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F. H. SCHMIDT

ON THE DIFFUSION OF STACK GASES IN THE ATMOSPHERE

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VOORWOORD

Het probleem der atmosferische verontreiniging neemt ten gevolge van de zich sedert het einde van de tweede wereldoorlog steeds verder uitbreidende industrialisatie geleidelijk ernstiger afmetingen aan. Daarnaast stelt het toenemend gebruik van kernenergie de mens ten aanzien van de zuiverheid van de dampkring voor geheel nieuwe vraagstukken.

Het is noodzakelijk, dat ook het K.N.M.I. zich in steeds sterkere mate bezighoudt met de bestudering van de verspreiding van verontreinigingen in de atmosfeer.

De bewerking van dit probleem door Dr. F. H. SCHMIDT is een poging om de theorie van de verspreiding algemener op te zetten dan tot dusverre gebeurde. Daarbij is veel aandacht besteed aan de gunstige invloed, die een hoge temperatuur van schoorsteengassen heeft op de concentraties aan de grond van de uit de schoorsteen afkomstige verontreiniging.

Bij de mathematische bewerkingen is een waardevolle bijdrage gegeven door drs. W. J. A. KUIPERS.

De Hoofddirecteur van het K.N.M.I.,

IR. C. J. WARNERS.

PREFACE

The gradual increase of industrialisation since World War II has created serious problems with regard to the atmospheric pollution. Mankind is, in addition, faced with analogous problems as a result of the rapidly developing use of atomic energy.

It has been necessary for the Royal Netherlands Meteorological Institute to pay due attention to the study of the diffusion of waste in the atmosphere and in this publication Dr. F. H. SCHMIDT has made an effort to develop a theory which is more general in character than those existing at present. The favourable influence which high temperatures of the stack effluents have on the lowering of the surface concentration, has found special consideration.

Mr. W. J. A. KUIPERS gave valuable assistance with the mathematical computations.

Director in Chief R.N.M.I.,

IR. C. J. WARNERS.

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CHAPTER I

INTRODUCTION

1.1 The problem

The diffusion of stack gases into the atmosphere is a phenomenon of steadily growing importance as industrial development is increasing all over the world.

Although the abatement of air pollution is a technical problem in the first place, it is closely connected with the state of the atmosphere. It is especially wind speed, vertical stability and the rate of turbulence in the lowest few hundred meters of the atmosphere that play a predominant role in the diffusion of pollution. Unfortunately there exists a strong correlation between these three factors, making an exact computation of the rate of diffusion more or less a fiction at present. This is due to the fact that the laws ruling atmospheric turbulence are insufficiently known.

On one hand several attempts have been made to describe turbulent diffusion in the atmosphere by starting from some assumption on the nature of atmospheric turbulence. The best known, giving the most satisfactory results, is that developed by SUTTON (1947, 1953). Although his results agree fairly well with empirical data, the theoretical basis of his concept seems to be rather weak.

On the other hand much theoretical work has been done to explain the behaviour of heated or non-heated jets escaping into the atmosphere in a vertical direction. No natural turbulence is introduced in these theories, but it is assumed that the rising jet itself is responsible for turbulent diffusion leading to its gradual broadening and, consequently, to a decrease of its vertical velocity and its temperature excess over the environment.

Both problems, the diffusion of pollution in a horizontal wind and the diffusion of momentum and heat in a rising jet have been treated as separate problems up to now. It is clear, however, that they are related, as stack gases to which the theories are applied, usually escape with a finite vertical velocity from the stack orifice, their path becoming only gradually inclined with respect to the vertical under the influence of the wind.

In the following an effort is made to treat both problems from the same point of view.

Starting from the assumption that the diffusion of properties when a jet of air enters the atmosphere is a phenomenon in principle independent of the main direction in which the jet moves, it will be attempted to describe all phenomena occurring with such jets by simple dimensional analysis, as was used by BATCHELOR (1954) in the case of vertical jets. It will then appear that, generally speaking, only approximated solutions are possible, to this effect that at short distances from the orifice the solutions do not hold good.

The great advantage of the method of dimensional analysis lies, however, in the fact that it is not necessary to introduce more or less ad hoc assumptions as to the nature of turbulence.

1.2 Scope of the present treatise

As the propagation of jets emerging vertically from a nozzle, whether or not under the influence of buoyancy forces, is the most simple as well as the most thoroughly investigated phenomenon, the theory concerning these jets wil be considered in the next chapter as completely as seems to be necessary. We will restrict ourselves to axially symmetric jets. Existing theories will be discussed and compared with the dimensional analysis presented here.

In the third chapter the diffusion, under the influence of the natural wind, of gases escaping from an orifice will be treated by applying the results of the second chapter.

Some possible applications to the practice of the diffusion of industrial waste gases will be mentioned in chapter IV.

CHAPTER II

THE VERTICAL PROPAGATION OF JETS IN A STAGNANT ENVIRONMENT

2.1 Survey of existing theories

As the case of a heated axially symmetric jet rising in a stagnant atmosphere is the most general one, we start with a survey of the theories developed to describe the behaviour of such jets. They lead to some conflicting results in the case of an indifferent environment. As to the behaviour of such jets in the case of stable stratification of the environment no satisfactory solutions exist at all. [1])

The obvious way would be to try to distinguish between the various theories by comparing them with experimental or empirical results. However, reliable measurements are difficult to obtain and some of the experimental results could be used by various authors as an argument for the correctness of their respective ideas.

There exist, generally speaking, three different concepts, that of W. SCHMIDT (1941), that of O. G. SUTTON (1950) and the most recent theory of PRIESTLEY and BALL (1955). They are all concerned with the stationary state obtained after an infinitely long time.

2.1.1 W. SCHMIDT considered the propagation of a heated jet, rising vertically in a stagnant indifferent environment. Density differences between the heated jet and the environment are neglected except where they influence buoyancy. This approximation has been adopted by all later authors; except near the nozzle in the case of very hot stack gases the approximation may be considered a sufficient one.

Another essential assumption made by all authors is that the surroundings of the rising jet are not disturbed, except for the horizontal converging motion connected with the entrainment of environmental fluid into the jet. In a later section we shall deal with this point in some detail.

W. SCHMIDT based his theory on PRANDTL's mixing length concept and assumed that mixing between the rising current and the surrounding air, resulting in an exchange of momentum and heat, takes place horizontally. The mixing length, l, was taken proportional to the "width" of the jet, R, generally the distance from the axis where the velocity or temperature excess have decreased to some fraction of their values in the axis itself. It then follows from

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[1]) See also the end of this chapter, section 2.5.4.

the assumtion of similarity that R increases linearly with z, the vertical distance to the nozzle from which the jet emerges, or

$$R = R_1 \frac{z}{z_1} \tag{2.1.1}$$

the suffix 1 indicating the values at some reference level.

The variation of the difference in potential temperature between the axis of the jet and the undisturbed environment is found to be:

$$\vartheta_m = \vartheta_{m\,1} \left(\frac{z}{z_1}\right)^{-5/3} \tag{2.1.2}$$

whereas the vertical velocity in the axis is given by

$$w_m = w_{m\,1} \left(\frac{z}{z_1}\right)^{-1/3} \tag{2.1.3}$$

The profiles of ϑ and w for $z =$ constant were obtained by a series development of the stream function. The jet was assumed to have a finite width R_b. It follows from (2.1.2) and (2.1.3) that ϑ_m and w_m both become ∞ for $z = 0$. As $R_b = 0$ for $z = 0$, no physical absurdity is involved. The point $z = 0$, $r = 0$ can be considered as the apparent origin of the jet. It lies below the orifice from which the jet emerges.

Results analogous to SCHMIDT's were obtained independently by ROUSE, YIH and HUMPHREYS (1952). These authors started from the vertical equation of motion, the equation of diffusion of heat and a third one "stating that the vertical gradient of the flux of kinetic energy is equal to the rate at which work is done by the buoyant force less the rate at which work is done by the turbulent shear". Instead of introducing a mixing length they took the turbulent shear proportional to w_m^2. Assuming dynamic similarity of the mean and turbulent motions for all values of z and applying dimensional analysis to the relations obtained, they also found the equations (2.1.1), (2.1.2) and (2.1.3).

The assumption of dynamic similarity was justified afterwards experimentally by comparing the profiles of w and ϑ for various values of z. ROUSE et al. found that these profiles could be approximated reasonably by error curves, namely $\vartheta = \vartheta_m \exp\left\{-71r^2/z^2\right\}$ and $w = w_m \exp\left\{-96r^2/z^2\right\}$ respectively. This means a radial decrease of ϑ and w to a value of $1/e$ of the maximum values, at $r = R_\vartheta = \dfrac{1}{8{,}4} z$ and $r = R_w = \dfrac{1}{9{,}8} z$ respectively. Although the experimental values scatter rather widely, especially for large values of r, there can be no doubt as to the fact that $R_\vartheta > R_w$. In words: the radial decrease of w is stronger than that of ϑ in every level. Finally, BATCHELOR (1954) in his Symons Memorial Lecture also applied dimensional analysis to the problem and again found the relations (2.1.1), (2.1.2) and (2.1.3).

2.1.2 A different approach was made by SUTTON (1950, 1953). His calculations were based on the assumption of a length l, proportional to the lateral extent of the jet as in W. SCHMIDT's theory. He used it in quite a different way to SCHMIDT, however. According to SUTTON the air entrained into the jet is proportional to $l \frac{dw_m}{dz}$ so that l cannot be considered as a mixing length in the true sense. SUTTON also did not introduce frictional forces or terms accounting for the turbulent heat exchange between the jet and its environment.

Strictly taken he only considered two equations expressing the conservation of mass and of heat in the expanding jet. It is not amazing, therefore, that he arrived at a solution that cannot be considered as a complete one:

$$R \sim z^n;\ w_m \sim z^{-{}^1/_3 n};\ \vartheta_m \sim z^{-{}^5/_3 n}, \tag{2.1.4}$$

n being a dimensionless constant to be determined from observation.

Putting $n = 1$ we find W. SCHMIDT's solution. According to SUTTON's observations on horizontal smoke trails 0,875 should be a better value, however, so that in reality his results are not in accordance with W. SCHMIDT's.

2.1.3 The latest attempt to obtain a theoretical solution for the problem of the propagation of a heated jet was made by PRIESTLEY and BALL (1955). They raised two main objections against the theories of W. SCHMIDT and SUTTON.

Firstly, these theories do not supply solutions for the case that $w_m = 0$ for finite values of z, so as to desbribe the case of heated air rising from the ground.

In the second place the preceding theories give no solution for the case of stable or unstable stratification. BATCHELOR only arrived at a very limited solution for the latter case.

PRIESTLEY and BALL started from the assumption that the cross-jet velocity profile is represented by the same function as the temperature profile, namely by the distribution function $\exp\left\{-r^2/R^2\right\}$. We mentioned already, that this assumption is in contradiction to experimental evidence. From this starting-point they could prove in a general way, i.e. without applying dimensional analysis, that the "radius" of the jet, R, depends linearly on z, i.e. $R = cz$. It is now possible to find the following solutions from the equations used by ROUSE et al.:

$$w_m = \left(\frac{A}{z} + \frac{B}{z^3}\right)^{1/3} \tag{2.1.5}$$

and

$$\vartheta_m = \frac{C}{z^2}\left(\frac{A}{z} + \frac{B}{z^3}\right)^{-1/3} \tag{2.1.6}$$

Here A and C are constants following from the physical properties of the jet and the surrounding fluid, whereas B is an additional integration constant. $B = 0$ leads to the classical solution given in (2.1.2) and (2.1.3).

For $z = z_0 = \sqrt{-\frac{B}{A}}$, w_m is found to be zero indeed, R being cz_0. However, ϑ_m is infinite there, and not like in SCHMIDT's and SUTTON's theories just in a point without physical reality but over a finite surface. It is clear that this result is unacceptable from a physical point of view. The theory fails in this case amongst others because density differences may no longer be neglected. Moreover, the equations used by PRIESTLEY and BALL are based on the assumption that horizontal pressure gradients may be neglected compared with turbulent frictional forces. These forces are assumed to depend on ${w_m}^2$ so that the neglect of the pressure forces is certainly incorrect for $w_m = 0$, and the authors cannot be considered to have succeeded in giving a solution for this special case.

As to the applicability of their theory to the propagation of jets in a stable or unstable atmosphere, PRIESTLEY and BALL assumed $R = cz$ also for this case. As the vertical velocity decreases to zero for a finite value of z, this assumption is not justified, however.

2.2 Experimental results

2.2.1 Laboratory experiments on the rising of heated jets in a stagnant environment are very difficult to carry out. Accurate measurements of velocities as well as of temperatures seem to be hardly possible, especially near the "boundary" of the jet. Very slight draughts disturb the current immediately.

Experiments have been carried out by W. SCHMIDT, ROUSE et al. and more recently by RAILSTON (1954). As far as can be judged, RAILSTON's experiments are the most accurate. It is worth while mentioning e.g. that he measured the width of the jet by observing the "Schlieren" caused by the turbulence in the heated air.

As to the decrease of ϑ_m and w_m with z, SCHMIDT and ROUSE et al. found a more or less close agreement with the $^5/_3$- and $^1/_3$-law. RAILSTON on the contrary found a better agreement with SUTTON's theoretical results.

There has been much criticism, however, of RAILSTON's interpretation of his experimental results.

PRIESTLEY and BALL showed that the law $R = c z^{0,91}$, found experimentally by RAILSTON, does not follow with great certainly from his measurements, whereas BATCHELOR did not exclude the possibility that the current is a laminar one just above the heat source in which case no unique relation between R and z can be expected. All things considered, the temperature decrease along the axis as measured by RAILSTON is certainly not in contradiction to the $z^{-5/3}$ law as follows from table 2.2.1.

Table 2.2.1

height above heat source, $(z-z_1)$ cm	5	10	15	25	35	45	55	65	70
ϑ_m measured °C	12,9	7,7	5,8	3,4	2,3	1,7	1,3	1,0	0,9
theoretical computation according to SUTTON/RAILSTON $z_1 = 10{,}2$ cm $n = 0{,}91$	12,20	7,93	5,66	3,40	2,32	1,72	1,33	1,07	0,97
theoretical computation according to W. SCHMIDT $z_1 = 13{,}7$ cm $n = 1{,}00$	11,42	7,70	5,58	3,40	2,32	1,70	1,31	1,04	0,94

The computations of the theoretical decay of ϑ_m according to SUTTON/RAILSTON with $n = 0{,}91$ and W. SCHMIDT with $n = 1$, were based on the measured value $\vartheta_m = 3{,}4$ °C at 25 cm above the heat source.

2.2.2 Special attention should be given to SUTTON's assumption that the spreading of an axially symmetric jet is better described by a law $R = R_1 \left(\frac{z}{z_1}\right)^n$ with $n < 1$ than by the linear law found by W. SCHMIDT.

Apparently SUTTON deduced the value $n = 0{,}875$ from observations made on horizontally stretching smoke plumes in the atmosphere. But in that case there is no question of a jet emerging into a stagnant environment but of pollution diffusing under the influence of natural wind. Although both phenomena are related in so far that they may be treated according to similar methods, there is no reason to assume a priori that the expansion of a jet follows the same law as the expansion of a quasi horizontal smoke plume.

On the contrary, there exists an important difference between the two phenomena. In the case of the jet emerging into a stagnant environment it is assumed that turbulence is created by the motion itself. This assumption is justified by the empirical fact that at large distances from the jet's axis the motion is almost a laminar one. The diffusion of pollution in a horizontal wind is mainly caused by the natural turbulence of the atmosphere and almost independent of the distance to the plume's axis (see chapter III).

But even if a vertically emerging jet were polluted in some way one should be very careful with the interpretation of visual observations. For if we assume (as is generally done) that the concentration of pollution in a jet depends in the same way on z and r as ϑ, it can easily be understood that the value of $R(z)$ determined from optical contrast-measurements must increase more slowly with growing distance from the nozzle, than if it had been determined from the temperature profile itself.

For contrast-measurements are only possible up to a certain lower limit and this limit will be found at gradually growing distance from the "boundary" of the jet for increasing distance from the nozzle. Similar effects may be responsible for the apparent justification of SUTTON's theory by RAILSTON's measurements of the 'width" of the jet by means of the Schlieren method.

Summarizing we may state that theoretical as well as experimental results seem to indicate that W. SCHMIDT's solution is to be preferred.

2.3 A new theory based on the complete equations of motion

2.3.1 It has been seen how various theories have been developed to describe the motion of a heated jet rising in a stagnant environment. In none of these has the complete set of equations of motion been used as the radial component equation was not considered.

It is worth while to try to find a solution for the axially symmetric jet without employing such special equations as those introduced by ROUSE c.s.

The motion in such a heated jet in a surrounding which is in indifferent equilibrium must be derived from the following equations:

the continuity equation:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho w)}{\partial z} + \frac{1}{r}\frac{\partial (\varrho r u)}{\partial r} = 0 \tag{2.3.1}$$

the equation for vertical motion:

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial r} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} - g \tag{2.3.2}$$

the equation for horizontal motion:

$$\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial r} = -\frac{1}{\varrho}\frac{\partial p}{\partial r} \tag{2.3.3}$$

and finally the equation expressing the conservation of heat:

$$\frac{\partial \Theta}{\partial t} + w\frac{\partial \Theta}{\partial z} + u\frac{\partial \Theta}{\partial r} = 0 \tag{2.3.4}$$

The symbols have the following meanings:

ϱ = density

w = total vertical velocity

u = total radial velocity

p = pressure

Θ = total potential temperature (or temperature in the case of fluids).

The influences of molecular effects such as viscosity and diffusivity have been neglected. The effects of turbulence are contained implicitely in the equations as:

$$w = \overline{w} + w'; \; u = \overline{u} + u'; \; \Theta = \Theta_\infty + \overline{\vartheta} + \vartheta' \tag{2.3.5}$$

where $\overline{w}$, $\overline{u}$ and $\overline{\vartheta}$ denote mean values of the velocity and the excess of temperature in the jet over the temperature of the environment Θ_∞, and where w', u' and ϑ' are the turbulent fluctuations.

ϱ and p have not been split up in a mean value and a fluctuation; ϱ will be treated as a constant, except where it influences the buoyancy, in which case density differences may well be considered due to temperature differences only (homotropy):

$$\varrho = a/\Theta \tag{2.3.6}$$

a being a constant.

The fact that ϱ is considered to be constant except where it gives rise to buoyancy forces means, of course, an approximation. In the equation of continuity for instance, $w\dfrac{\partial\varrho}{\partial z}$ and $u\dfrac{\partial\varrho}{\partial r}$ must remain smaller than $\varrho\dfrac{\partial w}{\partial z}$ and $\varrho\dfrac{\partial u}{\partial r}$, respectively, for at least an order of magnitude. Except in the immediate neighbourhood of orifices from which very hot gases emerge, this condition will generally be fulfilled, as can be easily verified.

Moreover, we consider the motion to be quasi-static, i.e.:

$$\frac{\partial p}{\partial z} = \frac{\partial p^*}{\partial z} = -g\varrho^*; \; \frac{\partial p}{\partial r} = \frac{\partial p^*}{\partial r} = 0, \tag{2.3.7}$$

the asterisk relating to quantities in the undisturbed environment.

With the above assumptions the righthand side of (2.3.2) becomes

$$-\frac{1}{\varrho}\frac{\partial p}{\partial z} - g = \frac{1}{\varrho}\frac{\partial p^*}{\partial z} - g = g\left[\frac{\varrho^* - \varrho}{\varrho}\right] = g\,\frac{\overline{\vartheta} + \vartheta'}{\Theta_\infty} \tag{2.3.8}$$

Restricting ourselves to the stationary case and taking Θ_∞ constant, we find the following equations for the mean motion:

$$\frac{\partial(r\overline{w})}{\partial z} + \frac{\partial(r\overline{u})}{\partial r} = 0 \tag{2.3.9}$$

$$\frac{\partial}{\partial z}(r\overline{w^2})+\frac{\partial}{\partial r}(r\overline{uw})=rg\frac{\bar{\vartheta}}{\Theta_\infty}-\frac{\partial}{\partial z}\{r\overline{(w')^2}\}-\frac{\partial}{\partial r}\{r\overline{u'w'}\} \quad (2.3.10)$$

$$\frac{\partial}{\partial z}(r\overline{uw})+\frac{\partial}{\partial r}(r\overline{u^2})=-\frac{\partial}{\partial z}\{r\overline{u'w'}\}-\frac{\partial}{\partial r}\{r\overline{(u')^2}\} \quad (2.3.11)$$

$$\frac{\partial}{\partial z}(r\bar{w}\bar{\vartheta})+\frac{\partial}{\partial r}(r\bar{u}\bar{\vartheta})=-\frac{\partial}{\partial z}\{r\overline{w'\vartheta'}\}-\frac{\partial}{\partial r}\{r\overline{u'\vartheta'}\} \quad (2.3.12)$$

2.3.2 It is possible to eliminate $\bar{u}$ from equation (2.3.9). Introducing the generally accepted and experimentally verified similarity hypothesis by putting

$$\bar{w}=w_m(z)f(\eta) \quad (2.3.13)$$

w_m, the mean vertical component velocity in the axis [1]), being a function of z only, and $\eta=\frac{r}{R}$, R being a measure for the width of the jet and assumed to depend on z only, we obtain:

$$\frac{\partial(r\bar{u})}{\partial r}=-\frac{\partial(r\bar{w})}{\partial z}=-r\left\{\frac{dw_m}{dz}f(\eta)-w_m\frac{df(\eta)}{d\eta}\frac{r}{R^3}\frac{dR}{dz}\right\}$$

or

$$r\bar{u}=-R^2\frac{dw_m}{dz}\{\eta\,\varphi(\eta)-\psi(\eta)\}+w_m\frac{dR}{dz}R\{\eta^2f(\eta)-2\eta\,\varphi(\eta)+2\psi(\eta)\} \quad (2.3.14)$$

As $\bar{u}$ must remain finite for $r\to 0$, $f(\eta)$ must be such a function of η that $\psi(\eta)$ is at least of the first order in η and, moreover, the additional integration constant must be zero. In 2.4 adequate functions $f(\eta)$ will be introduced.

We now suppose a relation for $\bar{\vartheta}$ to exist analogous to (2.3.13):

$$\bar{\vartheta}=\vartheta_m\,g(\eta) \quad (2.3.15)$$

As the total entrainment at infinite distance of the axis of the jet must be zero – certainly in a laboratory with fixed walls or in an atmosphere of a spherical earth – all values $(r\bar{w})$, etc, tend to zero for $r\to\infty$. As on the other hand all velocities and all temperature excesses are necessarily supposed to remain finite for $r\to 0$, introduction of (2.3.13), (2.3.14) and (2.3.15) into the equations (2.3.10), (2.3.11) and (2.3.12) and integration over r from $r=0$ to $r=\infty$ gives:

$$\frac{d}{dz}(I_1R^2w_m{}^2)=I_2\vartheta_mR^2-\frac{d}{dz}\int_0^\infty\{r\overline{(w')^2}\}dr \quad (2.3.16)$$

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[1]) The bar may be omitted here.

$$\frac{d}{dz}\left(I_3 R^2 \frac{dR}{dz} w_m{}^2 + I_4 R^3 \frac{dw_m}{dz} w_m\right) = -\frac{d}{dz}\int\limits_0^\infty \{r\overline{u'w'}\}\, dr \qquad (2.3.17)$$

$$\frac{d}{dz}(I_5 R^2 w_m \vartheta_m) = -\frac{d}{dz}\int\limits_0^\infty \{r\overline{w'\vartheta'}\}\, dr, \qquad (2.3.18)$$

I_1 to I_5 being constants, depending on $f(\eta)$ and $g(\eta)$. Partial differentiations with respect to z have been replaced by absolute differentiations as the quantities R, w_m and ϑ_m are functions of z only.

Assuming the absolute values of the fluctuations u' and w' to be w_m times functions of η, the absolute value of ϑ' to be ϑ_m times a function of η and applying the most simple dimensional analysis, assuming $R \sim z^\alpha$, $w_m \sim z^\beta$ and $\vartheta_m \sim z^\gamma$ we immediately find: $\alpha = 1$, $\beta = -1/3$ and $\gamma = -5/3$, i.e. the solution of W. SCHMIDT.

2.3.3 It will be clear that the simple dimensional analysis applied in the foregoing sections only supplies a special solution of the problem, as the three remaining differential equations (2.3.16), (2.3.17) and (2.3.18), two being of the first order and one of the second order, require four independent constants of integration.

Putting $R = R_1\left(\frac{z}{z_1}\right)$, $w_m = w_{m1}\left(\frac{z}{z_1}\right)^{-1/3}$ and $\vartheta_m = \vartheta_{m1}\left(\frac{z}{z_1}\right)^{-5/3}$ and integrating equation (2.3.18): $R^2 w_m \vartheta_m = \text{const. } Q$, Q being the vertical heatflux, we actually find four constants, namely R_1, w_{m1}, ϑ_{m1} and Q, z_1, being a reference height.

They are not mutually independent, however, as obviously $R_1{}^2 w_{m1} \vartheta_{m1} \sim Q$.

Equations (2.3.16) and (2.3.17) supply an opportunity to find a more general solution. Integrating (2.3.17) with respect to z, it is permissible to put the integration constant $= 0$, as no energies are involved except kinetic energy and frictional energy, so that we may write, again taking $\sqrt{\overline{(w')^2}} \sim w_m$, etc.:

$$I_3 R^2 \frac{dR}{dz} w_m{}^2 + I_4 R^3 \frac{dw_m}{dz} w_m = \text{Cst. } R^2 w_m{}^2 \qquad (2.3.19)$$

This equation has to be combined with the first equation of motion, from which ϑ_m can be eliminated:

$$\frac{d}{dz}(IR^2 w_m{}^2) = \frac{Q}{w_m} \qquad (2.3.20)$$

From these two equations w_m may be eliminated in turn and a second order equation in R results finally:

$$RR'' + aR' + b(R')^2 = c$$

where a prime denotes differentiation with respect to z and where a, b and c are constants. Putting $R = x$ and $R' = y$ we get a differential equation of the first order, namely

$$xy\frac{dy}{dx} + ay + by^2 = c \tag{2.3.21}$$

or

$$by^2 + ay - c = C_0\, x^{-2b}\, e^{h(y)} \tag{2.3.22}$$

where $h(y) = \int \frac{a dy}{by^2 + ay - c}$ and C_0 is a constant of integration. The general solution leads to very complicated results. $C_0 = 0$ immediately gives a special solution, however, namely

$$R' = y = -\frac{a}{2b} \pm \sqrt{a^2 + 4\,bc} \tag{2.3.23}$$

Or, $R \sim z$ is a solution that can be obtained in a more general way than by applying dimensional analysis.

This is the same result as obtained by PRIESTLEY and BALL. We arrived at it, however, independent of special assumptions about $f(\eta)$ and $g(\eta)$.

R_1/z_1 can be computed in principle, but it still depends on the intensity of the turbulent exchange so that it remains to be determined from experiment in general. Writing $R = \frac{R_1}{z_1} z + \text{cst.}$ only leads to another position of the origin on the z axis so that the additional constant of integration may be omitted.

Introduction of $R = R_1 \frac{z}{z_1}$ into equations (2.3.16) and (2.3.18) leads to

$$w_m = \left[I_I \frac{Q}{z} + \frac{B}{z^3}\right]^{1/3} \tag{2.3.24}$$

$$\vartheta_m = I_{II}\frac{Q}{z^2}\left[I_I \frac{Q}{z} + \frac{B}{z^3}\right]^{-1/3} \tag{2.3.25}$$

The radial w- and ϑ-profiles are implicitely contained in the numerical constants I_I and I_{II}. B is an integration constant.

It is again possible to define two constants w_{m1} and ϑ_{m1}, being the values of w_m and ϑ_m at reference height z_1. Together with Q, R_1 and B we have 5 constants here, equivalent, however, to 4 independent constants of integration. If we fix, for instance, w_{m1} and R_1, we find a relation between Q and B from the expression for w_m so that they can be considered as one independent constant only. ϑ_{m1} still may obtain an infinite range of values, however, so that four mutually independent constants remain.

The solution thus found contains some special cases. If B is taken zero we find the old power solution, which refers obviously to the case of free and unforced convection. An important result in this case is, that obviously

$$w_m \sim Q^{1/3} \text{ and } \vartheta_m \sim Q^{2/3} \tag{2.3.26}$$

Taking $Q = 0$, i.e. considering a jet rising in a stagnant atmosphere without the action of buoyancy forces, we find

$$R = \frac{R_1}{z_1} z \tag{2.3.27}$$

$$w_m = w_{m1} \left(\frac{z}{z_1}\right)^{-1} \tag{2.3.28}$$

$$\vartheta_m = 0 \tag{2.3.29}$$

This is the solution obtained by TOLLMIEN (1925) for the propagation of a jet of the same temperature as the environment, emerging with some initial velocity from a nozzle, so that the general solution connects his solution with that of W. SCHMIDT.

2.4 The velocity- and temperature profiles

2.4.1 The velocity- and temperature profiles have been found in two different ways.

In the first place they have been calculated by expansion in series, starting from the equation of continuity (W. SCHMIDT and TOLLMIEN). Other investigators adopted an error curve as a sufficient approximation to reality.

There exists an important difference between the two solutions, however. In applying expansion to series, it was assumed that the jet achieves a finite width, R_0, so that for $r = R_0$, $\overline{w}$ as well as $\frac{\partial \overline{w}}{\partial r}$ become $= 0$. The error curve, however, gives $\overline{w} = \frac{\partial \overline{w}}{\partial r} = 0$ only for $r = \infty$, so that according to the second group of solutions all environmental fluid is involved in the motion, albeit that for large values of r that motion is hardly perceptible.

Although at first sight the power series solution seems to be the best, it has, however, disadvantages. For values of $r > R_0$ the series do not give $\overline{w} = 0$ any longer.

This difficulty does not arise with the Gauss solution which may also be preferable from a physical point of view. For the stationary state will only occur an infinitely long time after the moment the motion started and it is plausible that the whole environment will be involved in the motion at that time.

2.4.2 The Gauss solution, however, also manifests an important shortcoming, as can be seen immediately by considering the equation of continuity (2.3.9). Integration over a horizontal plane gives:

$$\frac{\partial}{\partial z}\int_0^\infty 2\pi\, r\, \overline{w}\, dr = 2\pi\,[r\overline{u}]_0^\infty \tag{2.4.1}$$

As the influence of the jet must be zero at infinite distance from the axis, and as on the other hand $\overline{u}$ must remain finite for $r = 0$, this equation leads to the condition:

$$\int_0^\infty r\overline{w}\, dr = \text{cst.} \tag{2.4.2}$$

This condition is not fulfilled by the generally accepted error-distribution:

$$\overline{w} = w_m \exp\{-r^2/R^2\} \tag{2.4.3}$$

except in case $(R^2\, w_m)$ is independent of z. Starting from SUTTON's notation this would mean $2n - {}^1/_3\, n = 0$, $n = 0$ or R and w_m both independent of z. A solution of this kind only applies to the motion of a fluid in a tube of constant diameter.

Incidentally, it is clear that the representation of the velocity profile in a free jet by an error curve (2.4.3) leads to absurdities for $r = 0$.

For $\dfrac{\partial}{\partial r}(r\overline{u}) = -\dfrac{\partial}{\partial z}(r\overline{w}) = -\dfrac{\partial}{\partial z}(rw_m\, e^{-r^2/R^2}) = -\left[\dfrac{dw_m}{dz}\, r + w_m\, r^3\, \dfrac{2}{R^3}\dfrac{dR}{dz}\right] e^{-r^2/R^2}$

or

$$r\overline{u} = \left[\frac{1}{2}\, R^2\, \frac{dw_m}{dz} + w_m\, R\, \frac{dR}{dz}\left(\frac{r^2}{R^2} + 1\right)\right] e^{-r^2/R^2} \tag{2.4.4}$$

So $\overline{u} \to \infty$ for $r \to 0$ except when $w_m \sim R^{-2}$. But this is in contradiction to all experimental and theoretical evidence.

Indeed, ROUSE c.s. found $\overline{u} \to \infty$ for $r \to 0$ from their assumptions on the velocity profile, a physically inadmissable result.

It is possible to find a way out by starting from equation (2.4.2). From the properties of Γ-function it follows that this equation holds true for all velocity distributions:

$$\overline{w} = w_m \left[1 - \frac{\varkappa}{2}\left(\frac{r}{R}\right)^{\varkappa}\right] \exp\left\{-\left(\frac{r}{R}\right)^{\varkappa}\right\} \tag{2.4.5}$$

where $\varkappa$ is an arbitrary constant that must be > 1 as $\dfrac{\partial \overline{w}}{\partial r}$ must equal zero in the axis of the jet and where the constant in the righthand side of (2.4.2) equals zero. Figure 1 represents the $\bar{w}$-distributions for $\varkappa = 1, 2, 3$ and 4. It is also shown in the figure, that the error function $\exp\{-r^2/R_*{}^2\}$ offers a satisfactory approximation for the velocity distribution proposed here, when $\varkappa = 2$ and $R_* = 0{,}67\,R$.

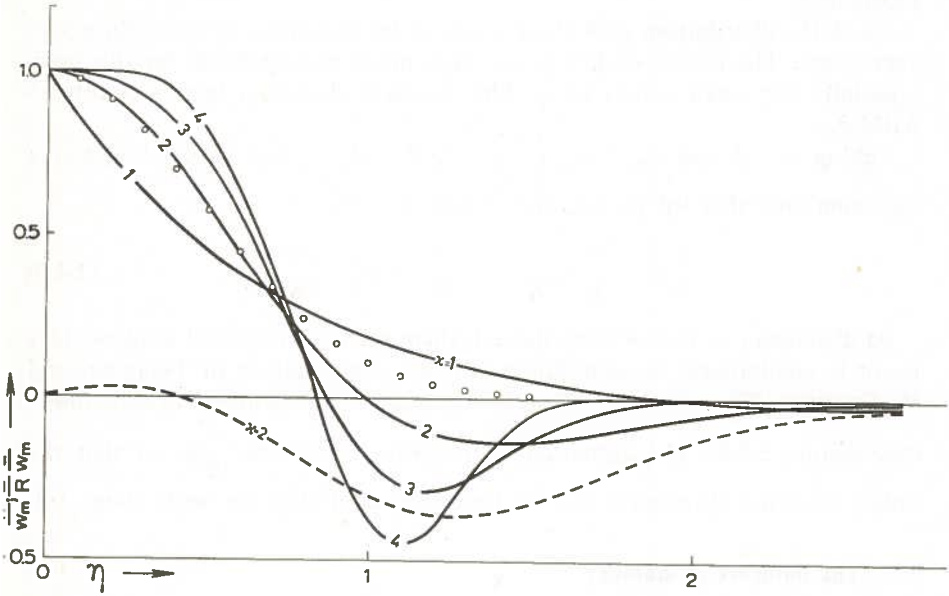

FIG. 1. Distribution of the vertical and horizontal velocities in a jet rising in a stagnant environment. The full curves represent $\frac{\bar{w}}{w_m}$, the broken curve $\frac{z}{R}\frac{\bar{u}}{w_m}$ for $x = 2$, and the dots an error curve respectively.

The velocity distributions constructed in this way may obviously be interpreted as follows.

The heated jet creates a more or less closed circulation in its immediate environment as it forces outer air to flow downwards. The descending current around the ascending jet has long ago been recognized in meteorology as the countercurrent. It is a phenomenon observed with certainty around cumuliform clouds (See PETTERSSEN, 1939 and F. H. SCHMIDT, 1947).

The vertical velocity becomes zero for $r = R\sqrt[\varkappa]{2/\varkappa}$ and for $r = \infty$. The maximum vertical velocity in the countercurrent occurs at $r = R\sqrt[\varkappa]{2/\varkappa + 1}$ and amounts to $-\varkappa/2\; w_m \exp\{-(2/\varkappa + 1)\}$. For $\varkappa = 2$ this value becomes $-0{,}135\; w_m$, so that it is not surprising that the countercurrent has not been observed. In the experiments of ROUSE c.s. for instance the maximum scatter of the velocity for any value of r/R amounted to more than 20% of the maximum value w_m, whereas only 2 out of a total of 69 measurements gave a velocity smaller than 0,1 w_m. The standard deviation of the measured values with respect to the theoretical curve giving the best fit, amounted to 0,075 w_m. It is improbable that a velocity less than twice the standard deviation should be recognized.

As to the distribution of $\bar{\vartheta}$ there seems to be no reason to reject the error-curve here. The results of ROUSE c.s. look much more reliable for this case, especially for small values of η. The standard deviation here amounts to 0,054 ϑ_m.

Taking $\varkappa = 2$ and $w_m = w_{m1}\left(\frac{z}{z_1}\right)^{-1/3}$, $R = R_1 \frac{z}{z_1}$ and proceeding along the same lines that led to equation (2.4.4), we obtain:

$$\frac{z}{R}\bar{u} = \left[\frac{1}{6} w_m \frac{r}{R} - w_m \frac{r^3}{R^3}\right] \exp\left\{-\left(\frac{r^2}{R^2}\right)\right\} \tag{2.4.6}$$

At distances $r < 0{,}42\ R$ from the axis there exists a horizontal outflow. This result is qualitatively in accordance with the computations of TOLLMIEN and W. SCHMIDT. The value of z/R amounts to about 8 according to experiment (See section 2.5.2). The dotted curve in figure 1 represents $\frac{z}{R}\bar{u}$, so that the radial velocities themselves can be found by multiplication with about 1/8.

2.5 The influence of stability

2.5.1 Apart from the effort made by PRIESTLEY and BALL no solution has been found as yet for the case of a stationary jet rising in a non-indifferent environment. This problem is of great importance for the diffusion of smoke plumes rising vertically in a stagnant atmosphere. By the following reasoning it is possible to find a solution which may be a reasonable approximation.

The rising jet as well as its descending countercurrent both tend to disturb the environment, the more so as the vertical motions show more or less strong turbulence. This will ultimately result in a thoroughly mixed atmosphere in which a constant potential temperature will occur. An analogous phenomenon is observed in the atmosphere when moderate or strong winds disturb a stable layer near the surface. The disturbed layer is then bounded at its upper level by an inversion.

Reasoning along these lines we will assume that the environment of the jet is mixed, so that an indifferent layer occurs bounded at its top by a temperature inversion.

Assuming further that the jet will be able to rise stationary as long as the temperature at its axis is higher than that of the environment, it is possible to find an upper limit for the height achieved by the jet in the case of stable stratification. Unstable stratification is not considered as instability only occurs in layers of limited vertical dimensions.

In the equations of motion (2.3.9)—(2.3.12) the introduction of a lapse rate leads to the following modifications of equations (2.3.10) and (2.3.12):

$$\frac{\partial}{\partial z}(r\overline{w}^2)+\frac{\partial}{\partial r}(r\overline{u}\,\overline{w})=rg\frac{\overline{\vartheta}}{\Theta_{\infty 1}+\Gamma(z-z_1)}-\frac{\partial}{\partial z}\{r\overline{(w')^2}\}-\frac{\partial}{\partial r}\{r\overline{u'w'}\} \quad (2.5.1)$$

and

$$\frac{\partial}{\partial z}(r\overline{w}\,\overline{\vartheta})+\frac{\partial}{\partial z}(r\,\overline{w}\,\Theta_\infty)+\frac{\partial}{\partial r}(r\overline{u}\,\overline{\vartheta})=-\frac{\partial}{\partial z}\{r\overline{w'\vartheta'}\}-\frac{\partial}{\partial r}\{r\overline{u'\vartheta'}\} \quad (2.5.2)$$

whereas (2.3.9) and (2.3.11) do not change. In (2.1.5) Γ denotes the lapse rate of Θ_∞ and z_1 the height of the orifice above the origin $z=0$.

Taking Γ constant and integrating over r from $r=0$ to $r=\infty$, we find that the second term in (2.5.2) disappears as a consequence of $\int_0^\infty r\overline{w}\,dr=0$, so that finally the substitution of $rg\dfrac{\overline{\vartheta}}{\Theta_\infty}$ by $rg\dfrac{\overline{\vartheta}}{\Theta_{\infty 1}+\Gamma(z-z_1)}$ in (2.3.10) is the only effect of the stable stratification.

Mathematically, therefore, the assumption of mixing by the jet comes to the introduction of a mean value of Θ_∞ over a layer of adequate height. As $\Gamma\,(z-z_1)$ is small compared with Θ_∞ in alle practical cases, the method proposed here may be considered as an acceptable first approximation.

Of course the jet will not totally lose its velocity in the level where its maximum temperature equals the environmental one, but it will be retarded and spread in the layer of which this level forms the lower boundary.

2.5.2 We will first try to find a relation between the properties of the jet and the lapse rate of the atmosphere for the case of a pure convection jet, i.e. a jet for which $w_m=w_{m1}\,(z/z_1)^{-1/3}$ and $\vartheta_m=\vartheta_{m1}\,(z/z_1)^{-5/3}$.

Figure 2 illustrates the supposed situation.

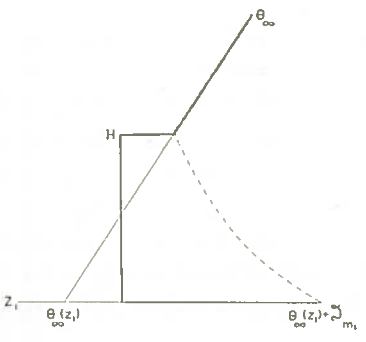

FIG. 2.
Illustration of the approximations, introduced in order to compute the maximum height a plume can achieve in a stable and stagnant atmosphere.

$\Theta_{\infty 1} = \Theta_\infty\ (z_1)$ represents the temperature of the undisturbed environment at the level of the orifice, ϑ_{m1} the temperature difference between the axis of the jet and the environment in the same level. The vertical part of the Θ_∞-curve represents the layer in the immediate surrounding of the jet where an indifferent equilibrium is created by vertical turbulent motions or where in the computations the variable value $\Theta_\infty\ (z)$ has been replaced by a mean value. The dotted line represents the decreasing value of the excess temperature in the axis of the jet as a function of height, according to W. SCHMIDT and assuming neutral equilibrium between z_1 and H.

We now find the following relation from which H, the height of the inversion can be determined:

$$\Theta_{\infty 1} + \Gamma(H - z_1) = \Theta_{\infty 1} + \frac{1}{2}\Gamma(H - z_1) + \\ + \left\{\vartheta_m(z_1) - \frac{1}{2}\Gamma(H - z_1)\right\}\left(\frac{H}{z_1}\right)^{-5/3} \tag{2.5.3}$$

or with $\zeta = \dfrac{H - z_1}{z_1}$

$$\Gamma\zeta = \left\{\frac{2\vartheta_{m1}}{z_1} - \Gamma\zeta\right\}(1+\zeta)^{-5/3} \tag{2.5.4}$$

where $\vartheta_m\ (z_1) = \vartheta_{m1}$ denotes the temperature excess in the axis of the jet over the undisturbed environment in the level of the orifice, as before.

The only variable remaining to be determined in equation (2.5.4) is z_1, the height of the orifice above the origin, determined by $R = 0$. It is reasonable to express z_1 in terms of the dimensions of the stack, e.g. of the diameter of the orifice, D. It is true, the flow in the orifice will not obey the law $\overline{w}_1 = w_{m1}\left(1 - \dfrac{r^2}{R_1^2}\right)e^{-r^2/R_1^2}$ nor will the temperature excess over the environment there be exactly described by $\overline{\vartheta}_1 = \vartheta_{m1}\, e^{-r^2/R_1^2}$, but exact laws lacking we will assume $R_1 = \frac{1}{2} D$ to be a sufficiently accurate approximation.

It is necessary to deduce the relation between R and z from laboratory experiments. ROUSE c.s. found from the velocity distribution $R_*{}^2 = {}^1/_{96}\, z^2$ or with $R_* = 0{,}67\ R$ (see section 2.4.2), $R \sim 0.153z$ and from the temperature distribution $R = \sqrt{{}^1/_{71}}\ z = 0.120z$. According to PRIESTLEY and BALL the temperature profile observed by RAILSTON can be described by $\overline{\vartheta} = \vartheta_m$ exp $(- r^2/2R_r{}^2)$ with $R_r = 0.09z$. This leads according to our notation with $2R_r{}^2 = R^2$ to $R = 0.127z$. Taking the value $z_1 = 13.7$ cm from table 2.2.1 we find $R = 0.133z$. We adopt $R = 0.125z$ as a reasonable approximation, or $z_1 \sim 4\ D$. Other proportionality factors could, of course, be employed without difficulty.

It is possible now to compute $\frac{H - z_1}{D}$ as a function of Γ and $\Psi = \frac{\vartheta_{m1}}{D}$. The result is given in figure 3.

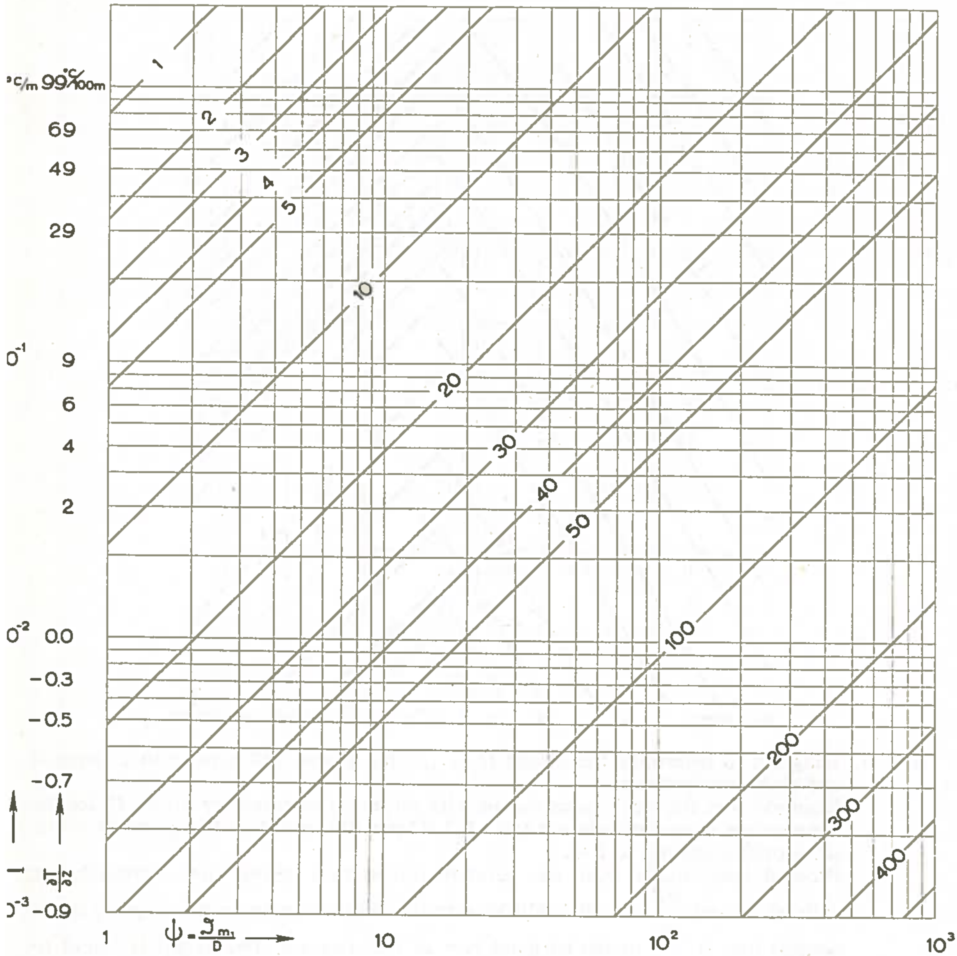

FIG. 3. $\frac{H - z_1}{D}$ as a function of Γ or $\frac{\partial T}{\partial z}$ and $\Psi = \frac{\vartheta_{m1}}{D}$

Figure 4 enables the height of the inversion above the orifice, $(H - z_1)$, to be found immediately, expressed in meters and as a function of ϑ_{m1}, Γ (expressed as $\frac{\partial \Theta_\infty}{\partial z}$ in °C/m or as the actual temperature gradient $-\frac{\partial T}{\partial z}$ expressed in °C per 100 m), and D, the diameter of the stack orifice.

Above the level found in this way the stack gases will show a relatively strong decrease of vertical velocity and eventually existing pollution will accumulate in a rather thin layer.

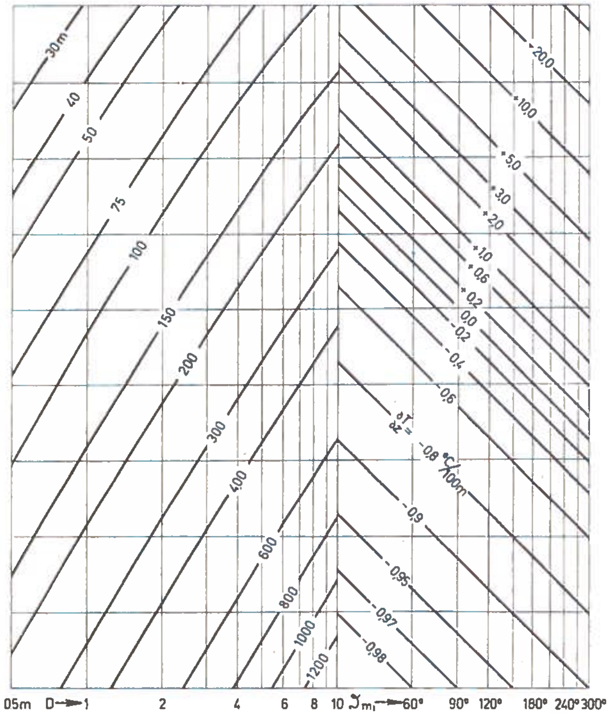

FIG. 4. Diagram to determine the height $H - z_1$, a hot plume will achieve in a stagnant and stable environment.
Example: Let the stack gases escape with an initial temperature of 90 °C, let the temperature lapse rate amount to — 0,2 °C per 100 m and let the diameter of the stack orifice amount to 1 m.
Proceed horizontally from the point of intersection of the lines corresponding with 90 °C and $\frac{\partial T}{\partial z} = -0{,}2$ °C/100 m in the richthand part of the diagram to the vertical line $D = 1$ in the lefthand part of the diagram. The height is found to be 100 m.

2.5.3 Generally a plume emerging from a stack will behave according to equations (2.3.24) and (2.3.25). This may be due to draught or to the addition of cold air to the stack gases in order to increase the exit-velocity.

It might be worth while trying to obtain an impression of the influence of the latter effect on the thickness of the layer in a stable atmosphere in which the jet is able to rise stationarily.

Adding a fraction φ of pure air with environmental temperature to a jet rising exclusively due to buoyancy, the maximum vertical velocity in the orifice will become

$$W_1 = w_{\vartheta 1}(1 + \varphi) \qquad (2.5.5)$$

$w_{\vartheta 1}$ being the original maximum velocity whereas the maximum (potential) temperature excess will decrease to

$$T_1 = \vartheta_{\vartheta 1}(1 + \varphi)^{-1} \qquad (2.5.6)$$

$\vartheta_{\vartheta 1}$ being the original maximum (potential) temperature excess.

Now equations (2.3.24) and (2.3.25) may be rewritten as follows:

$$W_m(z) = \left[w_{c1}^3 \frac{z_1}{z} + w_{f1}^3 \frac{z_1^3}{z^3}\right]^{1/3} \qquad (2.5.7)$$

and

$$T_m(z) = T_1 \frac{z_1^2}{z^2} \frac{(w_{c1}^3 + w_{f1}^3)^{1/3}}{\left(w_{c1}^3 \frac{z_1}{z} + w_{f1}^3 \frac{z_1^3}{z^3}\right)^{1/3}} \qquad (2.5.8)$$

where

w_c = max. vertical velocity due to buoyancy under the new circumstances.
w_f = max. vertical velocity due to extra forced convection.
W_m = total max. vertical velocity under the new circumstances.
T_m = max. (potential) temperature excess of the jet over the environment.

It should be kept in mind that both, w_c and w_f are hypothetical velocities, namely the velocity the jet would acquire if $\varphi = 0$, the temperature excess remaining T, and the velocity the jet would obtain if $T \to 0$ and so all free convection stopped, respectively. It is not surprising, therefore, that w_c and w_f are not additive quantities.

It is possible now to relate w_{c1}, the new maximum exit velocity in the orifice due to convection and $w_{\vartheta 1}$ the original maximum exit velocity. For in the orifice the following relations hold before and after the addition of cold air respectively:

$$R_1^2 \vartheta_{\vartheta 1} w_{\vartheta 1} = Q \qquad (2.5.9)$$

$$R_1^2 T_1 w_{c1} = Q' \qquad (2.5.10)$$

where $Q' < Q$. In fact after the addition of cold air

$$R_1^2 T_1 W_1 = Q \qquad (2.5.11)$$

As generally the temperature excess and the convectional part of the maximum vertical velocity are related according to

$$\vartheta_m \sim Q^{2/3}; \; w_m \sim Q^{1/3} \qquad (2.5.12)$$

(see section 2.3.4), we immediately find from (2.5.6), (2.5.9), (2.5.10) and (2.5.12):

$$Q' = Q\,(1 + \varphi)^{-3/2} \tag{2.5.13}$$

and

$$w_{c1} = w_{\vartheta 1}\,(1 + \vartheta)^{-1/2} \tag{2.5.14}$$

It then follows from (2.5.7) and (2.5.8) that

$$w_{\vartheta 1}{}^3\,(1 + \varphi)^3 = w_{\vartheta 1}{}^3\,(1 + \varphi)^{-3/2} + w_{f1}{}^3$$

or

$$w_{f1} = w_{\vartheta 1}\left\{(1 + \varphi)^3 - (1 + \varphi)^{-3/2}\right\}^{1/3} \tag{2.5.15}$$

or finally:

$$T_m(z) = \vartheta_{\vartheta 1}\,\frac{z_1}{z}\left\{(1 + \varphi)^{-3/2}\,\frac{z^2}{z_1{}^2} + [(1 + \varphi)^3 - (1 + \varphi)^{-3/2}]\right\}^{-1/3} \tag{2.5.16}$$

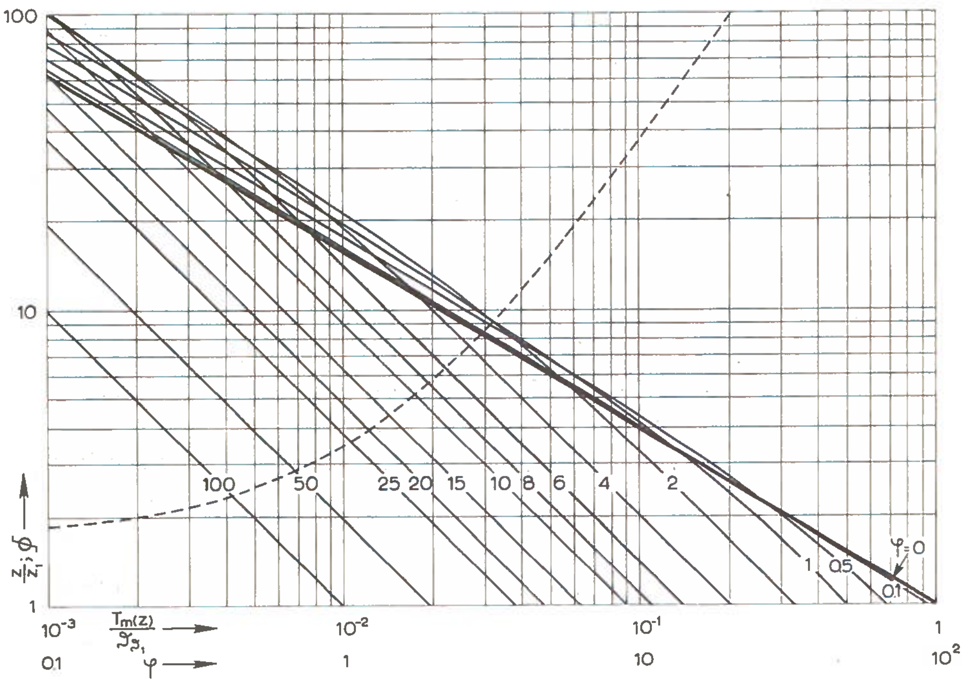

FIG. 5. $\dfrac{T_m(z)}{\vartheta_{\vartheta 1}}$ as a function of $\dfrac{z}{z_1}$ and φ (full curves). Φ as a function of φ (broken curve)

Figure 5 shows the results. It appears that for moderate to strong mixing of the original jet with cold air the temperature of the mixture decreases in the lower levels almost proportional to z^{-1}, whereas at greater heights the temperature decrease approaches the $-(^5/_3)$–law.

This means that above some level Φz_1, Φ being a number given by $\Phi^2 = \left\{\dfrac{(1+\varphi)^3 - (1+\varphi)^{-^3/_2}}{1-(1+\varphi)^{-^3/_2}}\right\}$, the mixed jet will be warmer than the original one. This level rises with the increasing value of φ.

In the case of a stable atmosphere, however, the influence of the temperature decrease below Φz_1 will be much more important than the influence of the temperature gain for $z > \Phi z_1$, so that in general the addition of cold air to the effluent stack gases with the intention of increasing their exit velocity, will influence unfavourably the maximum height the gases can attain in a stable atmosphere, i.e. under normal conditions. This is especially so in the case of strong temperature inversions.

2.5.4 Since the present treatise was finished the writer acquainted himself with an article by MORTON, TAYLOR and TURNER (1956) in which the rise of heated jets in a stable environment was discussed in an exact way.

They started their discussions from equations representing the conservation of volume, momentum and density and approximated the velocity profile by an error distribution.

Apart from the objections that may be raised against the accepted velocity profile their computations must be considered as the best available at the moment.

They found that below the level where ϑ_m becomes zero R is almost $\sim z$, but at greater heights the jet spreads rapidly, so that there is agreement with our qualitative assumptions in this respect. MORTON c.s. adopted $R = 0{,}112\ z$, a value slightly below the one used here.

They introduced dimensionless quantities and found that the height where ϑ_m becomes zero depends on $\Gamma^{-^3/_8}$ and on (*the flux of buoyancy from the point source*) $^{1/4}$. The latter quantity is difficult to handle in practical cases because the point source is always a virtual one, the actual source having finite dimensions in reality. But according to their theory the flux of buoyancy depends on z so that the flux in the virtual point source cannot be determined directly from the quantities relating to the real source.

In order to compare the results of MORTON c.s. with those obtained by the present approximation we will assume that the strength of the virtual source is proportional to $\vartheta_{m1}{}^{^3/_2}$. This means that according to MORTON c.s. H must be proportional to $\left(\dfrac{\vartheta_{m1}}{\Gamma}\right)^{3/8}$.

Figure 6 shows H as a function of ϑ_{m1}/Γ, constructed from figure 4 and for nozzle diameters of 0,5, 1, 2, 5 and 10 m. The lines pertain to $H \sim (\vartheta_{m1}/\Gamma)^{^3/_8}$. The agreement is excellent and shows that the present rough computation of H

is of about the same quality as MORTON's exact one. Slight deviations occur when ϑ_{m1}/Γ becomes $< 10^3$ i.e. in the case of very strong inversions and/or very small values of ϑ_{m1}. It is to be expected that these deviations will increase with the growing diameter of the real source.

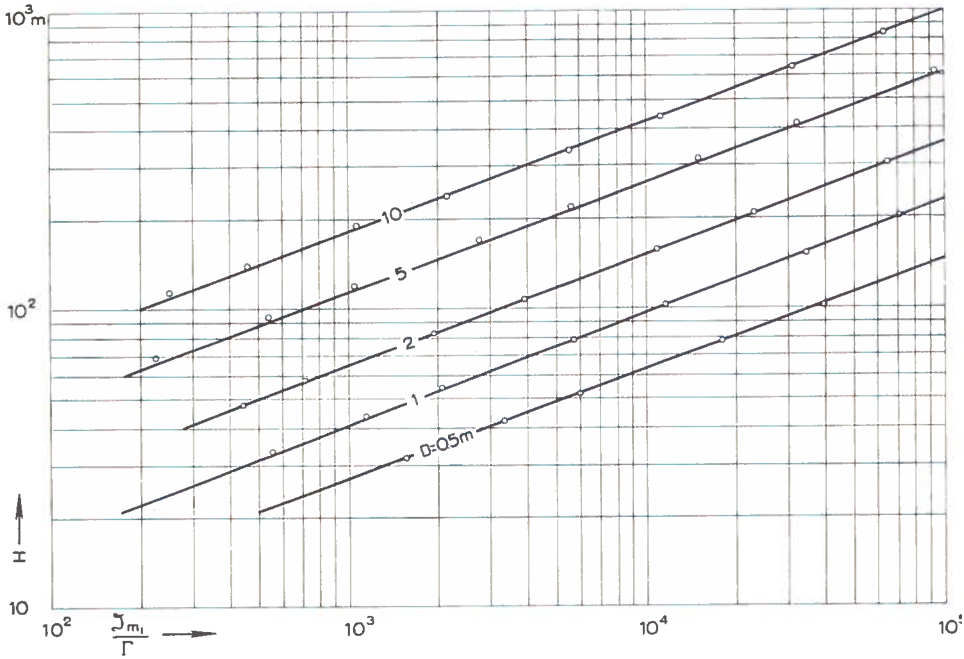

FIG. 6. H as a function of $\dfrac{\vartheta_{m1}}{\Gamma}$ and D. Comparison between the present results and those obtained by MORTON c.s. The dots are obtained from fig. 4, the lines from (2.5.18).

The height a plume escaping from a stack orifice will reach is given by:

$$(H - z_1) = C\,(\vartheta_{m1}/\Gamma)^{3/8} - 4D \tag{2.5.17}$$

where C is a constant depending on the diameter, D, of the orifice. It follows from figure 6 that (at least between $D = 0{,}5$ and $D = 10$ m) a good approximation is given by

$$H - z_1 \approx 3{,}1\, D^{5/8}\, \vartheta_{m1}{}^{3/8}\, \Gamma^{-3/8} - 4D \tag{2.5.18}$$

One should not forget, however, that the constants depend on $\dfrac{dR}{dz}$ for which quantity MORTON c.s. gave a slightly different value than was mentioned before. If carefully performed experiments provide us with reliable constants, equation (2.5.18) might become an easy tool in the prediction of the height at which a stack plume obtains the same temperature as the stable environment.

According to MORTON c.s. the stack's pollution will then accumulate in the layer between H and 1,32 H. MORTON's deductions cannot be applied without modifications to the case of jets emerging with a general velocity W as treated in section 2.5.3.

Moreover, these deductions seem to be more difficult to apply to some practical problems. It is not clear e.g. how an example like that of the burning town mentioned on page 21 of MORTON's paper can be tackled by this theory as in that case the virtual point source is at a distance below the earth's surface comparable with or even greater than $H - z_1$ so that the flux of buoyancy from that point source is much larger than the rate of release of heat in the town.

Although inferior to MORTON's treatment from a theoretical point of view, the present approximate solution of the problem seems to be at least as good from a practical point of view.

CHAPTER III

THE PROPAGATION OF JETS IN THE CASE OF WIND

3.1 The diffusion of "cold" jets

Up to now we have considered exclusively the propagation of a (heated) jet in a stagnant environment. This special case may be of some importance from a theoretical point of view, it is much less so for the practical problem of the diffusion of industrial pollution, in which the horizontal wind plays a predominant role. In general the stack gases will escape from the stack with some vertical velocity and the plume will bend gradually under the influence of the wind until it becomes practically horizontal.

The theory agreeing best with experimental data is without doubt that developed by SUTTON (1947, 1953). He started from the following assumptions:
1. The stack gases are immediately carried away quasi-horizontally by the wind. This assumption involves a discontinuity in the horizontal velocity of these gases, it being zero in the stack orifice and equal to $\bar{u}$, the mean wind velocity at an infinitely small distance from the stack.
2. The diffusion of the stack gases is exclusively due to the natural turbulence in the atmosphere.

Both assumptions are at variance with those underlying the theory developed in the preceding chapter where the velocity of the jet only gradually approaches the velocity of the environment (zero) and where the diffusion was due to the velocity difference between the jet and the environmental fluid. There seems to be no connection between the two problems.

3.2 Dimensional analysis of the diffusion of cold jets

It will now be essayed to arrive at an expression for the diffusion of stack gases in a horizontal wind along lines analogous to those developed in chapter II, i.e. without introducing any special assumption about the nature of turbulence in the lowest layers of the atmosphere. In order to do so we start from the following assumptions:
1. The air escaping from the stack orifice is accelerated more or less gradually by the wind, which is blowing with a constant mean velocity $\bar{U}$ in the x-direction. The total velocity in the positive x-direction can be writen $U + u$, u being a (negative) perturbation velocity decreasing with growing distance from the stack. Variations in wind direction are neglected.
2. The plume is considered to be axially symmetric. This assumption simplifies the computations, but it is not an essential one. It would be relatively easy to replace R^2 by $R_y^2 + R_z^2$ in the following developments.

First consider the equations of motion. They can be written as follows:

$$\frac{\partial(U+u)}{\partial t}+(U+u)\frac{\partial(U+u)}{\partial x}+(W+w)\frac{\partial(U+u)}{\partial r}=-\frac{1}{\varrho}\frac{\partial P}{\partial x}-\frac{1}{\varrho}\frac{\partial p}{\partial x} \tag{3.2.1}$$

$$\frac{\partial(W+w)}{\partial t}+(U+u)\frac{\partial(W+w)}{\partial x}+(W+w)\frac{\partial(W+w)}{\partial r}=-\frac{1}{\varrho}\frac{\partial P}{\partial r}-\frac{1}{\varrho}\frac{\partial p}{\partial r} \tag{3.2.2}$$

where g has been omitted in the second equation as it is neutralized by the vertical component of the general pressure gradient, and where $r^2 = y^2 + z^2$. All capitals refer to the general motion in the atmosphere, whereas the small letters are connected with the perturbation caused by the stack gases. Next add the equation of continuity:

$$\frac{\partial\varrho}{\partial t}+\frac{\partial[\varrho(U+u)]}{\partial x}+\frac{1}{r}\frac{\partial[r\varrho(W+w)]}{\partial r}=0 \tag{3.2.3}$$

As in the foregoing chapter the motion is assumed to be stationary. Now put:

$$U=\overline{U}+U', \qquad u=\overline{u}+u', \qquad W=W', \qquad w=\overline{w}+w',$$

with $\overline{U'}=\overline{W'}=\overline{u'}=\overline{w'}=0.$

$\varrho =$ constant.

Neglecting $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial r}$ with respect to the frictional forces, taking account of the equations for the undisturbed motion (abstracting from Coriolis terms) and connecting $\overline{u}$ and $\overline{w}$ as in the foregoing chapter, the following equations are obtained:

$$\begin{aligned}&\frac{\partial}{\partial x}\{r(2\overline{U}\,\overline{u}+\overline{u}^2)\}+\frac{\partial}{\partial r}\{r(\overline{U}\,\overline{w}+\overline{u}\,\overline{w})\}=\\&=-\frac{\partial}{\partial x}\{r\overline{(u')^2}+2r\overline{(u'\,U')}\}-\frac{\partial}{\partial r}\{r\overline{(W'\,u')}+r\overline{(U'\,w')}\}\end{aligned} \tag{3.2.4}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\{r(\overline{U}\,\overline{w}+\overline{u}\,\overline{w})\}+\frac{\partial}{\partial r}\{r(\overline{w})^2\}=-\frac{\partial}{\partial x}\{r\overline{(u'\,W')}+\\&+r\overline{(U'\,w')}+r\overline{(u'\,w')}\}-\frac{\partial}{\partial r}\{2r\overline{(W'\,w')}\}\end{aligned} \tag{3.2.5}$$

where

$$r\,\overline{w}=R^2\frac{\partial u_m}{\partial x}\varphi_1(\eta)+R\frac{\partial R}{\partial x}u_m\,\varphi_2(\eta) \tag{3.2.6}$$

with $\overline{u}=u_m\,\varphi(\eta)$ and $\eta=r/R$

Integrating over r from $r = 0$ to $r = \infty$, putting $u' \sim w' \sim u_m \varphi_3(\eta)$, taking $\int_0^\infty r\,\bar{u}\,dr = \text{cst}$ as in chapter II and restricting ourselves to values of x that are not too small so that $U \gg u_m$, we finally find:

$$\frac{d}{dx}[I_1 R^2 u_m{}^2] = -\frac{d}{dx}[I_2 R^2 u_m{}^2 + I_3 R^2 u_m \boldsymbol{U}'] \tag{3.2.7}$$

$$\frac{d}{dx}\left\{\left[I_4 R^3 \frac{du_m}{dx} + I_5 R^2 \frac{dR}{dx} u_m\right]\bar{U}\right\} = -\frac{d}{dx}[I_6 R^2 u_m{}^2 + I_7 R^2 u_m \boldsymbol{U}'] \tag{3.2.8}$$

where $\boldsymbol{U}'$ has been written for $\sqrt{\overline{(U')^2}}$ and where u_m and R are functions of x only. $I_1 - I_7$ are constants depending on the exact form of $\varphi(\eta)$ and on the correlation between the turbulent fluctuations of the original motion $(U, 0)$ and the perturbation motion (u, w).

Again endeavouring to find solutions for u_m and R from these at least formally correct equations by simple dimensional analysis, three cases may be discerned:

a. the atmosphere is in a very stable equilibrium or $\boldsymbol{U}' \sim 0$.

b. the atmosphere is very turbulent or $U' \gg u' \approx w'$, at least at sufficiently great distance from the stack.

c. the intermediate case for a moderately turbulent atmosphere.

3.2.1 *a.* In a very stable atmosphere (or at least in an atmosphere with insignificant natural turbulence) equations (3.2.7) and (3.2.8) reduce to

$$\frac{d}{dx}[I_1 R^2 u_m{}^2] = -\frac{d}{dx}[I_2 R^2 u_m{}^2]$$

$$\frac{d}{dx}\left\{\left[I_4 R^3 \frac{du_m}{dx} + I_5 R^2 \frac{dR}{dx} u_m\right]\bar{U}\right\} = -\frac{d}{dx}[I_6 R^2 u_m{}^2]$$

Taking $R \sim x^\alpha$ and $u_m \sim x^\beta$ dimensional analysis leads to $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ of $R = R_1\ (x/x_1)^{1/2}$, $u_m = u_{m1}\ (x/x_1)^{-1/2}$, R_1 and u_{m1} referring to some standard distance x_1, e.g. the distance of the stack orifice from $x = 0$ where $R = 0$. R_1 plays the role of some characteristic dimension and need in practice not be identical with the radius of the (horizontal) orifice although some relation will exist between the two.

The first equation holds identically.

The second can be written:

$$\frac{d}{dx}\left[-\frac{1}{2} I_4 \bar{U} R_1{}^3 u_{m1}\frac{1}{x_1} + \frac{1}{2} I_5 \bar{U} R_1{}^3 u_{m1}\frac{1}{x_1}\right] = -\frac{d}{dx}[I_6 R_1{}^2 u_{m1}{}^2]$$

and holds identically also, so that the solution obtained is a general one.

3.2.2 *b*. In an atmosphere with very strong turbulence it is assumed to be permissible to neglect the first terms in the righthand side of equations (3.2.7) and (3.2.8) and to take I_3 and I_7, both containing a measure for the correlation between u' and U' independent of x. Under these assumptions the equations reduce to:

$$\frac{d}{dx}[I_1 R^2 u_m^2] = -\frac{d}{dx}[I_3 R^2 u_m U']$$

and

$$\frac{d}{dx}\left\{\left[I_4 R^3 \frac{du_m}{dx} + I_5 R^2 \frac{dR}{dx} u_m\right] \bar{U}\right\} = -\frac{d}{dx}[I_7 R^2 u_m U']$$

Dimensional analysis leads to $u_m = \text{cst} = -\frac{I_3}{I_1} U'$ or $= 0$ and $R = R_1 \left(\frac{x}{x_1}\right)$. Besides $\frac{R_1}{x_1} = -\frac{I_7 U'}{I_5 \bar{U}}$, so that the R-cone is sharper the larger the mean velocity $\bar{U}$ compared with the mean turbulent velocity U'. The solution shows that there may remain a small residual velocity u_m at some distance from the stack orifice.

3.2.3 *c*. The discussion of the third case is difficult. Obviously it forms the transition from case *a* where the stack gases themselves disturb the general current and are the only cause of the turbulent diffusion to case *b* where the general turbulence is so well developed that the small disturbance due to the stack gases does not play a role. This means that not only the turbulent state of the atmosphere determines the turbulent diffusion in this case. but that also the properties of the stack gases and notably the dimensions of the plume are of importance.

Wind tunnel experiments show that behind a rectangular grid turbulence decays proportionally to the distance behind the grid. As an approximation we will assume here, that within certain limits of x the perturbation due to the stack gases can be accounted for by introducing a factor $\left(\frac{x}{x_2}\right)^{-\varepsilon}$ in I_3 and I_7, again neglecting the first terms of the righthand sides of the equations. Dimensional analysis shows that $u_m = u_{m1}\left(\frac{x}{x_1}\right)^{-\varepsilon}$ and $R = R_1\left(\frac{x}{x_1}\right)^{1-\varepsilon}$, where x_1 need not necessarily equal x_2.

Obviously ε can obtain values between 0 and $\frac{1}{2}$. The lower limit belongs to the case of strong natural turbulence (case *b*), the upper limit refers to a very stable atmosphere (case *a*).

As a matter of fact the assumption that I_3 and I_7 are proportional to a power of x can only be considered as approximately true. As an example an expression like $(I_{30}, I_{70})\ [1 + Ke^{-\varepsilon x}]\ R^2 u_m^2 \boldsymbol{U'}$ seems to be preferable. The power was introduced to make dimensional methods possible. At large distances from the stack solution b will give a better approximation. Introduction of the present solution into the equations shows that

$$u_m = -\frac{I_3}{I_1}\left(\frac{x_2}{x_1}\right)^{\varepsilon} \boldsymbol{U'} \text{ or } = 0 \text{ and } \frac{R_1}{x_1} = \frac{I_7}{\varepsilon I_4 - (1-\varepsilon) I_5}\left(\frac{x_2}{x_1}\right)^{\varepsilon} \frac{\boldsymbol{U'}}{\overline{U}}.$$

3.2.4 The present results will be applied to the problem of the diffusion of industrial pollution, originating from "cold" sources in the atmosphere.

The concentration of pollution χ as a function of $r = \sqrt{y^2 + z^2}$ can be approximated by an error function

$$\chi(x, r) = \chi(x, 0)\, e^{-r^2/R_*^2} \tag{3.2.9}$$

Although R_* need not equal R of the preceding sections, it is obvious that some relation exists between the two. No pollution being annihilated or created for $x > x_1$ and the total mean horizontal velocity being $\overline{U} + \overline{u} \sim \overline{U}$ we obtain:

$$\pi \int_0^\infty r\,\chi(x, r)\,\overline{U}\,dr = \pi R_*^2\,\chi(x, 0)\,\overline{U} = Q \tag{3.2.10}$$

where Q denotes the mass of pollution leaving the stack per unit time. Assuming R_* to be proportional to R, $R_* = kR$, we find for the general case (c):

$$\chi(x, r) = \frac{Q}{\pi k^2 \dfrac{R_1^2}{x_1^{2-2\varepsilon}}\,\overline{U}\,x^{2-2\varepsilon}}\, e^{-\dfrac{r^2}{k^2 \dfrac{R_1^2}{x_1^{2-2\varepsilon}} x^{2-2\varepsilon}}} \tag{3.2.11}$$

This is SUTTON's solution for the case of an axially symmetric plume.

SUTTON's generalized diffusion coefficient equals $k\,\dfrac{R_1}{x_1^{1-\varepsilon}}$ in our notation, whereas $n = 2\varepsilon$, or in his notation

$$\chi(x, r) = \frac{Q}{\pi C^2\,\overline{U}\,x^{2-n}}\, e^{-\dfrac{r^2}{C^2 x^{2-n}}} \tag{3.2.12}$$

so that $\varepsilon = {}^1/_8$ corresponds with the rate of diffusion ascribed to "normal conditions" by SUTTON. Assuming $k = 1$, e.g., $x_1 \sim 8R$, analogous to $z_1 \sim 4D$ in section 2.5.2, $R_1 = 1$ m, and $2\varepsilon = {}^1/_4$, we find $C \sim 0{,}16\ \mathrm{m}^{1/8}$, the same order of magnitude as given by SUTTON (1947). So it appears to be possible to understand at least qualitatively the behaviour of stack gases transported by a horizontal wind and diluted by turbulence without introducing SUTTON's hypothesis concerning natural turbulence in the atmosphere. As the power-law $x^{-\varepsilon}$ is only approximately valid for not too large values of x, a rather bad agreement with empirical results should be expected at large distances from the stack.

3.3 The influence of the vertical velocity of the jet.

Usually stack gases will escape from the stack with a vertical velocity decreasing with height so that the plume will bend gradually under the influence of the wind until it follows approximately a horizontal course.

This problem has not been solved in a general way up to now and it will probably take a long time to be approximated satisfactorily. SUTTON (1950) assumed that in the case of a heated jet rising due to buoyancy, the vertical component velocity w_m will depend on s, the path followed by the jet under the influence of $\overline{U}$ and $\overline{w}$ in the same way as w_m depends on z for $\overline{U} = 0$ i.e. $w_m \sim s^{-1/3}$. This solution includes that the natural turbulence of the wind is neglected. Moreover, SCHMIDT's $^1/_3$-law is based on the fact, that the buoyancy forces act along the axis of the jet which is not so in the case under consideration. CSANADY (1956) tried to apply SUTTON's theory to the measurements made by BOSANQUET c.s. (1950), but without very satistactory results. Recently PRIESTLEY (1956) tried to solve the problem starting from the theory laid down in the article by BALL and himself. His results will be referred to again later on.

We will now try to solve the problem at least approximately by again applying dimensional analysis. To begin with, it is assumed that the atmosphere is in indifferent equilibrium.

3.3.1 First consider the case of a jet emerging from an orifice due to its initial velocity i.e. without introducing buoyancy forces.

It is assumed that the total amount of vertical momentum is conserved and that the decrease of w_m due to the horizontal motion of the stack gases is larger than that due to the vertical motion by at least an order of magnitude, an assumption that will only hold approximately for not too short distances from the stack, and we write:

$$w_m \sim \tilde{w}_{m1}\left(\frac{s}{s_1}\right)^{-2+2\varepsilon} \sim \tilde{w}_{m1}\left(\frac{x}{x_1}\right)^{-2+2\varepsilon} \qquad (3.3.1)$$

in the notation of the preceding section where s denotes distance measured along the axis of the plume, and where $\tilde{w}_{m1}$ indicates that a small correction may have to be applied to the velocity w_{m1} occurring in reality as a consequence of the neglect of the decrease of w just above the orifice. In general $\tilde{w}_{m1} \sim w_{m1}$. x_1 can no longer be taken $= 8\ R_1$ now, but it will be proportional to z_1, the factor of proportionality depending on $\dfrac{\overline{U}}{w_{m1}}$. As an approximation we may assume $x_1 = \frac{\bar{U}}{\tilde{w}_{m1}} z_1 \approx 4 \frac{\bar{U}}{w_{m1}} D$, with D the diameter of the orifice. The height above the orifice obtained by the plume during the time interval $T - t_1$ amounts to

$$H - z_1 = \int_{t_1}^{T} \tilde{w}_{m1} \left(\frac{x}{x_1}\right)^{-2+2\varepsilon} dt = \int_{x_1}^{X} \frac{\tilde{w}_{m1}}{\bar{U}} \left(\frac{x}{x_1}\right)^{-2+2\varepsilon} dx \tag{3.3.2}$$

with $X - x_1 = \bar{U}(T - t_1)$.

In case $\varepsilon = {}^1/_2$, i.e. for lacking natural turbulence, the height increases logarithmically with growing distance from the stack. For smaller values of ε the plume only rises to a finite height:

$$H_\infty - z_1 = \frac{\tilde{w}_{m1}}{\bar{U}} \frac{x_1}{1-2\varepsilon} \approx \frac{4D}{1-2\varepsilon} \tag{3.3.3}$$

So we find, that $H_\infty - z_1$ decreases with growing turbulence i.e. decreasing ε and that it is proportional to the diameter of the orifice. For $\varepsilon = {}^1/_8$, and $R_1 = 1$ m, $H_\infty - z_1$ amounts to about 11 m, so that the gain in height over the orifice is generally small in the case of cold stack gases.

3.3.2 The case of a jet rising due to buoyancy in a horizontally moving atmosphere is more complicated, w_m depending on turbulence as well as on ϑ_m, the temperature excess of the jet over its environment.

The same reasoning as in 3.3.1 is applied to ϑ_m now:

$$\vartheta_m \sim \tilde{\vartheta}_{m1} \left(\frac{x}{x_1}\right)^{-2+2\varepsilon} \tag{3.3.4}$$

where ϑ_{m1} has a similar meaning to $\tilde{w}_{m1}$.

Since the stresses are proportional to $\overline{w^2}$, the vertical component of the equations of motion can be written formally as follows

$$I \frac{\partial}{\partial z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{w^2}\, dx\, dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{g}{\Theta_\infty} \bar{\vartheta}\, dx\, dy, \tag{3.3.5}$$

I being a constant depending on the proportion between inertia forces and frictional forces. The double integrals can be approximated by (see fig. 7).

$$A \frac{\sqrt{\bar{U}^2 + w_m^2}}{w_m} w_m^2 R_s^2 \text{ and } B \frac{g \vartheta_m}{\Theta_\infty} \frac{\sqrt{\bar{U}^2 + w_m^2}}{w_m} R_s^2,$$

A and B being inknown integration constants.

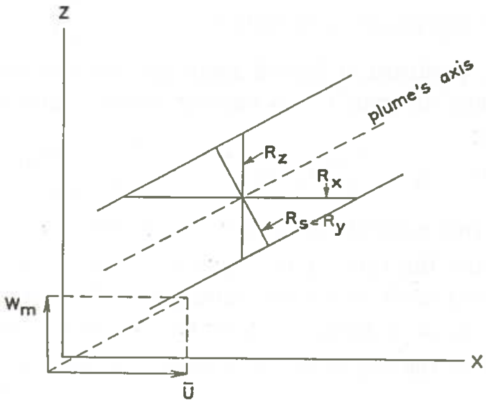

FIG. 7. Showing the relation between the various cross sections of a plume and the vertical and horizontal component velocities.

R_s denotes the characteristic distance from the axis of the axially symmetric plume in a surface perpendicular to s. Or, by neglecting w_m^2 with respect to $\bar{U}^2$ (again only permissible for not too small values of x):

$$I A \frac{\bar{U}}{w_m} \frac{\partial}{\partial x} \left\{ \frac{\bar{U}}{w_m} w_m^2 R_s^2 \right\} = B \frac{g}{\Theta_\infty} \vartheta_m \frac{\bar{U}}{w_m} R_s^2 \qquad (3.3.6)$$

With $\vartheta_m = \tilde{\vartheta}_{m1} \left(\frac{x}{x_1}\right)^{-2+2\varepsilon}$ and $R_s = R_{s1} \left(\frac{x}{x_1}\right)^{1-\varepsilon}$ one obtains:

$$w_m \sim \tilde{w}_{m1} \left(\frac{x}{x_1}\right)^{-1+2\varepsilon} \qquad (3.3.7)$$

where $\tilde{w}_{m1} = \frac{B}{IA} \frac{g\tilde{\vartheta}_{m1}}{\bar{U}} x_1$, again a hypothetical value, but probably not differing too much from the real value w_{m1}. Again putting $x_1 \sim \frac{\bar{U}}{\tilde{w}_{m1}} z_1$, we find $\tilde{w}_{m1} = \frac{B}{IA} \frac{g\,\tilde{\vartheta}_{m1}}{\Theta_\infty \tilde{w}_{m1}} z_1$ or $\tilde{w}_{m1}$ (:)$\sqrt{\tilde{\vartheta}_{m1}}$, as in the case of $\bar{U} = 0$. (See also section 2.3.4).

For the height reached by the plume during the time interval $T - t_1$

$$H - z_1 = \int_{t_1}^{T} \tilde{w}_{m1} \left(\frac{x}{x_1}\right)^{-1+2\varepsilon} dt \text{ or } H - z_1 = \frac{B}{IA} \frac{g\tilde{\vartheta}_{m1}}{\Theta_\infty \bar{U}^2} x_1^{2-2\varepsilon} \int_{x_1}^{X} x^{-1+2\varepsilon} dx =$$

$$= \frac{B}{IA} \frac{g\,\tilde{\vartheta}_{m1}}{2\varepsilon \bar{U}^2 \Theta_\infty} x_1^2 \left[\left(\frac{X}{x_1}\right)^{2\varepsilon} - 1\right] = \frac{1}{2\varepsilon} z_1 \left\{\left(\frac{X}{z_1}\right)^{2\varepsilon} \left(\frac{\tilde{w}_{m1}}{\bar{U}}\right)^{2\varepsilon} - 1\right\} \qquad (3.3.8)$$

is found.

Of course, this expression only makes sense for $\frac{X}{x_1} > 1$. As ε has values between 0 and $^1/_2$, inclusive, a heated stack gas will rise indefinitely in an indifferent atmosphere. In case $\varepsilon = 0$ (strong natural turbulence) the integral of (3.3.8) becomes:

$$H - z_1 = \frac{\tilde{w}_{m1}}{\overline{U}} x_1 \log \frac{X}{x_1} = z_1 \log \left(\frac{X}{z_1} \frac{\tilde{w}_{m1}}{\overline{U}} \right) \tag{3.3.9}$$

In case $\varepsilon = {}^1/_2$ (no natural turbulence) w_m becomes constant. As this case generally only occurs for very small values of $\overline{U}$ the approximations do not hold here. Solution (3.3.8) shows the same general characteristics as (3.3.2). A large value of z_1, i.e. a large diameter of the orifice increases the height otherwise reached by the plume in the atmosphere. Increasing $\overline{U}$ leads to decreasing $H - z_1$ for constant X. The effect of ε depends on $F = \frac{1}{2\varepsilon} [y^{2\varepsilon} - 1]$ with $y > 1$ and 2ε between 0 and 1. It can be shown that F increases with increasing ε, also in accordance with the results of (3.3.2).

On the other hand (3.3.8) shows that the gain of height of the plume increases with the increasing value of $\tilde{\vartheta}_{m1}$. Fig. 8 shows the trajectories of plumes for various values of ε.

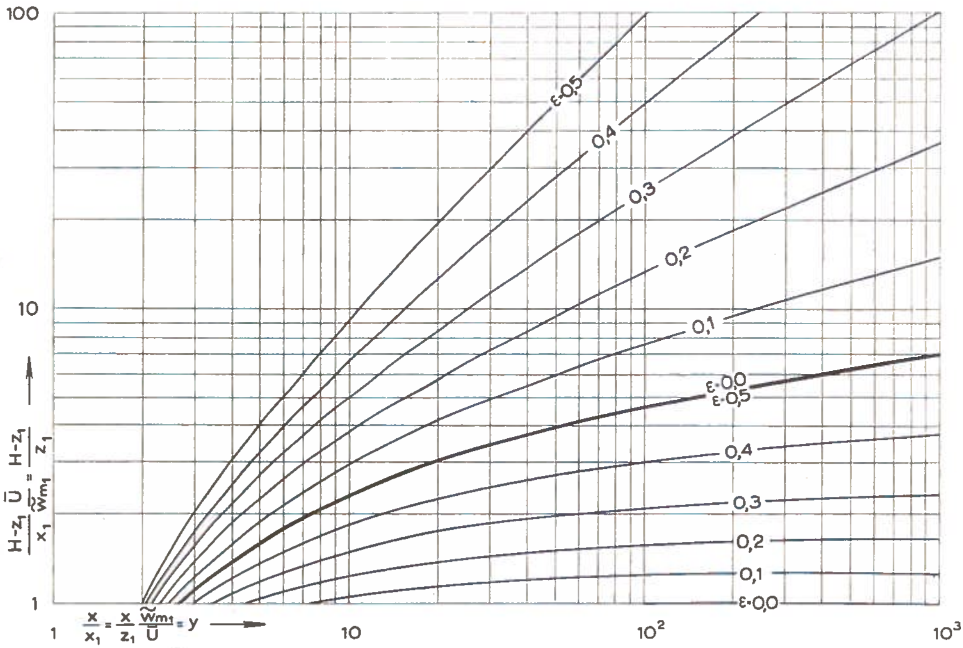

FIG. 8. Trajectories of plumes in an indifferent atmosphere as a function of ε. The upper curves pertain to the case of hot plumes, the lower curves to the case of cold plumes.

It appears that in the case of natural wind heated gases escaping from a stack achieve greater heights in an indifferent atmosphere than gases without any temperature excess. The height achieved by heated jets having an extra vertical velocity (see 2.5.3) will lie between both systems of curves. A return to this general problem will be made in chapter IV.

3.4 The influence of stability

The case of the propagation of a jet in a stably stratified atmosphere under the influence of natural wind is difficult to handle.

As far as the jet does not show any excess temperature towards its environment the best approximation may be to assume that the jet axis is horizontal and lies in the level of the orifice.

In case of a heated jet a reasoning analogous to that of chapter II may be applied. Strictly speaking it implies the assumption of a disturbed atmospheric layer in indifferent equilibrium and with potential temperature equal to the average in the originally undisturbed layer, but it may also be considered as a first approximation as before. Accordingly the jet will rise to a level H where $\tilde{\vartheta}_{m1}\left(\frac{x}{x_1}\right)^{-2+2\varepsilon}$ equals $\Theta_\infty(H)$, the temperature of the environmental air. It is then assumed that the axis of the jet will remain horizontal in the equilibrium level and that the turbulent diffusion will take place according to (3.2.11) with some value of ε, a function in general of stability. Or, in analogy with 2.5.2:

$$\frac{1}{2}\Gamma\frac{B}{IA}\frac{g}{2\varepsilon\,\bar{U}^2\Theta_\infty}x_1^2\left[\left(\frac{X}{x_1}\right)^{2\varepsilon}-1\right]= \\ =\left\{1-\frac{1}{2}\Gamma\frac{B}{IA}\frac{g}{2\varepsilon\,\bar{U}^2\Theta_\infty}x_1^2\left[\left(\frac{X}{x_1}\right)^{2\varepsilon}-1\right]\right\}\left(\frac{X}{x_1}\right)^{-2+2\varepsilon} \tag{3.4.1}$$

Substituting p for $\frac{1}{2}\frac{B}{IA}\frac{g}{\bar{U}^2\Theta_\infty}x_1^2=\frac{1}{2}\frac{B}{IA}\frac{g}{\tilde{w}_{m1}^2\Theta_\infty}z_1^2=\frac{1}{2}\frac{z_1}{\tilde{\vartheta}_{m1}}$ and y for $\frac{X}{x_1}$

$$\frac{1}{2\varepsilon}\Gamma p\,(y^{2\varepsilon}-1)=\left\{1-\Gamma\frac{p}{2\varepsilon}(y^{2\varepsilon}-1)\right\}y^{-2+2\varepsilon}$$

or $$y^2=\frac{\left\{1-\Gamma\frac{p}{2\varepsilon}(y^{2\varepsilon}-1)\right\}y^{2\varepsilon}}{\Gamma\frac{p}{2\varepsilon}(y^{2\varepsilon}-1)}$$ is obtained.

For all values $y \gg 1$, we find that $\Gamma\frac{p}{2\varepsilon}(y^{2\varepsilon}-1)$ must be smaller than 1 by at least one order of magnitude, so that the value of y follows as a good approximation from:

$$\frac{1}{2\varepsilon}\, y^2(1-y^{-2\varepsilon}) = \frac{1}{\Gamma p} \tag{3.4.2}$$

For very strong turbulence ($\varepsilon \to 0$) $H - z_1$ is given by (3.3.9)

or $$y^2\, \Gamma p \log y = [1 - \Gamma p \log y]$$

Again it can be shown that $\Gamma p \log y \ll 1$ or y can be computed from

$$y^2 \log y = 1/\Gamma p \tag{3.4.3}$$

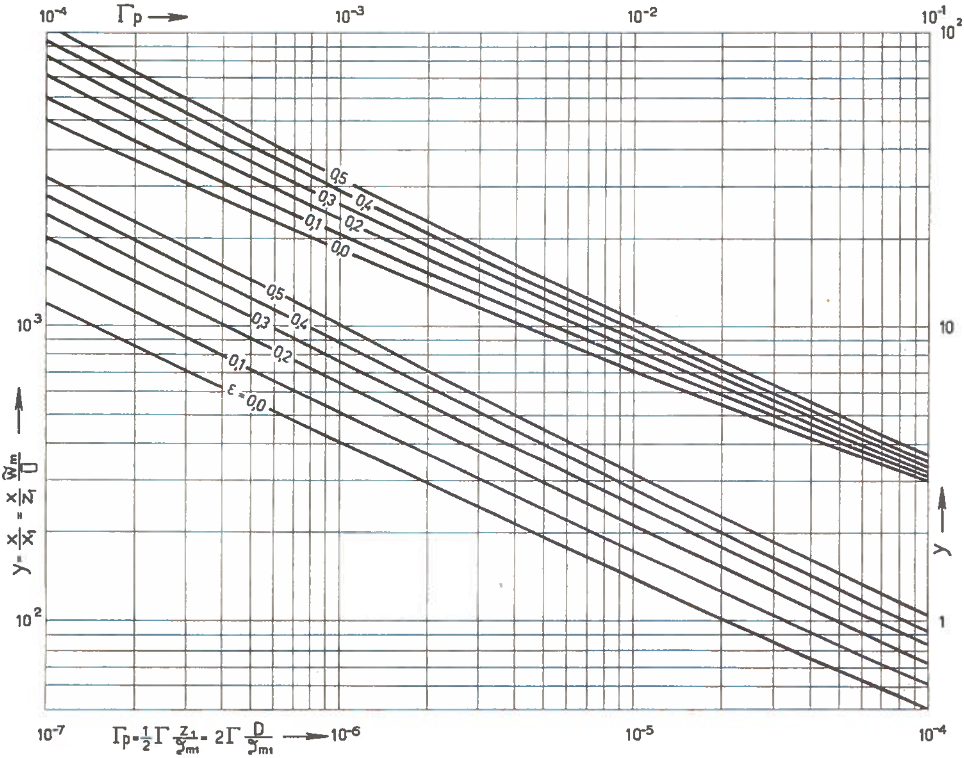

FIG. 9. Distance from stack at which a plume achieves its maximum height above the orifice. Scales at the left and the bottom of the figure pertain to the lowest group of curves.

Fig. 9 shows y as a function of Γp and ε. Large values of ε (little natural turbulence) correspond with large values of y, at least theoretically. As was remarked already the approximations do not hold for cases without significant natural turbulence. From the values of y the ultimate height reached by the plume is found by (3.3.8):

$$H - z_1 = \frac{\tilde{\vartheta}_{m1}}{\varepsilon} p\,(y^{2\varepsilon} - 1) = \frac{1}{2\varepsilon} z_1\,(y^{2\varepsilon} - 1) \tag{3.4.4}$$

or by (3.3.9):

$$H - z_1 = 2\tilde{\vartheta}_{m1}\, p \log y = z_1 \log y \tag{3.4.5}$$

$H - z_1$ depends on $\tilde{\vartheta}_{m1}$, g/Θ_∞, $\bar{U}$, z_1, ε and Γ. The role played by the various quantities is difficult to appreciate. It is clear that $H - z_1$ increases with $\tilde{\vartheta}_{m1}$ as y increases with increasing $\tilde{\vartheta}_{m1}$. The influence of g/Θ_∞ is small, g being a constant and Θ_∞ only varying over a limited range.

z_1 occurs in (3.4.2) as well as in (3.4.4). Approximately we can state that y is proportional to $p^{-1/2}$ (from 3.4.2) and $H - z_1 \sim p^{1-\varepsilon}$, so that $H - z_1$ increases with $z_1{}^{1-\varepsilon}$. So we find again that a large value of z_1 i.e. a large diameter of the orifice increases the maximum height otherwise reached by the plume in a stable atmosphere. On the other hand $\bar{U}$ seems to be without significance for the ultimate height, the plume will achieve, apart from its influence on ε and from the fact that with growing $\bar{U}$ the maximum height is reached at greater distance from the stack.

The same reasoning shows that $H - z_1 \sim \Gamma^{-\varepsilon}$ so that the height of the plume is reduced by growing stability as was to be expected. The influence of ε has been discussed already: the larger ε the larger y, the greater the height the plume reaches.

z_1 and ε being known, $H - z_1$ can be obtained from fig. 8 by taking the horizontal distance equal to the value of y found from fig. 9. The procedure is rather complicated as many parameters ought to be known. It might be possible, however, to use the relations developed here qualitatively when planning industrial stacks.

CHAPTER IV

METEOROLOGICAL APPLICATIONS

4.1 The influence of the earth's surface

The foregoing results may be used in studies concerning the inconvenience caused by the diffusion of industrial and domestic gases in the atmosphere.

Starting from SUTTON's theoretical results, obtained in the present study by making use of dimensional analysis, it can first be stated, that the surface of the earth disturbs the distribution of χ as given by (3.2.12).

There are two extreme possibilities:

a) All pollution is reflected by the earth's surface, causing an increase of χ in every level $z > 0$.

b) All pollution is adsorbed and annihilated by the surface, resulting in a decrease of χ for $z > 0$.

SUTTON approximated the first case by introducing a mirror-orifice below the earth's surface.

Writing $y^2 + z^2$ for r^2 but maintaining the axial symmetry of the plume for reasons of simplicity the final solution becomes:

$$\chi(x, y, z) = \frac{Q\, e^{-y^2/C^2 x^{2-2\varepsilon}}}{\pi C^2 \overline{U} x^{2-2\varepsilon}} \left\{ e^{-\frac{(z+h)^2}{C^2 x^{2-2\varepsilon}}} + e^{-\frac{(z-h)^2}{C^2 x^{2-2\varepsilon}}} \right\} \tag{4.1.1}$$

with h the height of the stack orifice above the earth's surface and $C = k \dfrac{R_1}{x_1{}^{1-\varepsilon}}$ according to the developments in 3.3.4. Accordingly we may write for the case of an absolutely adsorbing surface

$$\chi(x, y, z) = \frac{Q\, e^{-y^2/C^2 x^{2-2\varepsilon}}}{\pi C^2 \overline{U} x^{2-2\varepsilon}} \left\{ -e^{-\frac{(z+h)^2}{C^2 x^{2-2\varepsilon}}} + e^{-\frac{(z-h)^2}{C^2 x^{2-2\varepsilon}}} \right\} \tag{4.1.2}$$

Both solutions must be considered as qualitatively correct only, since the perturbation of the motion of the air due to the surface has not been taken into account. As to the concentration χ near the earth's surface both solutions result in a variation of Q.

4.2 The influence of thermal stratification on the concentration of pollution near the earth's surface

In practice the concentration of pollution at ground level depends on various parameters.

If equation (3.2.11) is considered first, i.e. the influence of buoyancy or any initial vertical motion of the stack gases neglected, it will be seen that k, R_1, x_1, ε and $\bar{U}$ are the quantities which determine $\chi(x, r)$.

The maximum concentration that can occur at ground level is generally the most important quantity to be determined. For fixed values of $\bar{U}$, R_1, x_1, ε and k it occurs at a distance

$$x_{max}(y = z = 0) = \left[\frac{h^2}{k^2\, R_1^2/x_1^{2-2\varepsilon}}\right]^{1/2-2\varepsilon} = x_1\left(\frac{h}{k R_1}\right)^{1/1-\varepsilon} \tag{4.2.1}$$

from the stack.

It amounts to

$$\chi_{max}(x_{max}, y = z = 0) = \frac{Q}{e\pi\,\bar{U}\,h^2} \tag{4.2.2}$$

The results are surprising as they show that the maximum concentration χ_{max} appears to be independent of the intensity of atmospheric turbulence! Only the distance from the stack where the maximum occurs depends on ε. The stronger turbulence, the smaller ε, and the shorter the distance from the stack of the maximum concentration. In the extreme case of very strong turbulence, or $\varepsilon = 0$, x_{max} equals $\frac{hx_1}{kR_1}$ or with $\frac{x_1}{R_1} = 8$: $x_{max} \sim \frac{8}{k}h$; in the case of absence of natural turbulence ($\varepsilon = {}^1/_2$) $x_{max} = x_1\left(\frac{h}{kR_1}\right)^2 \sim \frac{8h^2}{k^2R_1}$.

However, as was stated in 3.2 the assumption that the plume is axially symmetric is a simplification, not permissible in reality in many cases.

Sutton, therefore, introduced two diffusion coefficients C_y and C_z or according to our notation $k_y\frac{R_{y1}}{x_{y1}^{2-2\varepsilon_y}}$ and $k_z\frac{R_{z1}}{x_{z1}^{2-z\varepsilon_z}}$. As he further assumed $\varepsilon_y = \varepsilon_z = \varepsilon$ different intensities of diffusion in the y and z-directions are ascribed to differences in the values of k, R_1 and x_1 exclusively. However it seems to be a better approximation to reality to consider the plume to be almost axially symmetric at the orifice i.e. to put $k_y = k_z = k$, $R_{y1} = R_{z1} = R_1$ and $x_{y1} = x_{z1} = x_1$ and consequently to connect the anisotropy of turbulence with $\varepsilon_y \neq \varepsilon_z$,

or

$$\chi(x, y, z) = \frac{Q}{\pi\, C_*^2\, \bar{U}\left(\frac{x}{x_1}\right)^{2-\varepsilon_y-\varepsilon_z}} \exp\left\{-\frac{y^2}{C_*^2\left(\frac{x}{x_1}\right)^{2-2\varepsilon_y}} - \frac{(z-h)^2}{C_*^2\left(\frac{x}{x_1}\right)^{2-2\varepsilon_z}}\right\} \tag{4.2.3}$$

with $C_* = kR_1$.

From (4.2.3) the following expressions for x_{max} and χ_{max} can be derived:

$$x_{max} = x_1 \left\{ \frac{h}{kR_1} \sqrt{\frac{2-2\varepsilon_z}{2-\varepsilon_y-\varepsilon_z}} \right\}^{1/1-\varepsilon_z} \tag{4.2.4}$$

and

$$\chi_{max} = \frac{Q \exp \left\{ -\frac{1-{}^1/_2(\varepsilon_y+\varepsilon_z)}{1-\varepsilon_z} \right\}}{\pi \bar{U} h^{\frac{2-\varepsilon_y-\varepsilon_z}{1-\varepsilon_z}} (kR_1)^{\frac{\varepsilon_y-\varepsilon_z}{1-\varepsilon_z}} \left(\frac{2-2\varepsilon_z}{2-\varepsilon_y-\varepsilon_z} \right)^{\frac{1-{}^1/_2(\varepsilon_y+\varepsilon_z)}{1-\varepsilon_z}}} \tag{4.2.5}$$

so that both, x_{max} and χ_{max} are functions of ε_y and ε_z. For $\varepsilon_y = \varepsilon_z$ (4.2.4) and (4.2.3) become identical with (4.2.1) and (4.2.2) respectively.

As the extreme case of a very stable atmosphere we may assume $\varepsilon_y = 0$, $\varepsilon_z = {}^1/_2$ i.e. exclusively horizontal turbulent motions or

$$x_{max} = x_1 \left\{ \frac{h}{kR_1} \sqrt{\frac{2}{3}} \right\}^2 = \frac{2}{3} x_1 \left(\frac{h}{kR_1} \right)^2 \tag{4.2.5}$$

and

$$\chi_{max} = \frac{Q\, e^{-{}^3/_2}}{\pi \bar{U} h^3 (kR_1)^{-1} \left(\frac{2}{3}\right)^{{}^3/_2}} = \frac{Q}{e \pi \bar{U} h^2} \frac{kR_1}{0{,}9\, h} \tag{4.2.6}$$

Comparing (4.2.6) with SUTTON's result:

$$\chi_{max} = \frac{Q}{e \pi \bar{U} h^2} \left(\frac{C_z}{C_y} \right) \tag{4.2.7}$$

(apart from the factor 2, to be introduced into both expressions if the earth is considered to act as a mirror for pollution) we see a remarkable difference between the two expressions for $\chi_{|max}$.

According to SUTTON, χ_{max} at ground level is always reduced by stable atmospheric conditions ($C_z < C_y$). According to our assumptions the variation with stability of the maximum concentration of pollution at surface level depends on the ratio between R_1 and h. For normal stacks where, as $k \sim 1$ always $kR_1 < 0.9\ h$ stability reduces χ_{max} (x_{max}, 0,0), but when pollution originates from very low orifices, low at least in comparison with kR_1, stability of the atmosphere increases the value of χ_{max} at ground level.

But this is exactly the situation in large cities or industrial areas under stable conditions, where h is small compared with the radius of a stack orifice equivalent to the total area of all stack orifices in the region, so that our approach seems to be in better accordance with reality.

4.3 Gas attacks

"Gas attacks", high concentrations of pollution near the earth's surface occuring during relatively short periods, also called **fumigation** (WEXLER, 1955), may result from single or only a few stacks. From (4.2.6) as well as from (4.2.7) it follows, that temporarily high concentrations of pollution near the surface of the earth will generally not be connected with very stable situations in the lowest levels. On the contrary, stability tends to reduce χ near the ground when $\frac{kR_1}{0{,}9\,h} < 1$ (apart from the influence of eventually descending air currents due to the obstacles formed by buildings or of rain).

The phenomenon may be explained by an extension of SUTTON's mirror-method. If we assume that at a height h^* above the orifice of the stack a strong inversion acting almost as an absolute barrier for the polluted air is present the concentration between the surface of the earth and that inversion is given by

$$\chi(x, y, z) = \frac{Q \exp\left\{-\frac{y^2}{C^2 x^{2-2\varepsilon}}\right\}}{\pi \bar{U}^2 C^2 x^{2-2\varepsilon}} \cdot \tag{4.3.1}$$

$$\cdot \left\{ \sum_{\mu=0}^{\infty} \left[\exp\left\{-\frac{[(z-h)-2\mu(h+h^*)]^2}{C^2 x^{2-2\varepsilon}}\right\} + \exp\left\{-\frac{[(z+h)+2\mu(h+h^*)]^2}{C^2 x^{2-2\varepsilon}}\right\}\right] + \right.$$

$$\left. + \sum_{\nu=1}^{\infty} \left[\exp\left\{-\frac{[(z+h)-2\nu(h+h^*)]^2}{C^2 x^{2-2\varepsilon}}\right\} + \exp\left\{-\frac{[(z-h)+2\nu(h+h^*)]^2}{C^2 x^{2-2\varepsilon}}\right\}\right]\right\}$$

where ε has been taken independent of y and z.

For $z = y = 0$, the line in which the maximum concentration at the earth's surface is found,

$$\chi(x, 0{,}0) = \frac{2Q}{\pi \bar{U} C^2 x^{2-2\varepsilon}} \left\{ \exp\left\{-\frac{h^2}{C^2 x^{2-2\varepsilon}}\right\} + \right.$$

$$\left. + \sum_{\mu=1}^{\infty} \exp\left\{-\frac{[h+2\mu(h+h^*)]^2}{C^2 x^{2-2\varepsilon}}\right\} + \sum_{\nu=1}^{\infty} \exp\left\{-\frac{[h-2\nu(h+h^*)]^2}{C^2 x^{2-2\varepsilon}}\right\}\right\} \tag{4.3.2}$$

is obtained.

It is clear, that both series show a maximum for $h^* = 0$ i.e. when the inversion lies immediately above the stack orifice. (4.3.2) then reduces to:

$$\chi(x, 0{,}0) = \frac{4Q}{\pi \bar{U} C^2 x^{2-2\varepsilon}} \sum_{\mu=0}^{\infty} \exp\left\{-\frac{(2\mu+1)^2 h^2}{C^2 x^{2-2\varepsilon}}\right\} \tag{4.3.3}$$

Or, the concentration at the ground rises to at least twice its normal value. At great distance from the stack, the extra terms increase relatively in importance.

Fig. 10 shows as an example the concentration χ $(y = z = 0)$ as a function of distance from the stack, both with and without an inversion at $z = h = 100$ m. χ has been expressed in $0.04 \dfrac{Q}{e \pi \bar{U} h^2}$ as unit, ε has been taken $= 0$ for reasons of simplicity or $C \sim 10^{-1}$.

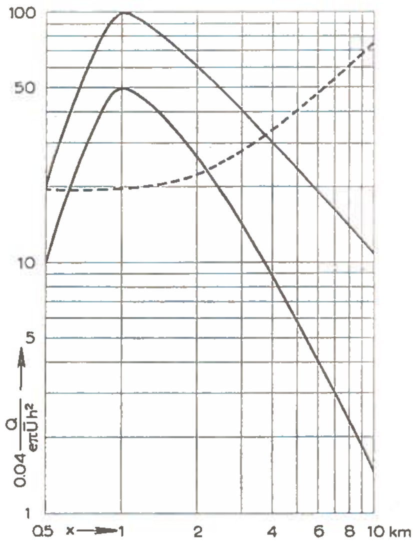

FIG. 10. χ $(y = z = 0)$ with and without an inversion at the stack height h = 100 m. The dotted curve represents the proportion between both values of χ times 10.

The case corresponds with the occurrence of a turbulence inversion. It appears that under those circumstances the concentration decreases much more slowly with distance than it does when no inversion is present at the stack orifice.

The result is that, apart from descending air motions and the influence of precipitation, $\chi_{\max}$ may reach the value $\dfrac{4Q}{e \pi \bar{U} h^2}$ at a distance of about 10 times the stack height h in this unfavourable case. This distance 10 h is also mentioned in literature as a result of observation. According to HOLLAND (1953) ground concentrations may rise to even higher values. This will occur when $\varepsilon_y > \varepsilon_z$.

So that, from a general meteorological point of view, it can be stated that strongly developed ground inversions are dangerous in great cities, whereas in the case of one or a few isolated stacks turbulence inversions just above the orifices create the most dangerous situations. Low wind speed increases the hazards in both cases.

4.4 Examples

4.4.1 During the first decade of December 1952 the concentration of pollution increased to an exceptionally high one in the London area, resulting in about 4000 casualties.

The synoptic situation was such that the centre of an anticyclone was lying just over SE England. Winds were very slight and a strong inversion developed over the region.

Table 4.4.1 gives the inversions and isothermal layers in Hemsby during the first ten days of the month as well as the mean wind velocity in knots at London Airport.

Table 4.4.1

Dec.	0300z		1500z		mean wind velocity in knts.
	layer in m.	temp in °C.	layer in m.	temp in °C.	
1	0–160	+3/+4	—	—	10
2	—	—	—	—	4
3	0–110	+6/+7	—	—	8
4	0–240	+5/+6	—	—	4
5	0–140	+2/+3	—	—	3
	140–290	+3			
6	0–130	—1/+3	0–170	+4	0
	130–290	+3			
7	0–130	0/+4	0–290	+4	0
8	0–200	—1/+4	0–180	+3	0
9	0–700	0/+5	200–430	+1/+6	1
10	0–260	+3/+7	—	—	9

It is clear that December 6, 7 and 8 have been the crucial days with inversions or isothermal layers from the surface up to more than 100 m and with absolute calm.

Figure 11 shows the mean daily vertical temperature lapse for the period December 1—11 inclusive. The curves are constructed from the 0300z and

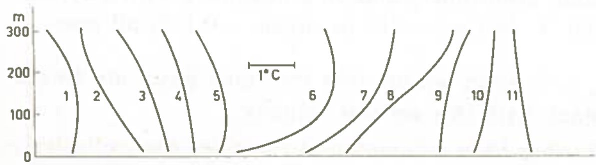

FIG. 11. Temperature lapse in the lowest 300 m at Hemsby during the period December 1-11, 1952.

1500z ascents. Again December 6, 7 and 8 show the most unfavourable situation.

Similar situations may be considered responsible for the high concentrations of pollution in other cities.

It might be worth while to give special warnings when strong inversions such as those encountered during the great smog of 1952 are likely to occur.

4.4.2 The exact description of the meteorological situation during so-called "gas-attacks" is rather difficult to give.

In the first place gas attacks are, in general, of a relatively short duration. Secondly, much of the harm done to vegetation e.g. may be due to the washout of pollution by rain so that it is often difficult to indicate the exact cause of the damage done. Thirdly, with modern radiosounds it is almost impossible to determine the exact structure of the lowest 100 m of the atmosphere.

A wellknown example of the hazards connected with an excessive concentration of atmospheric pollution due to a low lying inversion is the catastrophy in the Meuse valley in December 1930. Pressure was high over central Europe and in the region a weak to moderate wind was blowing from easterly directions. According to the very few but relatively accurate airplane and cable balloon observations an inversion of about 4 °C was present over most of central and western Europe at about 250—300 m above sea level. This value corresponds roughly with the height of the hills bordering the valleys in the polluted region. See also FIRKET (1936).

4.5 The influence of buoyancy

As has been shown and has been recognized in practice (see e.g. SCORER, 1955) positive buoyancy of the gases escaping from a stack is an important aid in diminishing the nuisance of pollution at ground level.

We will demonstrate the effect by computting the groundconcentration of pollution escaping from a single stack according to the developments in chapter III and under the following circumstances:

$w_{m1} = \overline{U} = 5$ m sec^{-1} $z_1 = 10$ m (or $D \sim 2{,}5$ m)

$\widetilde{\vartheta}_{m1} = 0$, and 200 °C respectively and $\Gamma = 0$ and $\Gamma = 4.10^{-3}$ °C m^{-1}, corresponding to an indifferent atmosphere and an atmosphere with a vertical temperature lapse rate of 0,6 °C/100 m. ε will be taken $= 0{,}1$ in all cases.

In the case $\widetilde{\vartheta}_{m1} = 0$, $\widetilde{w}_{m1} = 5$ m/sec means that the stack gases are forced in some way to leave the stack with this vertical velocity.

In the case $\widetilde{\vartheta}_{m1} = 200$ °C it has been assumed that the gases rise exclusively due to buoyancy and that the approximate relation of 3. 4. 2 may be applied:

$$\tilde{w}_{m1} = \sqrt{\frac{B}{I\,A}\,\frac{g}{\Theta_\infty}\,\tilde{\vartheta}_{m1}\,z_1} \sim 8\sqrt{\frac{B}{I\,A}}\ \text{m sec}^{-1}$$

There remains of course an uncertainty as $I\,A$ and B are not known. However, one of the stacks of the steel plants in Velzen (Netherlands) has $w_{m1} = 8$ m sec^{-1} with $z_1 = 10$ m and $\vartheta_{m1} = 300°$, so that with $\tilde{\vartheta}_{m1} = 200$ °C the value of 5 m sec^{-1}, for $\tilde{w}_{m1}$, may be of the right order of magnitude.

We now calculate the concentration along the line $y = 0$, $z = 0$, x being the wind direction. In order to do so, we apply equation (3.2.11) assuming C to be $= 0{,}1$ and taking for r the height of the plume which is a function of x. As the axis of the plume is not horizontal in general, a small error is involved in this procedure. The picture as a whole will, however, give a satisfactory impression of the influence of buoyancy in the case of an indifferent and normally stratified atmosphere.

From figure 8 the height of the axis of the plume as a function of $\frac{x}{z_1}$ is read immediately for $\Gamma = 0$ and $\varepsilon = 0{,}1$. The lower curve relates to $\tilde{\vartheta}_{m1} = 0$, the upper one to $\tilde{\vartheta}_{m1} = 200$.

In the case of stable stratification the cold gases ($\tilde{\vartheta}_{m1} = 0$) will remain in the level of the stack orifice, whereas the heated gases will reach an equilibrium height to be determined from figure 9. In our case $^1/_2\,\Gamma\,\frac{z_1}{\tilde{\vartheta}_{m1}}$ amounts to 10^{-4} so that for $\varepsilon = 0{,}1$ the equilibrium height is reached for $y = 60$, or the plume rises to 64 m above the orifice.

Table 4.5.1 gives the height $H - z_1$ of the plume above the stack orifice as function of $y^* = x/z_1$.

Table 4.5.1

y^*	$\Gamma=0$, $\tilde{\vartheta}_{m1}=0$	$\Gamma=0$, $\tilde{\vartheta}_{m1}=200$	$\Gamma=4.10^{-3}$, $\tilde{\vartheta}_{m1}=0$	$\Gamma=4.10^{-3}$, $\tilde{\vartheta}_{m1}=200$
10	10,5 m	29 m	0 m	29
20	11,3	41	0	41
50	12,0	60	0	60
60	12,1	64	0	64
100	12,3	75	0	64
200	12,5	94	0	64
500	12,5	123	0	64
1000	12,5	147	0	64

It is clear from the table that buoyancy leads to greater heights of the plume, i.e. to a decrease of air pollution near the ground.

Generally speaking, the influence of $\widetilde{\vartheta}_{m1}$ will be more important, the lower the stack (z_1 assumed to be constant and independent of the height of the stack for the moment). But even with a stack of 100 m, a moderately high one, the concentration of pollution at ground level is diminished appreciably by buoyancy.

Figure 12 shows the results, obtained by adding the stack height h (50 m and 100 m) to the values of the table and then applying (3.2.11), after a factor 2 has been added in the numerator to take account of the influence of the earth's surface. The values of χ $(x, 0,0)$ are given in percentages of the maximum value occuring in the case of the stable atmosphere with $\widetilde{\vartheta}_{m1} = 0$ and $h = 50$ m.

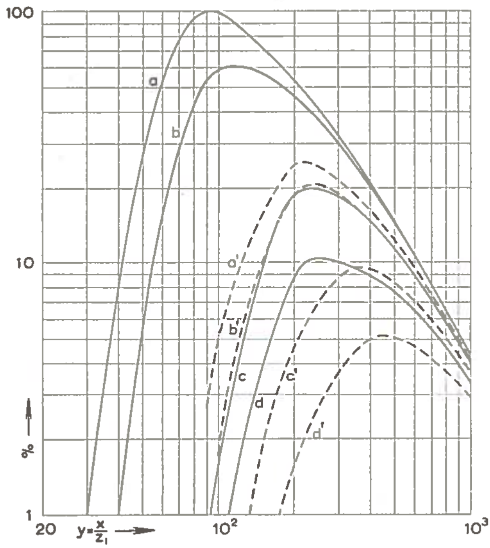

FIG. 12. Distribution of χ $(x, 0,0)$ under various circumstances.
a and a': $\Gamma = 4.10^{-3}$ °C m^{-1}; $\widetilde{\vartheta}_{m1} = 0$ °C
b and b': $\Gamma = 0$ °C m^{-1}; $\widetilde{\vartheta}_{m1} = 0$ °C
c and c': $\Gamma = 4.10^{-3}$ °C m^{-1}; $\widetilde{\vartheta}_{m1} = 200$ °C
d and d': $\Gamma = 0$ °C m^{-1}; $\widetilde{\vartheta}_{m1} = 200$ °C

The full curves a - d pertain to a stack height of 50 m, the dotted curves a' - d' to a stack height of 100 m. All curves give values relative to case a.

The favourable influence of buoyancy is evident. With both stack heights the maximum concentration decreases due to the high temperature of the stack gases, and at the same time the place where the maximum occurs is found at greater distance from the stack.

Comparing the curves c and c', we see that in the case of a normally stratified atmosphere the influence of $\widetilde{\vartheta}_{m1}$ decreases with growing stack height.

Washing (and consequent cooling) of the stack gases in the case of a stack height of 50 m would be sensible if at least 80 % of the pollution (SO_2 e.g.) could be removed. SCORER (1955) pointed out, however, that the escaping gases would contain a great amount of rather large droplets then. Evaporation of these droplets would tend to cool the gases below the temparature of the atmosphere, leading to great concentration of pollution at short distance from the stack.

4.6 Comparison with empirical data

In trying to compare the theory on the effective stack height developed here with empirical results, we are confronted with a number of difficulties.

Firstly, reliable measurements of the heights of plumes in dependence on distance from the stack and the parameters occurring in the various equations are completely lacking. The only measurements available are those made by BOSANQUET, CAREY and HALTON (1950). They do not give any information as to the diameter of the stack orifice. Probably their observations relate to an almost indifferent atmosphere. For more detailed information they refer to appendices which are, however, not at our disposal.

Secondly, it is clear from their results that, apart from two cases, the rising of the stack gases is due to initial forced velocity as well as to buoyancy, so that the obtained heights cannot be derived in the most simple way from figure 8.

Finally, the distances from the stack for which measurements have been made are rather short, 244 m (800 ft) being the maximum, i.e. far nearer to the stack than the distance where the maximum concentration of pollution is found in general.

To find solutions in the case of "mixed" exit-velocities of the stack gases we start from (2.5.7) and write for the total velocity in the orifice:

$$W_{m1} = [w_{f1}{}^3 + w_{c1}{}^3]^{1/3} \qquad (4.6.1)$$

or with

$$w_f = w_{f1}\left(\frac{x}{x_1}\right)^{-2+2\varepsilon} \quad \text{and} \quad w_c = w_{c1}\left(\frac{x}{x_1}\right)^{-1+2\varepsilon}:$$

$$W_m = \left\{ w_{f1}{}^3\left(\frac{x}{x_1}\right)^{-6+6\varepsilon} + w_{c1}{}^3\left(\frac{x}{x_1}\right)^{-3+6\varepsilon}\right\}^{1/3} \qquad (4.6.2)$$

Putting $w_{c1} = \alpha W_{m1}$, with $0 \leqslant \alpha \leqslant 1$ and $\dfrac{x}{x_1} = y$, the height above the orifice reached by the plume can be written as:

$$H(Y) - z_1 = \frac{W_{m1}}{U}\, x_1 \int_1^Y y^{-2+2\varepsilon}\{1 + \alpha^3(y^3 - 1)\}^{1/3}\, dy \qquad (4.6.3)$$

Table 4.6.1

Plant No.	w_{m1} (ft/sec)	ϑ_{m1} (°C)	z_1 (ft)	$\overline{U}$ (ft/sec)	α	ε	Rise of plume (ft) at stated distance (ft) downwind of stack							
							100	150	200	300	400	600	800	
1	31	180	26	14	0,3	0,40	60		99		156	185		Observed
							59		94		153	196		Schmidt
							54		91		135	154		Priestley
1	31	180	26	33	0,3	0,25	29		35		52	70	84	O
							29		40		56	68	81	S
							24,5		39,5		58	66	69	P
1	31	180	26	26,5	0,3	0,27	28		40		63	80	101	O
							31		44		65	81	99	S
							29,5		48,5		72	82	86	P
1	31	180	26	19	0,3	0,36	55		75		121	165	200	O
							48		75		118	159	198	S
							42		67		101	114	120	P
2	28	25	16	10	~0,0	~0,5	46	53	58	67				O
							46	53,5	58	64				S
							41	53	61	79				P
2	28	25	16	13,3	~0,0	0,42	36	38	41	47				O
							34	38	42	45				S
							32	42	49	62				P
3	35,5	149	28	20,4	0,8(?)	$0{,}07^{5}$	46		66	82	92	103		O
							46		67	82	91	104		S
							40		64	82	94	108		P

It appears that this expression can be approximated sufficiently by:

$$H(Y) - z_1 = z_1 \int_1^{1/\alpha} y^{-2+2\varepsilon}\, dy + \alpha z_1 \int_{1/\alpha}^{Y} y^{-1+2\varepsilon}\, dy \tag{4.6.4}$$

For $Y \leqslant 1/\alpha$ only the first term of the righthand side need be used.

BOSANQUET c.s. also divided the initial exit-velocity of the jet into a "velocity"-part and a "buoyancy"-part but assumed that both parts are additive. This assumption cannot be considered correct.

Recently PRIESTLEY (1956) developed a theory concerning the propagation of stack gases in the case of natural wind.

His results are based on the assumption that the considerations by PRIESTLEY and BALL (1955) also hold approximately in the present case. At some distance from the stack the vertical motion of the plume is assumed to be represented by a simple equation in which the friction term is given by a constant, dependent on the velocity of the natural wind, multiplied by the vertical velocity in the plume. PRIESTLEY produced a table in which the observational results of BOSANQUET c.s. are compared with various theoretically computed values of the heights reached by the plume.

We reproduce part of this table adding our own results as well as the values of R_1, α and ε which had to be adopted to find close agreement. Both α and ε are determined sharply by the empirical results, so that only one combination (α, ε) gives sufficient agreement with data. The values of the constants used by PRIESTLEY are omitted as well as his values for the ceiling of the plumes, such a ceiling occurring only in the case $\alpha = 0$ and $\varepsilon < 0{,}5$ according to our theory. BOSANQUET's and SUTTON's theoretical results are also omitted from the table as they are inferior to PRIESTLEY's results in general. The values of R_1, the radius of the plume at the level of the orifice has been determined along the lines of PRIESTLEY's paper. They may be by about 10 % too large. In the cases of stack no. 2 the velocity is supposed to be independent of buoyancy up to a distance of 300 ft from the stack.

It will be seen that these results are better than, or at least as good as, PRIESTLEY's. They are also superior to those given recently by MEADE (1956).

ε seems to be rather large except in the last case. This may be due to the fact that reliable observations up to 800 ft from the stack can only be obtained in the case of weak turbulence. On the other hand SUTTON's value 1/8 was found from the place where the maximum concentration at ground level occurs. This distance x_{max} is far in excess of 800 ft for ordinary stacks. As ε may vary with distance from the stack (see section 3.3.3) the present results need not be in contradiction with the commonly used value $\varepsilon = 1/8$. Moreover, in determining x_{max} the influence of variations of wind direction cannot be omitted whereas the observations of the rise of plumes can be made independent of these variations. Observations at Oak Ridge (HOLLAND, 1953) also indicated that ε is in excess of SUTTON's value, namely 0,25—0,35 in the average between 100 and 400 ft. In general ε seems to increase with growing elevation, another point that may explain the lack of agreement between the value of ε found from the rise of plumes and the one derived from concentration measurements.

It is very satisfying that in the four examples relating to stack no. 1, α appears to have the same value. It is surprising, however, that in the last case with larger exit-velocity of the jet and smaller value of $\widetilde{\vartheta}_{m1}$, α should be larger than in the case of stack no. 1, as a relatively smaller value of w_{c1} would be expected. The result may be due to differences in stack height and friction within the stack, however. It seems to be impossible to solve this problem without having exact details of the processes occuring in both stacks.

Summarizing we can state that by adopting reasonable values for α and ε it is possible to explain the empirical results of BOSANQUET c.s. almost completely. This is a strong argument in favour of the present theory.

SUMMARY

The examples of the foregoing sections show clearly how complicated the problem of the diffusion of stack gases in the atmosphere is. Properties of the stack as well as those of the atmosphere play a predominant role in a way that is very difficult to survey. It is not surprising, therefore, that so many theories have been developed, each of them having its advantages as well as its shortcomings.

The theory presented here seems to have some advantages over other theories. The most important one may lie in the fact that it connects the theory of the diffusion of pollution in an atmosphere with natural wind, with the theory of the diffusion of temperature and velocity in a jet rising in a stagnant environment. Moreover, no special assumptions with respect to the nature of atmospheric turbulence are needed.

Further the influence of buoyancy on the rise of the plume seems to be better described than by any of the other existing theories.

Finally the influence of thermal stratification was obtained, be it only approximately.

Any essential improvement in the theory on the diffusion of stack gases seems to depend on the availability of reliable observations. Such observations should involve the thermal stratification of the atmosphere, the degree of atmospheric turbulence as well as the thermal properties of the stack gases.

When planning the erection of new industrial stacks a thorough investigation of all the possibilities by technologists, biologists and meteorologists could result in avoiding unexpected and undesirable complications.

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