the surface of the core, we obtain here the following expression for the Rayleigh wave

$$\Phi = -2\pi i \frac{(\nu_R+\frac{1}{2})P_{\nu_R}\{\cos(\pi-\vartheta)\}}{h\varrho\sin\nu_R\pi}\left\{\frac{2\nu(\nu+1)(y_2+m^2)}{\partial D/\partial\nu}\right\}_{\nu_R}\cdot\frac{\zeta^{(2)}_{\nu_R}(h\varrho_1)}{\zeta^{(2)}_{\nu_R}(h\varrho)}\frac{\zeta^{(2)}_{\nu_R}(hr)}{\zeta^{(2)}_{\nu_R}(h\varrho)} \tag{16}$$

where $\nu_R = h\varrho\sin a$ and a satisfies the equation of Rayleigh

$$\alpha^2/\beta^2 . \sin 2a \sin 2b + \cos^2 2b = 0.$$

As the last factor in (16) is proportional to $\exp\{i h(2\varrho-\varrho_1-r)\cos a\}$ and $\cos a$ is positive imaginary this wave will be observable at the surface if the depth $\varrho-\varrho_1$ of the focus is not too great.

The amplitude A of the longitudinal wave $\zeta_n^{(2)}(hr)$ is expressed by a fraction with the denominator

$$+D(x_2,y_2)E(x_1,y_1,x')\zeta^{(2)}_{h_0}\zeta^{(2)}_{k_0}\zeta^{(1)}_{h_c}\zeta^{(1)}_{k_c}-D(x_2,y_1)E(x_1,y_2,x')\zeta^{(2)}_{h_0}\zeta^{(1)}_{k_0}\zeta^{(1)}_{h_c}\zeta^{(2)}_{k_c}-$$

$$-D(x_1,y_2)E(x_2,y_1,x')\zeta^{(1)}_{h_0}\zeta^{(2)}_{k_0}\zeta^{(2)}_{h_c}\zeta^{(1)}_{k_c}+D(x_1,y_1)E(x_2,y_2,x')\zeta^{(1)}_{h_0}\zeta^{(1)}_{k_0}\zeta^{(2)}_{h_c}\zeta^{(2)}_{k_c}$$

$$-D(x_2-x_1,o)E(o,y_1-y_2,x')\{\zeta^{(1)}_{h_0}\zeta^{(2)}_{h_0}\zeta^{(1)}_{k_c}\zeta^{(2)}_{k_c}+\zeta^{(1)}_{k_0}\zeta^{(2)}_{k_0}\zeta^{(1)}_{h_c}\zeta^{(2)}_{h_c}\} \qquad (16)$$

and the nominator

$$-a\,D(x_1,y_2)E(x_1,y_1,x')\zeta^{(1)}_{h_0}\zeta^{(2)}_{k_0}\zeta^{(1)}_{h_c}\zeta^{(1)}_{k_c}+a\,D(x_1,y_1)E(x_1,y_2,x')\zeta^{(1)}_{h_0}\zeta^{(1)}_{k_0}\zeta^{(1)}_{h_c}\zeta^{(2)}_{k_c}$$

$$+b\,D(x_1,y_2)E(x_2,y_1,x')\zeta^{(1)}_{h_0}\zeta^{(2)}_{k_0}\zeta^{(2)}_{h_c}\zeta^{(1)}_{k_c}-b\,D(x_1,y_1)E(x_2,y_2,x')\zeta^{(1)}_{h_0}\zeta^{(1)}_{k_0}\zeta^{(2)}_{h_c}\zeta^{(2)}_{k_c}$$

$$+b\,D(x_2-x_1,o)E(o,y_1-y_2,x')\zeta^{(1)}_{h_0}\zeta^{(2)}_{k_0}\zeta^{(1)}_{h_c}\zeta^{(2)}_{k_c}.$$

The functions D and E are connected with the coefficients of reflection at the free surface and at the surface of the core respectively by means of the relations

$$\frac{D(x_1,y_2)}{D(x_2,y_2)}=-(ll),\quad \frac{D(x_2,y_1)}{D(x_2,y_2)}=-(tt),\quad \frac{D(x_1,y_1)}{D(x_2,y_2)}=(ll)(tt)-(lt)(tl)$$

$$\frac{D(x_2-x_1,o)}{D(x_2,y_2)}=-(n+1).(lt),\quad \frac{D(o,y_2-y_1)}{D(x_2,y_2)}=-\frac{(tl)}{n+1}.$$

$$\frac{E(x_2,y_1,x_2')}{E(x_1,y_1,x_2')}=-(LL),\ \frac{E(x_1,y_2,x_2')}{E(x_1,y_1,x_2')}=-(TT),\ \frac{E(x_2,y_2,x_2')}{E(x_1,y_1,x_2')}=(LL)(TT)-(LT)(TL)$$

$$\frac{E(x_1,y_1,x_1')}{E(x_1,y_1,x_2')}=-(L'L'),\ \frac{E(x_1-x_2,o,x_2')}{E(x_1,y_1,x_2')}=-(n+1).(LT),\ \frac{E(o,y_1-y_2,x_2')}{E(x_1,y_1,x_2')}=-\frac{(TL)}{n+1}.$$

Using the coefficients of refraction at the core's surface (see page 22) the quantities $E(x_2, y_1, x')$ can be expressed in these elementary coefficients; for example:

$$\frac{E(x_2,y_1,x')}{E(x_1,y_1,x')}=-(LL)-(LL')(L'L)Q \text{ with } Q=\frac{\zeta^{(1)}_{h_c'}/\zeta^{(2)}_{h_c'}}{1-(L'L')\,\zeta^{(1)}_{h_c'}/\zeta^{(2)}_{h_c'}}.$$

The amplitude A is then equal to a fraction with the denominator:

$$1-(ll)\frac{\zeta^{(1)}_{h_0}}{\zeta^{(2)}_{h_0}}\{(LL)+(LL')(L'L)Q\}\frac{\zeta^{(2}_{h_c}}{\zeta^{(1)}_{h_c}}-(lt)\frac{\zeta^{(1)}_{h_0}}{\zeta^{(2)}_{k_0}}\{(TL)+(TL')(L'T)Q\}\frac{\zeta^{(2)}_{k_c}}{\zeta^{(1)}_{h_c}}-$$

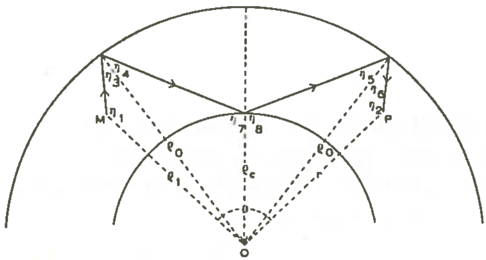
$$-(tt)\frac{\zeta^{(1)}_{k_0}}{\zeta^{(2)}_{k_0}}\{(TT)+(TL')(L'T)Q\}\frac{\zeta^{(2)}_{k_c}}{\zeta^{(1)}_{k_c}}-(tl)\frac{\zeta^{(1)}_{k_0}}{\zeta^{(2)}_{h_0}}\{(LT)+(LL')(L'T)Q\}\frac{\zeta^{2)}_{h_c}}{\zeta^{(1)}_{k_c}}$$

$$\ldots l)(tt)-(lt)(tl)\}\frac{\zeta^{(1)}_{h_0}}{\zeta^{(2)}_{h_0}}\frac{\zeta^{(1)}_{k_0}}{\zeta^{(2)}_{k_0}}\Big[\{(LL)(TT)-(LT)(TL)\}+\{(LL')(L'L)+(TL')(L'T)\}Q\Big]\frac{\zeta^{(2)}_{h_c}}{\zeta^{(1)}_{h_c}}\frac{\zeta^{(2)}_{k_c}}{\zeta^{(1)}_{k_c}}.$$

The perigrinations inside the core are defined by the terms with the factors $\left\{(L'L')\frac{\zeta^{(1)}_{h'_c}}{\zeta^{(2)}_{h'_c}}\right\}^m$ where $m =$ a whole number, occurring in the development of Q. These ray-paths are not indicated in fig. 23.

By means of the method used in § 4 the secondary movement connected with the primary wave $e^{-i\omega t + ihR}/i\,h\,R$ may be derived; the discontinuous part of the partial waves out of which this primary wave can be constructed gives rise to the above wave system, whereas the continuous part will be extinguished by the corresponding secondary wave (the solution of the boundary equations for the incident wave is obviously $A = B = D = A' = 0$, $C' = -1$). The saddle-point approximation then yields the waves going to and fro between the boundaries of the mantle, traveling part of their way inside the core and possibly changing at the points of reflection or refraction from longitudinal into transverse waves or vice versa; in short every wave conceivable in the ray picture of the wave system will be found with the exception of the waves which are repeatedly reflected by the outer surface only (*pPP*, *PPP*, etc. waves).

These waves, however, are contained in the terms $(ll)^2\,(LL)$, $(ll)\,(lT)\,(tl)$, a.s.o., as will be seen from an examination of the saddle-point equations; for example in the



PPcp wave

FIG. 24

case of a wave with coefficient $(ll)^2\,(LL)$ these equations are $v/h = \varrho_1 \sin \eta_1 = \varrho_0 \sin \eta_3 = \ldots\ldots = r \sin \eta_2$ and $\Sigma\,\eta_i + \vartheta = 4\pi$. These are satisfied by three values of v, two of which are larger than $h\varrho_0$; the values of η_7 and η_8 (see fig. 24) corresponding to that pair of roots are $\pi/2 \pm$ an imaginary number, which by substitution in (LL) gives $(LL) = 1$. By some reduction of the saddle-point integral we easily obtain the expressions for the *pPP* and *PPP* waves. The third root v is smaller than $h\varrho_0$ if the point P is in the lit area of the *PPcP* wave and determines the wave which is really reflected at the core's surface. For points inside the shadow zone this root too is larger than $h\varrho_0$ and the corresponding wave is then equal to *pPP* but of opposite sign; then the only remaining wave is *PPP*.

The saddle-point method leads us therefore to the complete ray picture of the propagation of a seismic disturbance in the mantle of the Earth; this is quite understandable as the equation $\partial\psi/\partial v = 0$ is the mathematical expression of the principle of Fermat.

field, which is commonly indicated as a normal mode or an eigen-vibration. At a given value of the wave-parameter ω we obtain an infinite number of roots n each determining such a mode. An example of this kind of wave we have already met in the case of the diffracted *PcP* wave, which may be interpretated as an eigen-vibration of the space outside the sphere. A characteristic property of that wave which is shared by the normal modes of a spherical layer is that the wave is progressive in the ϑ direction whereas it is standing in the radial direction. Analogous to the eigen-vibrations in acoustical instruments or in wave guides the amplitude of these vibrations is equal to zero at a finite number of surfaces $r =$ a constant. As for each root n of $N = 0$ the number of nodal surfaces is constant it is possible to arrange the roots and the corresponding residues according to an increasing number of these surfaces.

In the second method each term of the development is characterised by two whole numbers namely the exponents in the composite geometric series which are interpretated as the number of reflections (and refractions) the corresponding wave has suffered. The variable n is determined by the saddle-point equations and as these equations are, contrary to the equation $N = 0$, independent of the wave parameter ω we obtain a series of non-dispersive waves.

The total field is therefore interpretable in three different ways:

1. as the superposition of waves originating from a bipole, quadrupole, ... 2^n pole with their centre in O.
2. as consisting of waves traveling only in the meridional direction, with 1, 2, ... n nodal surfaces.
3. as a system of rays each determined by the number of reflections and refractions suffered at the boundaries.

After this rapid survey of the methods at our disposal we consider their applicability to the problem of seismic wave propagation. As the observed periods of these movements are generally smaller than say 30 seconds the radius of the core is many times larger than the wavelength; hence the original series is of no use at all. An exception forms the motion with the extremely long period of 57 minutes observed by BENIOFF (1954) by means of his linear strain seismograph with long period galvanometer; this wave will be sufficiently approximated by the Rayleigh scattering term $n = 1$.

The ordinary seismic waves with periods < 30 sec. may be represented by each of the two remaining series and although the series of non-dispersive waves is preferable as it is easier interpretable the usefulness of these series depends completely on their convergence. At this point we have to introduce another aspect of the seismic propagation; the primary disturbance is in actual conditions of course not a periodic function $e^{-i\omega t}$ but some function $F(t)$ which is different from zero during a finite time, e.g. between $t = 0$ and $t = \tau$. Representing this function by an integral

$$F(t) = \int_{-\infty}^{+\infty} f(\omega)\, e^{-i\omega t}\, d\omega$$

§ 13. LIMITATIONS TO THE THEORY

It remains to consider those phenomena which the saddle-point method is unable to deal with; the first example is the interference of waves frequently reflected at the outer surface of the mantle, in which case we have to apply the residue method. The poles ν of the integrand are the roots of the equation

$$1-(ll)\frac{\zeta_\nu^{(1)}(h\varrho_0)}{\zeta_\nu^{(2)}(h\varrho_0)}-(tt)\frac{\zeta_\nu^{(1)}(k\varrho_0)}{\zeta_\nu^{(2)}(k\varrho_0)}-\{(lt)(tl)-(ll)(tt)\}\frac{\zeta_\nu^{(1)}(h\varrho_0)}{\zeta_\nu^{(2)}(h\varrho_0)}\frac{\zeta_\nu^{(1)}(k\varrho_0)}{\zeta_\nu^{(2)}(k\varrho_0)}=0$$

where (ll), etc. are in first approximation the coefficients of reflection at a plane boundary.

Without entering into a discussion of the interference pattern we only remark that this phenomenon is certainly ended before the arrival of the transverse wave which has suffered an infinite number of reflections, in other words before a wave traveling with a velocity of 4.35 km/sec. along the surface of the mantle arrives. As the beginning of the surface waves travels with the same velocity the interference precedes the long waves observed in seismograms. These latter waves are unexplainable on the basis of the model of the Earth assumed in this investigation; their occurrence is due to a further differentiation of the mantle.

The geometric optical method fails in the second place for waves with a length comparable to the linear dimensions of the elastic system; this happens in the system under consideration only for waves with a length of the order of 3000 km which amounts to periods of about 5 minutes. These waves have to be investigated by means of the normal mode (or residue) method, as has been done by Ewing and Press (1954).

The equation determining the values ν of the eigen-vibrations of the system is obtained by putting the denominator (16) of § 11 equal to zero. Instead of using this very complicated equation Ewing and Press based their calculations on the period-equation of a plane solid layer resting on a semi-infinite liquid body; it is doubtful whether this model is usable as the wavelength is about equal to both the thickness of the mantle and the radius of the core. Probably a better although still a rather rough approximation will be obtained when the Hankel functions in the exact equation are approximated by the exponential functions (12, 13).

A final remark concerns the model we have used in this investigation; the two most notable shortcomings are the absence of the crust and the assumption of homogeneity. The consequence of the neglection of the stratification of the outer parts of the mantle is rather serious as the theory does not yield the surface waves. However, a calculation of the effect of the crust would be unnecessarily complicated by considering the sphericity; as the radial dimension is very small in comparison to the radius of the Earth it is quite sufficient to assume a stratified plane model. Again the saddle point method is unsuitable for investigating surface waves, since the length

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