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J. G. J. SCHOLTE

RAYLEIGH WAVES IN ISOTROPIC AND ANISOTROPIC ELASTIC MEDIA

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STAATSDRUKKERIJ- EN UITGEVERIJBEDRIJF / 'S-GRAVENHAGE

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PREFACE

The theory of the Rayleigh wave equation is often put forward in an unnecessarily complicated way which tends to obscure the significance of this equation. In this paper Dr. J. G. J. Scholte, scientific collaborator at the Geophysical Division of this Institute, derives this equation for an isotropic medium in a simple way by using the elementary requirements a surface wave has to fulfill.

The same method is then applied to an investigation concerning the occurrence of these waves in the different classes of crystalline bodies; this procedure was also followed for waves travelling along the boundary between two isotropic media.

In the course of this study some properties of polarised waves travelling in crystals have been derived which may be of some importance to experimental research.

Dr. J. Veldkamp, director of the Geophysical Division, gave valuable contribution by reading the manuscript and discussing it with the Author.

Mr. J. A. As, scientific assistant, carried out the numerical calculations.

The Director in Chief
Royal Netherlands Meteorological Institute

IR. C. J. WARNERS

INTRODUCTION

In his fundamental paper 'On waves propagated along the plane surface of an elastic solid' RAYLEIGH (1887) concerned himself with the question whether in an elastic, isotropic, semi-infinite, solid body a wave system could exist which decreased exponentially with the distance to the boundary plane. The conclusion he arrived at was that such a system is possible in any isotropic medium and that it travels in a direction parallel to the boundary with a velocity ε which is smaller than the velocity of transverse waves; the quantity ε satisfies an equation of the third degree, the Rayleigh equation, which determines ε as a function of the constant of Poisson for that medium.

RAYLEIGH's expectation that 'it is not impossible that these surface waves here investigated play an important part in earthquakes' has been fully realized; it is therefore not very surprising that many authors have generalized this theory in order to ascertain the existence and the properties of these waves in more general circumstances.

In an attempt to determine the influence of a superficial layer on these waves LOVE (1911) was led to a surface wave travelling along the interface between two media both of which he assumed to be incompressible. This investigation was taken up again by STONELEY (1925) who studied this kind of wave in the case of two compressible media; the equation to be satisfied by the velocity of these waves has been discussed by SEZAWA and KANAI (1939), by CAGNIARD (1939) and by the present author (1942).

Another generalization has been effectuated by HOMMA (1942) and by STONELEY (1943) who both examined Rayleigh waves in media which are isotropic only in the horizontal directions but which show different properties in the vertical direction ('transverse isotropic' bodies). This subject has again been taken up by SATÔ (1950), who made an attempt to prove that in these media a Rayleigh system is possible for any values of the elastic constants.

The next advance on this way of generalizing the suppositions relating to the elastic properties of the medium was made once more by STONELEY (1955), who took up the problem of Rayleigh waves in cubic crystalline material. The algebraic reductions becoming very complicated in this case the author solved the velocity equation by means of an electronic computer.

In this paper the theory will be reconsidered for various reasons; first of all the theory as outlined by RAYLEIGH – and afterwards represented in the same way in several textbooks on seismology – is unnecessarily intricate, which tends to obscure the essential features of this wave system and the corresponding velocity equation. It is thought useful to represent this theory more simply; moreover a better understanding of the Rayleigh system of an isotropic medium facilitates the treatment of analogous systems in more complicated media.

Secondly the author is of the opinion that the subject of Rayleigh waves in elastic media has been developed far enough to warrant a more or less comprehensive review of the results reached up till now; consequently although some aspects of this paper are here published for the first time some questions already discussed are again dealt with.

The subject matter has been divided into five parts; in the first chapter the ordinary Rayleigh equation has been derived and discussed. In the next chapter the theory has been extended to the case of a horizontally isotropic medium followed in chapter III by a discussion of the Rayleigh equation for different classes of crystals. An extension of the method used in these chapters enables us in the 4th chapter to derive the Stoneley wave equation concerning the wave travelling along the boundary between two media; in the same chapter this equation will be discussed.

In the last chapter the practical application of the theory to seismology has been pointed out; as some parts of the theory, although not of direct interest to the main subject of this paper, may be of some importance to the experimental investigation of wave propagation in crystals, finally some remarks relevant to this subject are included.

I. ISOTROPIC MEDIUM

We consider plane waves travelling in an isotropic semi-infinite body which borders at $z = 0$ on the vacuum. As in this medium each horizontal direction is equivalent it is possible without loss of generality to choose the line of intersection of the plane $z = 0$ and the plane of constant phase as the axis Y. The z-coordinate will be supposed to be positive for points inside the medium.

The wave is then expressed by functions of x, z and t which are of the form

$$\exp i\nu \left(t - \frac{x \sin \vartheta_i \pm z \cos \vartheta_i}{c_i}\right),$$

where c_i represents the longitudinal velocity $c_1 = \sqrt{(\lambda+2\,\mu)/\varrho}$ or the transverse one $c_2 = \sqrt{\mu/\varrho}$; ν is the frequency and ϑ_i determines the direction.

For any given positive real value of $\sin \vartheta_i/c_i$ four waves exist which are usually called the longitudinal (c_1) and transverse (c_2) incident ($+ \cos \vartheta_i$) and reflected ($- \cos \vartheta_i$) wave. At the boundary $z = 0$ the stress disappears for every value of x and t; it follows that this can only be brought about by waves corresponding to the same values of $\sin \vartheta_i/c_i$ and ν.

Rayleigh's requirement that the surface wave system must decrease exponentially with increasing distance z amounts to two separate conditions to be fulfilled by ϑ_i: in the first place an exponential variation with z is only possible if $\sin \vartheta_i > 1$; hence the phase velocity $\varepsilon = c_i/\sin \vartheta_i$ with which these waves travel in the direction x must be smaller than the velocities c_i.

Secondly the desideratum that this variation must be a decrease leads immediately to the exclusion of two of the four waves determined by ε, namely those waves with positive imaginary $\cos \vartheta_i$; the system therefore consists of only two plane waves.

This two-wave system must satisfy the boundary condition, which brings us to two further conclusions:

1. as $T_{zy} = \mu\, \partial v/\partial z = 0$ the component v of the (transverse) wave in the y-direction must vanish: the transverse wave is apparently of the SV type;
2. the tensions exerted by the two waves on the free boundary must cancel each other; this is only possible if these two tensions are directed along the same line. If ϑ_i is chosen in such a way that this condition is fulfilled it is easy to accomplish the cancellation by an adjustment of the amplitudes of the two waves. The equation expressing the colinearity of the two tensions is called the Rayleigh equation; together with the inequality $\sin \vartheta > 1$ this equation determines the Rayleigh system.

The derivation of this equation amounts to the calculation of the direction of the stresses on the boundary caused by the two waves; the components of this stress are given by

$$T_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \text{ and } T_{zz} = \lambda \operatorname{div} + 2\,\mu \frac{\partial w}{\partial z}.$$

The components u and w of the motion are, in the case of a longitudinal wave, represented by $u = A_1 \sin \vartheta_1 \,.\, \varphi_1$ and $w = A_1 \cos \vartheta_1 \,.\, \varphi_1$ where A_1 is a constant and φ_1 is the phase-factor

$$\exp i\nu \left(t - \frac{x \sin \vartheta_1 + z \cos \vartheta_1}{c_1} \right) .$$

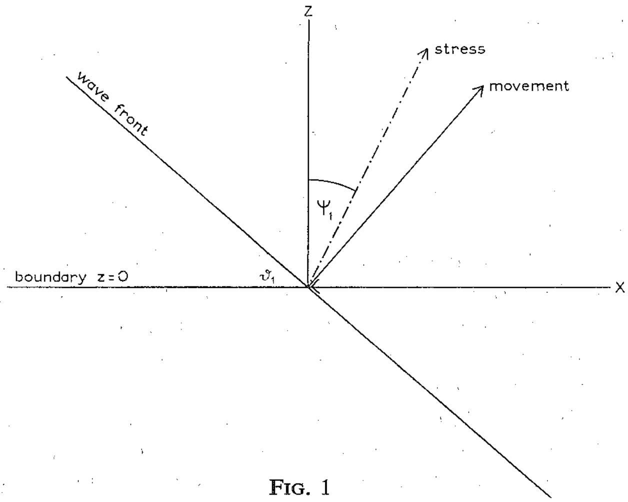

FIG. 1

It follows that

$$T_{zx} = - \frac{i\nu}{c_1} A_1 \,.\, \mu \sin 2\,\vartheta_1 \,.\, \varphi_1 \qquad T_{zz} = - \frac{i\nu}{c_1} A_1 \left(\lambda + 2\,\mu \cos^2\vartheta_1\right) .\, \varphi_1 \,, \qquad (1)$$

hence $\dfrac{T_{zx}}{T_{zz}} = \dfrac{\sin 2\,\vartheta_1}{c_1^2/c_2^2 - 2 \sin^2\vartheta_1}$.

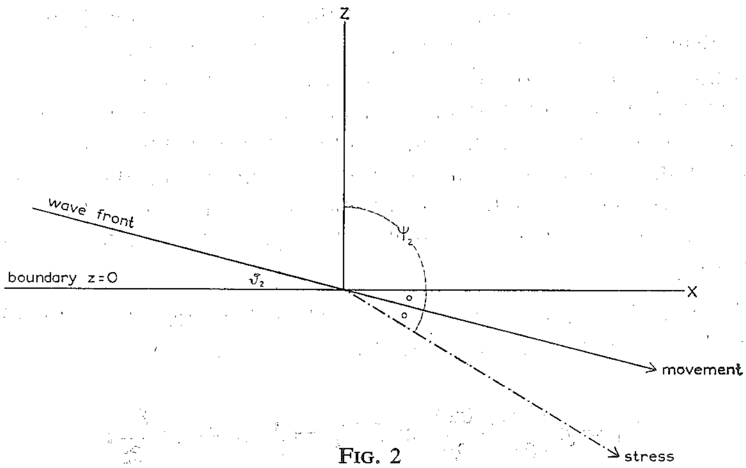

FIG. 2

In the case of a transverse SV wave the components of the motion are $u = A_2 \cos \vartheta_2 \,.\, \varphi_2$ and $w = -A_2 \sin \vartheta_2 \,.\, \varphi_2$ with $\varphi_2 = \exp iv \left(t - \dfrac{x \sin \vartheta_2 + z \cos \vartheta_2}{c_2}\right)$;

then $T_{zx} = -\dfrac{iv}{c_2} A_2 \,.\, \mu \cos 2\vartheta_2 \,.\, \varphi_2$ and $T_{zz} = +\dfrac{iv}{c_2} A_2 \,.\, \mu \sin 2\vartheta_2 \,.\, \varphi_2$ (2)

therefore $$\frac{T_{zx}}{T_{zz}} = -\cot 2\vartheta_2 \,.$$

A two-wave system is determined by $\dfrac{c_1}{\sin \vartheta_1} = \dfrac{c_2}{\sin \vartheta_2} = \varepsilon$ and by the equation

$$\frac{\sin 2\vartheta_1}{c_1^2/c_2^2 - 2\sin^2 \vartheta_1} = \pm \cot 2\vartheta_2 \,,$$

which expresses the equality of the directions of the stresses on the plane $z = 0$ caused by incident or reflected longitudinal and transverse waves.

Elimination of ϑ_1 and ϑ_2 yields the Rayleigh equation in the usual form of a cubic equation of ε^2:

$$\varepsilon^6 c_1^2 - 8\,\varepsilon^4 c_1^2 c_2^2 + 8\,\varepsilon^2 c_2^4 (3\,c_1^2 - 2\,c_2^2) - 16\,c_2^6 (c_1^2 - c_2^2) = 0.$$

As the value of the left-hand side of the Rayleigh equation is negative for $\varepsilon = 0$ and positive for $\varepsilon = c_2$ one *) of the roots ε^2 is real and smaller than c_2^2. The quantity

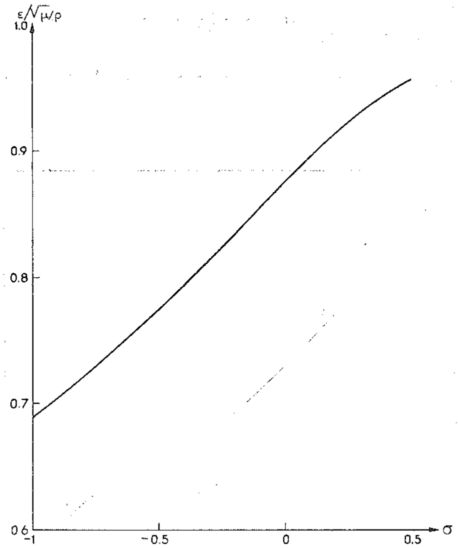

FIG. 3

* It has to be proved that only one root ε^2 is smaller than c_2^2; as this proof will be given in the next chapter in the case of a more general medium it has been omitted here.

$\varepsilon/\sqrt{\mu/\varrho}$, where ε satisfies the above equation, is a function of c_1/c_2 or of the constant σ of Poisson, as $\frac{c_1}{c_2} = \left(\frac{2-2\sigma}{1-2\sigma}\right)^{1/2}$.

In fig. 3 this relation is graphically represented for all admissible values of σ, which extend from $\sigma = -1$ (infinitely great rigidity) to $\sigma = 1/2$ (incompressible medium). Starting with the value 0.689 at $\sigma = -1$ the quantity $\varepsilon/\sqrt{\mu/\varrho}$ increases monotonously with increasing σ to 0.955 at $\sigma = 1/2$.

The quantities $\cos \vartheta_1$ and $\cos \vartheta_2$ being imaginary the components T_{zx} and T_{zz} of the stress differ 90° in phase, both in the longitudinal and in the transverse case. Each stress vector describes in the time $2\pi/\nu$ an ellipse of which the ratio between the two main axes is equal to the absolute value of T_{zx}/T_{zz}. The possibility of these two rotating stresses cancelling each other occurs only if the two ellipses are similar, which condition is expressed by the Rayleigh equation when $\sin \vartheta_i > 1$.

The other two roots of the Rayleigh equation also define a two-wave system, namely a system consisting of a longitudinal incident and a transverse reflected wave (or *vice versa*). The occurrence of such a system is easily shown to be possible by considering the directions of the stresses on the boundary for real values of ϑ.

If $\vartheta_1 = 0$ the stress of the longitudinal waves is directed along the z-axis; with increasing value of ϑ_1 the angle ψ_1 between the stress and this axis increases too and reaches at $\sin \vartheta_1 = \sqrt{\frac{c_1^2}{2(c_1^2 - c_2^2)}}$ its maximum value (equal to arc sin $\frac{c_2^2}{c_1^2 - c_2^2}$). At greater values of ϑ_1, the stress vector turns back towards the z-axis which is reached at $\vartheta_1 = \pi/2$.

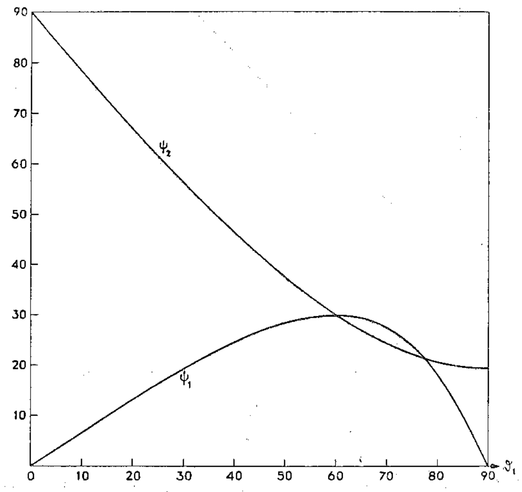

FIG. 4

The stress of the transverse wave is, when $\vartheta_1 = 0$, directed along the x-axis; with increasing ϑ_1 the angle ψ_2 between this stress and the z-axis decreases monotonously from $\pi/2$ to the value arc sin $(c_1^2 - 2\, c_2^2)/c_1^2$.

This behaviour is shown by the curves of fig. 4, which are drawn for $\sigma = {}^1/_4$; in this case the two stresses are colinear for two real values of ϑ_1. These two values approach each other at greater values of σ and coincide (at $\vartheta_1 = 80°42'$) when $\sigma = 0.3$; at still greater values of σ the two angles become complex, so that then no special reflection is possible.

II. TRANSVERSELY ISOTROPIC MEDIUM

§ 1. Derivation of the Rayleigh equation

Following the example of STONELEY we generalize this theory by considering next a medium which is isotropic only with respect to all directions parallel to some plane $z = 0$; such a body has been described by LOVE (1926) as being 'transversely isotropic'.

The strain-energy function of this medium has to be invariant to a transformation of the x and y axis; it may be shown that this property leads to the following expression of this function as given by LOVE:

$$2\,W = A\,(e_{xx}{}^2+e_{yy}{}^2) + C\,e_{zz}{}^2 + 2\,F\,(e_{xx}+e_{yy})\,e_{zz} + 2\,(A-2\,N)\,e_{xx}\,e_{yy} + L\,(e_{yz}{}^2+e_{xz}{}^2) + N\,e_{xy}{}^2.$$

This function must be definite positive; writing it as a sum of squares we obtain

$$2\,W = C\left\{e_{zz} + \frac{F}{C}\,(e_{xx}+e_{yy})\right\}^2 + \left(A-\frac{F^2}{C}\right)\left\{e_{xx} + \frac{A-2\,N-F^2/C}{A-F^2/C}\,e_{yy}\right\}^2 + 4\,N\,\frac{A-N-F^2/C}{A-F^2/C}\,e_{yy}{}^2 + L\,e_{xz}{}^2 + L\,e_{yz}{}^2 + N\,e_{xy}{}^2.$$

We conclude that C, N and L must be positive and $A > F^2/C + N$.

In the usual way the equations of motion are derived:

$$\varrho\,\frac{\partial^2 u}{\partial t^2} = \left(A\,\frac{\partial^2}{\partial x^2} + N\,\frac{\partial^2}{\partial y^2} + L\,\frac{\partial^2}{\partial z^2}\right) u + (A-N)\,\frac{\partial^2 v}{\partial x \partial y} + (F+L)\,\frac{\partial^2 w}{\partial x \partial z}$$

$$\varrho\,\frac{\partial^2 v}{\partial t^2} = \left(N\,\frac{\partial^2}{\partial x^2} + A\,\frac{\partial^2}{\partial y^2} + L\,\frac{\partial^2}{\partial z^2}\right) v + (A-N)\,\frac{\partial^2 u}{\partial x \partial y} + (F+L)\,\frac{\partial^2 w}{\partial y \partial z}$$

$$\varrho\,\frac{\partial^2 w}{\partial t^2} = \left(L\,\frac{\partial^2}{\partial x^2} + L\,\frac{\partial^2}{\partial y^2} + C\,\frac{\partial^2}{\partial z^2}\right) w + (F+L)\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z}\right).$$

For the same reason as in the case of an isotropic medium it is permissible without loss of generality to express the phase factor of plane waves travelling in this body by a function which does not depend on y:

$$\exp\,iv\left(t - \frac{x \sin\vartheta + z \cos\vartheta}{c}\right).$$

Denoting the amplitudes of the components in the x, y and z direction by p, q and r we obtain by substitution into the equations of motion:

$$p\,(A \sin^2\vartheta + L \cos^2\vartheta - \varrho\,c^2) + r\,(F+L) \sin\vartheta \cos\vartheta = 0$$
$$q\,(N \sin^2\vartheta + L \cos^2\vartheta - \varrho\,c^2) = 0$$
$$p\,(F+L) \sin\vartheta \cos\vartheta + r\,(L \sin^2\vartheta + C \cos^2\vartheta - \varrho\,c^2) = 0.$$

It follows that the y component is independent of the movements in the x and z direction; therefore, again as in the previous case, this component does not appear in a Rayleigh wave system.

The two equations for p and r are only solvable if

$$(A \sin^2\vartheta + L \cos^2\vartheta - \varrho c^2)(L \sin^2\vartheta + C \cos^2\vartheta - \varrho c^2) = (F+L)^2 \sin^2\vartheta \cos^2\vartheta \qquad (3)$$

which equation determines the velocity c of the waves.

We now introduce the parameter $\varepsilon = c/\sin\vartheta$ by means of which the equation becomes

$$LC \cot^4\vartheta + \{L(L-\varrho\varepsilon^2) + C(A-\varrho\varepsilon^2) - (F+L)^2\}\cot^2\vartheta + (L-\varrho\varepsilon^2)(A-\varrho\varepsilon^2) = 0 \qquad (4)$$

from which it is evident that at a given value of ε four waves are possible, defined by the roots $\pm \cot\vartheta_1$ and $\pm \cot\vartheta_2$. These waves may again be interpretated as incident and reflected waves, but the distinction between longitudinal and transverse waves is no longer applicable.

The ratio p/r between the two components of each wave is obtained by substituting these values of $\cot\vartheta$ into the expression

$$\frac{p}{r} = -\frac{L-\varrho\varepsilon^2 + C\cot^2\vartheta}{(L+F)\cot\vartheta}.$$

In order to obtain the equation to be satisfied by ε in the case of a two-wave system we have to calculate the ratio T_{xz}/T_{zz};

as $T_{zx} = -i\frac{v}{c}L(p\cos\vartheta + r\sin\vartheta)$ and $T_{zz} = -i\frac{v}{c}(Fp\sin\vartheta + Cr\cos\vartheta)$

we obtain by means of the expression of p/r:

$$T_{zx} = -\frac{ivr}{\varepsilon}\cdot\frac{L(F+\varrho\varepsilon^2) - LC\cot^2\vartheta}{L+F} \quad \text{and} \quad T_{zz} = +\frac{ivr}{\varepsilon}\cdot\frac{F(L-\varrho\varepsilon^2) - LC\cot^2\vartheta}{(L+F)\cot\vartheta}$$

whence

$$\frac{T_{zx}}{T_{zz}} = -\frac{L(F+\varrho\varepsilon^2) - LC\cot^2\vartheta}{F(L-\varrho\varepsilon^2) - LC\cot^2\vartheta}\cot\vartheta.$$

The Rayleigh equation, which expresses the colinearity of the two stresses on the boundary is therefore:

$$\frac{F+\varrho\varepsilon^2 - C\cot^2\vartheta_1}{F(L-\varrho\varepsilon^2) - LC\cot^2\vartheta_1}\cot\vartheta_1 = \pm\frac{F+\varrho\varepsilon^2 - C\cot^2\vartheta_2}{F(L-\varrho\varepsilon^2) - LC\cot^2\vartheta_2}\cot\vartheta_2.$$

Using the properties of the roots of the quadratic (4) this equation is readily reduced to

$$(L-\varrho\varepsilon^2)(AC-F^2-C\varrho\varepsilon^2) \pm \varrho\varepsilon^2\sqrt{LC(L-\varrho\varepsilon^2)(A-\varrho\varepsilon^2)} = 0 \qquad (5)$$

which may also be written in the more familiar form

$$\varrho^3\varepsilon^6 C(C-L) + \varrho^2\varepsilon^4 C(AL-2P-CL) + \varrho\varepsilon^2 P(P+2CL) - LP^2 = 0 \qquad (6)$$

where $P = AC-F^2$.

§ 2. Discussion of the equation

Before starting the discussion of the roots ε^2 of this equation it is well to remember that these roots determine any two-wave system; such a system is a Rayleigh wave if moreover the corresponding values of $\cot \vartheta_1$ and $\cot \vartheta_2$ are negative imaginary. In all other cases the roots of the Rayleigh equation define reflections at the boundary at which only one reflected wave occurs.

1. As is obvious from equation (5) the real roots ε^2 must be either smaller than A/ϱ and L/ϱ or greater than these two velocities. With regard to the second possibility we remark that as in that case the coefficient of $\cot^2\vartheta$ in the quadratic (4) is negative the quantity $\cot^2\vartheta_1 + \cot^2\vartheta_2$ must be positive. Hence $\cot \vartheta_1$ and $\cot \vartheta_2$ are either real or complex so that the roots $\varepsilon^2 > A/\varrho$ and L/ϱ define a case of special reflection and not a Rayleigh system.

2. From the same equation it will be seen that there are four values of $\varrho\varepsilon^2$ which will be of importance to this discussion, namely $\varrho\varepsilon^2 = 0, A, L$ and $A - F^2/C\ (= P/C)$. The value of the left-hand side of equation (6) is

at $\varrho\varepsilon^2 = 0$: $\quad -LP^2$

$\varrho\varepsilon^2 = A$: $\quad F^4(A-L)$

$\varrho\varepsilon^2 = L$: $\quad CL^3(A-L)$

$\varrho\varepsilon^2 = P/C$: $\quad P^2LF^2/C^2$.

It follows that at least one root is situated in the interval $0 < \varrho\varepsilon^2 < P/C$, a conclusion which has been obtained by Satô (1950).

3. The question whether only one or all three roots are confined to this interval may be decided by the introduction of the new variable $\varphi = \frac{P/C}{\varrho\varepsilon^2} - 1$. The interval $0 < \varrho\varepsilon^2 < P/C$ is then changed into $\varphi > 0$ and the equation into:

$$\varphi^3 + \varphi^2 \frac{CL-P}{CL} - \varphi \frac{A}{C} - \frac{F^2}{C} = 0.$$

The coefficient of φ and the last term being negative it follows that this equation yields one and only positive root; consequently the Rayleigh equation has one and only one root ε^2 satisfying the inequality $0 < \varrho\varepsilon^2 < P/C$.

4. This root is also smaller than L/ϱ, which has of course only to be demonstrated if $L < P/C$. In that case $L < A$ so the left-hand side of equation (6) is positive for $\varrho\varepsilon^2 = L$; as it is positive too for $\varrho\varepsilon^2 = P/C$ either two roots are situated between L/ϱ and $P/\varrho C$ or none at all. The first alternative is ruled out by the conclusion of the previous section, hence the root must be smaller than L/ϱ.

5. With respect to the two remaining roots of the Rayleigh equation the following properties are easily derived:

if $A < L$ one of these roots ε^2 is smaller than A/ϱ (and of course $> P/\varrho C$), and the other one is greater than L/ϱ if $L < C$ and negative if $L > C$.

If $A > L$ the situation is more complicated:

when $L > C$ one root ε^2 is greater than A/ϱ and the other one is negative;

when $L < C$ we have to distinguish two separate possibilities:

if $L > P/C$ both roots are either complex, or greater than A/ϱ, or smaller than L/ϱ;

if $L < P/C$ the last possibility does not exist.

§ 3. The existence of a Rayleigh system

Ignoring those roots which certainly are connected with systems of special reflection ($\varrho\varepsilon^2 > A$ and $> L$) we shall to some extent deal with the roots which may be relevant to a Rayleigh wave. These are:

1. the root $\varrho\varepsilon^2 < P/C$ which occurs for every value of the elastic constants.
2. the root $\varrho\varepsilon^2$ between P/C and A, which only exists if $A < L$.
3. two roots $\varrho\varepsilon^2$ between P/C and L, which possibly exist if A and C both are $> L > P/C$.

Contrary to the case of the isotropic medium it is by no means certain that these roots although smaller than A and L determine a Rayleigh system; this is only true if the corresponding values of $\cot^2\vartheta_1$ and $\cot^2\vartheta_2$ are negative real, which is *a priori* quite uncertain.

It is not our intention to enter into an exhaustive discussion of this question but we shall merely point out some elementary conditions to be satisfied by the elastic constants if $\cot^2\vartheta_1$ and $\cot^2\vartheta_2$ are negative real.

In that case the coefficient of $\cot^2\vartheta$ in equation (4) must be positive, or:

$$L(L-\varrho\varepsilon^2) + C(A-\varrho\varepsilon^2) - (F+L)^2 > 0.$$

The value of this quantity in the limiting points of the intervals to which the roots $\varrho\varepsilon^2$ are confined are:

at $\varrho\varepsilon^2 = 0$:	$P - 2FL$
$\varrho\varepsilon^2 = A$:	$-AL - F^2 - 2FL$
$\varrho\varepsilon^2 = L$:	$-C(L-P/C) - 2FL - L^2$
$\varrho\varepsilon^2 = P/C$:	$-PL/C - 2FL.$

It appears that as far as the roots of the second and third kind are concerned this quantity will be positive at any point of their intervals only for sufficiently great negative values of F. Even then it is not certain that those roots $\varrho\varepsilon^2$ determine a Rayleigh system as it is possible that $\cot^2\vartheta_1$ and $\cot^2\vartheta_2$ are conjugate complex (with a negative real part). However, we shall not pursue this matter any further.

Turning finally to the root $\varrho\varepsilon^2 < P/C$ we illustrate the above remarks by dealing in some detail with the wave system corresponding to this root.

The discussion of the Rayleigh wave in an isotropic medium is usually limited to two special values of the parameter μ/λ, namely $\mu/\lambda = 1$ and $\mu/\lambda = 0$; continuing this tradition we have chosen two cases of transverse isotropic media which resemble these two isotropic bodies.

1. For an isotropic medium with $\mu/\lambda = 1$ the elastic constants are in our notation $A = C = 3\,F = 3\,L\,(= 3\,\mu)$; a similar anisotropic medium is for instance defined by $A = C = 3\,F (= 3\,\mu)$ and $L = p\mu\,(p > 0)$.

2. An incompressible isotropic medium is defined by $A = C = F{+}2\,\mu$, $L = \mu$ and $F \rightarrow \infty$; we change these relations rather arbitrarily into $A = F{+}2\,\mu$, $C = F{+}2\,p\mu\,(p > -1)$, $L = \mu$ and $F \rightarrow \infty$.

In order to avoid any misunderstanding we emphasize that these two media are only intended as examples of anisotropic bodies, which are more or less similar to the two classical types of isotropic media.

In the first example the equation determining $\cot\vartheta$ becomes

$$3\,p\cot^4\vartheta + \{(8-2\,p) - x^2(3+p)\}\cot^2\vartheta + (p-x^2)\,(3-x^2) = 0$$

where $x^2 = \varrho\varepsilon^2/\mu$; the Rayleigh equation is then

$$3\,x^6(3-p) - 48\,x^4 + 16\,x^2(4+3\,p) - 64\,p = 0.$$

The condition $\varrho\varepsilon^2 < P/C$ becomes here $\varrho\varepsilon^2 < 2^2/_3\,\mu$ or $x < 1.633$; in table I and fig. 5 the results of a numerical calculation are given.

Table I

p	$x = \varepsilon/\sqrt{\mu/\varrho}$	$y = i\cot\vartheta_1$	$y = i\cot\vartheta_2$
0	0	0	∞
0.100	0.316	0.0082	4.993
0.201	0.447	0.0226	3.345
0.304	0.548	0.0442	2.581
0.411	0.632	0.0705	2.170
0.523	0.707	0.1060	1.816
0.642	0.775	0.1492	1.527
0.771	0.837	0.2090	1.268
0.915	0.894	0.3012	1.008
1.000	0.919	0.3933	0.8475 (isotropic medium)
1.07177	0.941	0.5917	0.5917

It appears that although for every positive value of p a real root $x < 1.633$ exists, a Rayleigh wave is only possible if $p < 1.07177$.

At greater values of p the values of $\cot\vartheta$ are complex imaginary and then the corresponding wave is not a Rayleigh wave, as it is semi-sinusoidal in the z-direction.

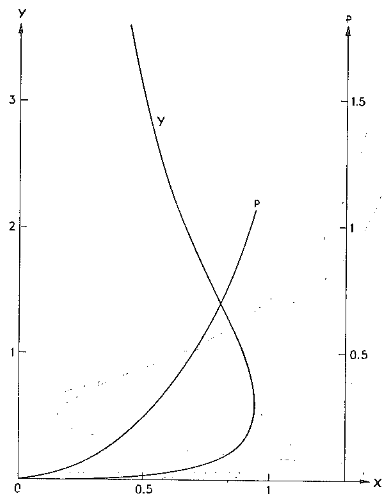

FIG. 5

In the second case the two equations are

$$\cot^4\vartheta + (2\,p - x^2)\cot^2\vartheta + (1 - x^2) = 0$$

where again $x^2 = \varrho\varepsilon^2/\mu$; and

$$x^6 - 4\,(1+p)\,x^4 + 4\,(1+p)\,(1+2\,p)\,x^2 - 4\,(1+p)^2 = 0.$$

The condition $\varrho\varepsilon^2 < P/C$ leads to $x^2 < 2\,(p+1)$; the results are shown in table I and fig. 6.

Table II

p	$x = \varepsilon/\sqrt{\mu/\varrho}$	$y = i \cot \vartheta_1$	$y = i \cot \vartheta_2$
∞	1	0	∞
9.474	0.999	0.0118	4.236
5.707	0.997	0.0248	3.228
4.445	0.995	0.0356	2.810
3.430	0.992	0.0516	2.423
2.747	0.989	0.0706	2.124
2.172	0.984	0.0981	1.835
1.880	0.980	0.1198	1.669
1.532	0.973	0.1590	1.447
1.344	0.968	0.1910	1.309
1.107	0.960	0.2527	1.108
1.000	0.955	0.2956	1.000 (isotropic medium)
0.972	0.954	0.3099	0.968
0.851	0.947	0.3979	0.804
7/9	0.943	0.5774	0.5774

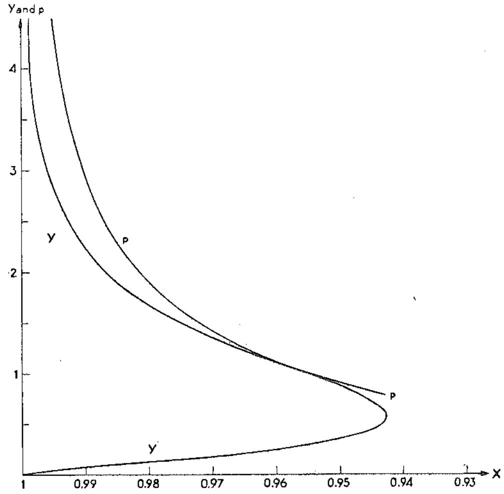

FIG. 6

In this case too a real root x exists for every permissible value of p ($p > -1$), but a Rayleigh system is only possible in this incompressible medium if $p > 7/9$; at smaller values of p the corresponding values of $\cot \vartheta$ are complex imaginary.

SATÔ (1950) also considered the proof of the property of equation (6) that one and only one of its roots is contained in the interval $0 < \varrho\varepsilon^2 < P/C$; not succeeding completely as he himself remarked in a note to his paper this problem was again treated by DÜRBAUM (1956) who arrived at a proof which differs considerably from the proof given here.

At the end of his paper SATÔ reaches the conclusion that 'one and only one sort of Rayleigh waves exists, in horizontally isotropic but vertically aeolotropic semi-infinite elastic medium'; this is not quite true: although for all possible values of the elastic constants a root $\varrho\varepsilon^2$ between 0 and P/C exists the corresponding wave system is not in every case a Rayleigh wave, as demonstrated above.

III. GENERAL ANISOTROPIC MEDIUM

§ 1. Plane waves in a crystalline medium

A perfectly elastic medium (or a Hooke body) is defined by a strain-energy function which is a homogeneous quadratic function of the strain-components $e_{xx}, e_{xy}, \ldots$ with constant coefficients; in the most general case this function contains 21 terms:

$$
\begin{aligned}
W = {}^1/_2\, c_{11}\, e_{xx}^2 &+ c_{12}\, e_{xx}\, e_{yy} + c_{13}\, e_{xx}\, e_{zz} + c_{14}\, e_{xx}\, e_{yz} + c_{15}\, e_{xx}\, e_{xz} + c_{16}\, e_{xx}\, e_{xy} \\
&+ {}^1/_2\, c_{22}\, e_{yy}^2 + c_{23}\, e_{yy}\, e_{zz} + c_{24}\, e_{yy}\, e_{yz} + c_{25}\, e_{yy}\, e_{xz} + c_{26}\, e_{yy}\, e_{xy} \\
&+ {}^1/_2\, c_{33}\, e_{zz}^2 + c_{34}\, e_{zz}\, e_{yz} + c_{35}\, e_{zz}\, e_{xz} + c_{36}\, e_{zz}\, e_{xy} \\
&+ {}^1/_2\, c_{44}\, e_{yz}^2 + c_{45}\, e_{yz}\, e_{xz} + c_{46}\, e_{yz}\, e_{xy} \\
&+ {}^1/_2\, c_{55}\, e_{xz}^2 + c_{56}\, e_{xz}\, e_{xy} \\
&+ {}^1/_2\, c_{66}\, e_{xy}^2.
\end{aligned}
$$

The coefficients c_{ij} are not completely arbitrary, as W is positive for every value of the strain-components; it follows that all under-determinants of the determinant

$$
\begin{matrix}
c_{11} & \cdot & \cdot & \cdot & \cdot & c_{16} \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
c_{16} & \cdot & \cdot & \cdot & \cdot & c_{66}
\end{matrix}
$$

which are symmetrical with respect to the diagonal must be positive.

The equations of motion in this medium are:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e_{xx}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial e_{xy}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial e_{xz}}\right) &= \varrho\, \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e_{yx}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial e_{yy}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial e_{yz}}\right) &= \varrho\, \frac{\partial^2 v}{\partial t^2} \\
\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e_{zx}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial e_{zy}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial e_{zz}}\right) &= \varrho\, \frac{\partial^2 w}{\partial t^2}.
\end{aligned}
$$

We try to satisfy these equations by a plane wave with the phase-factor

$$
\sin \nu \left(t - \frac{x \cos \varphi \sin \vartheta + y \sin \varphi \sin \vartheta + z \cos \vartheta}{c}\right)
$$

with the amplitudes p, q and r in the x, y and z directions; φ and ϑ define the direction of the wave-normal.

Substitution of this motion into the strain-energy function gives

$$2\,W = \frac{\nu^2}{\varepsilon^2}\,\overline{W}\cos^2\nu\left(t - \frac{x\cos\varphi\sin\vartheta + y\sin\varphi\sin\vartheta + z\cos\vartheta}{c}\right),$$

where $\varepsilon = c/\sin\vartheta$ and $\overline{W}$ is a homogeneous quadratic function of p, q and r:

$$\overline{W} = a_{11}\,p^2 + 2\,a_{12}\,pq + 2\,a_{13}\,pr + a_{22}\,q^2 + 2\,a_{23}\,qr + a_{33}\,r^2$$

with the coefficients:

$$\begin{aligned}
a_{11} &= (c_{11}\cos^2\varphi + 2\,c_{16}\sin\varphi\cos\varphi + c_{66}\sin^2\varphi) + 2\,(c_{15}\cos\varphi + c_{56}\sin\varphi)\cot\vartheta + c_{55}\cot^2\vartheta\\
a_{22} &= (c_{66}\cos^2\varphi + 2\,c_{26}\sin\varphi\cos\varphi + c_{22}\sin^2\varphi) + 2\,(c_{46}\cos\varphi + c_{24}\sin\varphi)\cot\vartheta + c_{44}\cot^2\vartheta\\
a_{33} &= (c_{55}\cos^2\varphi + 2\,c_{45}\sin\varphi\cos\varphi + c_{44}\sin^2\varphi) + 2\,(c_{35}\cos\varphi + c_{34}\sin\varphi)\cot\vartheta + c_{33}\cot^2\vartheta\\
a_{12} &= \{c_{16}\cos^2\varphi + (c_{12}+c_{66})\sin\varphi\cos\varphi + c_{26}\sin^2\varphi\} + \{(c_{14}+c_{56})\cos\varphi + (c_{25}+c_{46})\sin\varphi\}\cot\vartheta + c_{45}\cot^2\vartheta\\
a_{13} &= \{c_{15}\cos^2\varphi + (c_{14}+c_{56})\sin\varphi\cos\varphi + c_{46}\sin^2\varphi\} + \{(c_{13}+c_{55})\cos\varphi + (c_{36}+c_{45})\sin\varphi\}\cot\vartheta + c_{35}\cot^2\vartheta\\
a_{23} &= \{c_{56}\cos^2\varphi + (c_{25}+c_{46})\sin\varphi\cos\varphi + c_{24}\sin^2\varphi\} + \{(c_{36}+c_{45})\cos\varphi + (c_{23}+c_{44})\sin\varphi\}\cot\vartheta + c_{34}\cot^2\vartheta
\end{aligned} \tag{7}$$

Substitution into the equations of motion leads to

$$\begin{aligned}
(a_{11}-\varrho\varepsilon^2)\,p + a_{12}\,q + a_{13}\,r &= 0\\
a_{12}\,p + (a_{22}-\varrho\varepsilon^2)\,q + a_{23}\,r &= 0\\
a_{13}\,p + a_{23}\,q + (a_{33}-\varrho\varepsilon^2)\,r &= 0
\end{aligned} \tag{8}$$

Evidently a non-zero solution of this set of equations is only possible if $\varrho\varepsilon^2$ satisfies the equation:

$$\begin{vmatrix} a_{11}-\varrho\varepsilon^2 & a_{12} & a_{13}\\ a_{12} & a_{22}-\varrho\varepsilon^2 & a_{23}\\ a_{13} & a_{23} & a_{33}-\varrho\varepsilon^2 \end{vmatrix} = 0 \tag{9}$$

which is the S-equation of the quadratic function $\overline{W}$.

As is well-known from the theory of quadratic surfaces the S-equation yields three real roots $\varrho\varepsilon^2(=\varrho c^2/\sin^2\vartheta)$ for any real value of a_{ij}.
Moreover if this equation has been solved it is easy by a transformation of the p, q and r axes to change $\overline{W}$ into the form

$$\overline{W} = \varrho\varepsilon_1^2\,p'^2 + \varrho\varepsilon_2^2\,q'^2 + \varrho\varepsilon_3^2\,r'^2.$$

This quantity being positive for all real values of p', q' and r' the roots $\varrho\varepsilon_i^2$ of the S-equation are positive; hence the velocities $c_i = \varepsilon_i\sin\vartheta$ are real.

Consequently in an elastic medium three different waves will be propagated in any given direction φ, ϑ; the velocities c_i of these waves are in general all three unequal to each other and the three displacements associated with these waves are perpendicular to each other, being directed along the principal axes of the quadratic surface $\overline{W}$ = a constant. However, these movements are generally obliquely directed with respect to the wave normal φ, ϑ; longitudinal and transverse waves only occur in some rather specialised media (see § 3).

§ 2. The Rayleigh equation

The main purpose of this paper is the investigation of the possible existence of Rayleigh waves in a semi-infinite elastic body; therefore we shall not develop the above theory any further but turn our attention to Rayleigh's two requirements as stated on page 11.

Accordingly the first question to be answered is whether in such a body waves occur which vary exponentially in some direction, for instance in the z-direction; in that case $\cos\vartheta$ is imaginary and $\sin\vartheta > 1$.

As the roots $\varrho\varepsilon^2$ of equation (9) have to be real the coefficients of this equation are also real and this is only possible if the equation does not contain any uneven powers of $\cot\vartheta$ ($\cot\vartheta$ being imaginary). After some laborious reductions it appears that the coefficients of $\cot\vartheta$ and $\cot^3\vartheta$ vanish for every value of φ if

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0.$$

In order to ascertain which materials fulfill this condition we have merely to inspect the strain-energy functions of the different classes of crystals; these functions are represented in their most simple form if the z-axis is chosen in the direction of the principal axis of symmetry of the crystal (if such an axis exists). Moreover when there is a plane of symmetry through this axis this plane is taken as the xz-plane; when such a plane is absent but a digonal axis perpendicular to the principal axis occurs that axis is taken as y-axis. These expressions which have been derived by VOIGT (1900) may be found in LOVE's 'Treatise' and are given below in a schematic representation:

1. triclinic system: all 21 coefficients c_{ij} are unequal to zero

2. monoclinic system:

$$\begin{array}{cccccc}
{}^1\!/_2\, c_{11} & c_{12} & c_{13} & \cdot & \cdot & c_{16} \\
 & {}^1\!/_2\, c_{22} & c_{23} & \cdot & \cdot & c_{26} \\
 & & {}^1\!/_2\, c_{33} & \cdot & \cdot & c_{36} \\
 & & & {}^1\!/_2\, c_{44} & c_{45} & \cdot \\
 & & & & {}^1\!/_2\, c_{55} & \cdot \\
 & & & & & {}^1\!/_2\, c_{66}
\end{array}$$

3. rhombic system:

$$\begin{array}{cccccc}
{}^1\!/_2\, c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot \\
 & {}^1\!/_2\, c_{22} & c_{23} & \cdot & \cdot & \cdot \\
 & & {}^1\!/_2\, c_{33} & \cdot & \cdot & \cdot \\
 & & & {}^1\!/_2\, c_{44} & \cdot & \cdot \\
 & & & & {}^1\!/_2\, c_{55} & \cdot \\
 & & & & & {}^1\!/_2\, c_{66}
\end{array}$$

4. hexagonal and rhombohedral system:

$$\begin{array}{cccccc}
{}^1\!/_2\, c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & \cdot \\
 & {}^1\!/_2\, c_{11} & c_{13} & -c_{14} & -c_{15} & \cdot \\
 & & {}^1\!/_2\, c_{33} & \cdot & \cdot & \cdot \\
 & & & {}^1\!/_2\, c_{44} & \cdot & -c_{15} \\
 & & & & {}^1\!/_2\, c_{44} & c_{14} \\
 & & & & & {}^1\!/_2\, (c_{11}-c_{12})
\end{array}
\left\{\begin{array}{ll}
c_{14} \text{ and } c_{15} \neq 0 & \text{in the rhombohedral classes I, II} \\
c_{14}=0,\ c_{15}\neq 0 & \text{in the rhombohedral classes III, IV, V} \\
c_{14}=c_{15}=0 & \text{in all other classes.}
\end{array}\right.$$

5. tetragonal system:

$$\begin{array}{cccccc} {}^1\!/_2\, c_{11} & c_{12} & c_{13} & \cdot & \cdot & c_{16} \\ & {}^1\!/_2\, c_{11} & c_{13} & \cdot & \cdot & -c_{16} \\ & & {}^1\!/_2\, c_{33} & \cdot & \cdot & \cdot \\ & & & {}^1\!/_2\, c_{44} & \cdot & \cdot \\ & & & & {}^1\!/_2\, c_{44} & \cdot \\ & & & & & {}^1\!/_2\, c_{66} \end{array}$$

$c_{16} \neq 0$ in the tetragonal classes III, IV, VII
$c_{16} = 0$ in the remaining classes.

6. cubic system:

$$\begin{array}{cccccc} {}^1\!/_2\, c_{11} & c_{12} & c_{12} & \cdot & \cdot & \cdot \\ & {}^1\!/_2\, c_{11} & c_{12} & \cdot & \cdot & \cdot \\ & & {}^1\!/_2\, c_{11} & \cdot & \cdot & \cdot \\ & & & {}^1\!/_2\, c_{44} & \cdot & \cdot \\ & & & & {}^1\!/_2\, c_{44} & \cdot \\ & & & & & {}^1\!/_2\, c_{44} \end{array}$$

The three crystallographic axes are the x, y, z axis.

We see that the above condition is not met by the triclinic crystals and by the crystals of the rhombohedral classes I, II, III, IV and V; in these materials a Rayleigh system is impossible.

Before starting with Rayleigh's second requirement (a decrease of the wave with increasing distance to the surface) we first make some rather elementary remarks concerning the reflection at a free surface.

The wave-systems in any body bounded by a free surface $z = 0$ must satisfy the condition that the stress on that plane is identically zero. As each plane wave exerts a stress on $z = 0$ proportional to its amplitude several waves will be necessary in order to render the total stress equal to zero for every value of x, y and t. The stress of each wave being also proportional to a phase-factor containing the parameters φ and ε, these parameters must be the same for all waves of the system. It follows that these waves are determined by the roots $\cot \vartheta$ of equation (9) for some given value of φ and ε; this equation is of the sixth degree in $\cot \vartheta$.

Again the total stress components are homogeneous linear functions of the amplitudes of the waves forming the wave-system; the condition that the three components T_{zx}, T_{zy} and T_{zz} vanish at $z = 0$ is therefore expressed by three equations which are linear in the amplitudes. As a solution of this set of equations generally requires 4 amplitudes a wave-system in a semi-infinite body is (generally) composed of 4 plane waves which may be chosen in an arbitrary way from the 6 possible waves determined by the 6 roots of equation (9).

Rayleigh's second requirement demanding the exponential decrease with increasing distance to the boundary restricts this choice to the negative imaginary roots $\cot \vartheta$. Equation (9) is for those materials in which an exponentially varying wave is possible of the third degree in $\cot^2\vartheta$; if the roots $\cot \vartheta$ of (9) are imaginary only three of them are negative imaginary. Consequently a Rayleigh system consists of 3 waves instead of the 4 waves generally occurring in a semi-infinite body.

Such a special system is only possible if the determinant of the coefficients of the amplitudes appearing in the three boundary-conditions ($T_{zx} = T_{zy} = T_{zz} = 0$) is equal to zero. This equation will be called the Rayleigh equation of the elastic body.
As the cancelling of the stresses exerted by each of the three waves of such a special system can only be accomplished if these stresses are located in the same plane the Rayleigh equation expresses the coplanarity of the stresses of any set of three waves satisfying equation (9).
Such a set forms a Rayleigh system if the values of cot ϑ of these waves are negative imaginary; in other cases it constitutes a system of special reflection.

In order to obtain the Rayleigh equation we have to calculate the direction of the stress exerted by a plane wave on the boundary plane. From equations (8) we find the ratio's $p : q : r$, which determine the direction of the displacement-vector; substitution of these ratio's into $\frac{\partial W}{\partial e_{xz}} : \frac{\partial W}{\partial e_{yz}} : \frac{\partial W}{\partial e_{zz}}$ gives the ratio's $T_{xz} : T_{yz} : T_{zz}$ defining the direction of the stress on $z = 0$.

The Rayleigh equation expressing the coplanarity of the stress-vectors T^{I}, T^{II} and T^{III} is

$$\begin{vmatrix} T^{\mathrm{I}}{}_{xz} & T^{\mathrm{I}}{}_{yz} & T^{\mathrm{I}}{}_{zz} \\ T^{\mathrm{II}}{}_{xz} & T^{\mathrm{II}}{}_{yz} & T^{\mathrm{II}}{}_{zz} \\ T^{\mathrm{III}}{}_{xz} & T^{\mathrm{III}}{}_{yz} & T^{\mathrm{III}}{}_{zz} \end{vmatrix} = 0$$

where the indices I, II and III indicate that three different values of cot ϑ have to be substituted, namely three roots cot ϑ of equation (9). These roots are functions of ε and φ, so that the Rayleigh equation determines the velocity ε as a function of the azimuth φ.

§ 3. Polarised wave systems

A general discussion of this equation is obviously too complicated a task to be undertaken; even in the case of a cubic crystal a numerical computation of ε for some chosen values of φ proves to be so laborious that a further elaboration would only be warranted if it should be of some practical interest (STONELEY, 1955).
In the case of an isotropic medium this complexity does not exist; this circumstance is due to the property that in such a medium wave-systems are possible which are polarised in a vertical plane (P and SV wave). Both the component of the displacement and of the stress on $z = 0$ in the direction perpendicular to the plane of polarisation are identically zero. The boundary conditions are then expressed by two equations instead of three equations needed in the case of anisotropic media (one equation disappearing identically). Apart from these two kinds of waves a third kind (SH) is possible, of which both the displacement and the stress are perpendicular to a vertical plane; the boundary conditions are reduced to only one (namely $T_{zy} = 0$).

Evidently the same simplification will also appear in an anisotropic body for waves polarised in the same way. As we have seen the displacement vectors of three waves travelling in the same direction form an orthogonal triad; the occurrence of waves with a horizontal displacement vector implies the existence of two waves vibrating in a vertical plane. Therefore as far as the polarisation of the motion is concerned we only have to investigate in which circumstances a wave analogous to a *SH* wave is possible.

We choose an arbitrary vertical plane $y = x \tan \psi$ which makes an angle ψ with the xz-plane as plane of polarisation; the horizontal components of the displacement parallel (p') and perpendicular (q') to that plane are expressed in the corresponding components p and q by:

$$p' = p \cos \psi + q \sin \psi$$
$$q' = -p \sin \psi + q \cos \psi.$$

Equations (8) become:

$$p' \{ (a_{11} - \varrho \varepsilon^2) \cos \psi + a_{12} \sin \psi \} + q' \{ -(a_{11} - \varrho \varepsilon^2) \sin \psi + a_{12} \cos \psi \} + r\, a_{13} = 0$$
$$p' \{ (a_{22} - \varrho \varepsilon^2) \sin \psi + a_{12} \cos \psi \} + q' \{ + (a_{22} - \varrho \varepsilon^2) \cos \psi - a_{12} \sin \psi \} + r\, a_{23} = 0$$
$$p' (a_{13} \cos \psi + a_{23} \sin \psi) + q' (- a_{13} \sin \psi + a_{23} \cos \psi) + r\, (a_{33} - \varrho \varepsilon^2) = 0$$

or in the easily derivable forms:

$$p' \{ {}^1\!/_2 (a_{11} - a_{22}) \sin 2 \psi - a_{12} \cos 2 \psi \} + q' \{ {}^1\!/_2 (a_{11} - a_{22}) \cos 2\psi + a_{12} \sin 2 \psi - {}^1\!/_2 (a_{11} + a_{22}) + \varrho \varepsilon^2 \} + r\, (a_{13} \sin \psi - a_{23} \cos \psi) = 0$$

$$p' \{ {}^1\!/_2 (a_{11} - a_{22}) \cos 2 \psi + a_{12} \sin 2 \psi + {}^1\!/_2 (a_{11} + a_{22}) - \varrho \varepsilon^2 \} - q' \{ {}^1\!/_2 (a_{11} - a_{22}) \sin 2 \psi - a_{12} \cos 2 \psi \} + r\, (a_{13} \cos \psi + a_{23} \sin \psi) = 0$$

$$p' (a_{13} \cos \psi + a_{23} \sin \psi) - q' (a_{13} \sin \psi - a_{23} \cos \psi) + r\, (a_{33} - \varrho \varepsilon^2) = 0 \qquad (10)$$

In order to obtain a solution where only q' is different from zero it is necessary that

$${}^1\!/_2 (a_{11} - a_{22}) \sin 2 \psi - a_{12} \cos 2 \psi = 0 \quad \text{and} \quad a_{13} \sin \psi - a_{23} \cos \psi = 0$$

or

$$\tan 2 \psi = \frac{2\, a_{12}}{a_{11} - a_{22}} \quad \text{and} \quad \tan \psi = \frac{a_{23}}{a_{13}} \qquad (11)$$

In the directions φ, ϑ satisfying these two equations waves travel with only a horizontal component q' perpendicular to the plane $y/x = a_{23}/a_{13}$; in the same directions two other waves are possible with movements parallel to that plane (quasi *P* and *SV* waves).

Again we must determine the conditions to be fulfilled when the stress on the plane $z = 0$ is also polarised perpendicular to the plane $y/x = \tan \psi$.

The relevant stress-components are, if we omit the phase-factors:

$$T_{xz} = (c_{15} \cos \varphi + c_{56} \sin \varphi + c_{55} \cot \vartheta)\, p + (c_{56} \cos \varphi + c_{25} \sin \varphi + c_{45} \cot \vartheta)\, q + (c_{55} \cos \varphi + c_{45} \sin \varphi + c_{35} \cot \vartheta)\, r$$
$$T_{yz} = (c_{14} \cos \varphi + c_{46} \sin \varphi + c_{45} \cot \vartheta)\, p + (c_{46} \cos \varphi + c_{24} \sin \varphi + c_{44} \cot \vartheta)\, q + (c_{45} \cos \varphi + c_{44} \sin \varphi + c_{34} \cot \vartheta)\, r$$
$$T_{zz} = (c_{13} \cos \varphi + c_{36} \sin \varphi + c_{35} \cot \vartheta)\, p + (c_{36} \cos \varphi + c_{23} \sin \varphi + c_{34} \cot \vartheta)\, q + (c_{35} \cos \varphi + c_{34} \sin \varphi + c_{33} \cot \vartheta)\, r$$

The coefficients of p, q and r are parts of the functions a_{ij}; abbreviating the above expressions we use a notation which indicates this connection:

$$T_{xz} = a_{11}^{(1)}\, p + a_{12}^{(2)}\, q + a_{13}^{(1)}\, r$$
$$T_{yz} = a_{12}^{(1)}\, p + a_{22}^{(1)}\, q + a_{23}^{(2)}\, r$$
$$T_{zz} = a_{13}^{(2)}\, p + a_{23}^{(1)}\, q + a_{33}^{(1)}\, r.$$

We denote the stress-components in the directions of p' and q' by $T_{x'z}$ and $T_{y'z}$; written as functions of p', q' and r the components become:

$$T_{x'z} = p'\{a_{11}^{(1)}\cos^2\psi + (a_{12}^{(1)}+a_{12}^{(2)})\sin\psi\cos\psi + a_{22}^{(1)}\sin^2\psi\} +$$
$$q'\{a_{12}^{(2)}\cos^2\psi + (a_{22}^{(1)}-a_{11}^{(1)})\sin\psi\cos\psi - a_{12}^{(1)}\sin^2\psi\} + r\,(a_{13}^{(1)}\cos\psi + a_{23}^{(2)}\sin\psi)$$
$$T_{y'z} = p'\{a_{12}^{(1)}\cos^2\psi + (a_{22}^{(1)}-a_{11}^{(1)})\sin\psi\cos\psi - a_{12}^{(2)}\sin^2\psi\} +$$
$$q'\{a_{22}^{(1)}\cos^2\psi - (a_{12}^{(1)}+a_{12}^{(2)})\sin\psi\cos\psi + a_{11}^{(1)}\sin^2\psi\} + r\,(-a_{13}^{(1)}\sin\psi + a_{23}^{(2)}\cos\psi)$$
$$T_{zz} = p'\,(a_{13}^{(2)}\cos\psi + a_{23}^{(1)}\sin\psi) + q'\,(-a_{13}^{(2)}\sin\psi + a_{23}^{(1)}\cos\psi) + r\,a_{33}^{(1)}.$$

It follows that for waves with $q' = 0$ the corresponding stress-component $T_{y'z}$ will vanish too if

$$a_{12}^{(1)} - (a_{11}^{(1)}-a_{22}^{(1)})\tan\psi - a_{12}^{(2)}\tan^2\psi = 0 \quad\text{and}\quad \tan\psi = a_{23}^{(2)}/a_{13}^{(1)}. \qquad (12)$$

In the directions φ, ϑ satisfying (11) and (12) waves are possible which resemble the P and SV waves in isotropic media in this respect that both the displacement and the stress on the plane $z = 0$ are situated in a vertical plane.

Waves analogous to SH waves appear if $p' = r = 0$ and $T_{x'z} = T_{zz} = 0$; such waves are possible in the directions φ, ϑ determined by (11) and by the equations:

$$a_{12}^{(2)} - (a_{11}^{(1)}-a_{22}^{(1)})\tan\psi - a_{12}^{(1)}\tan^2\psi = 0 \quad\text{and}\quad \tan\psi = a_{23}^{(1)}/a_{13}^{(2)}. \qquad (13)$$

Suppose now that a wave of the quasi P or SV type is reflected at the boundary; the reflected waves travel in directions ϑ which are determined by the S-equation (9). The roots of this equation will generally not satisfy the equations determining the direction of polarised waves, so that the reflected waves are not of the quasi P or SV kind. Thus in an anisotropic medium such a polarised wave system is in general impossible.

However, this objection does not exist if equations (11) and (12) are independent of ϑ, in other words if polarised waves are possible in all directions in some vertical plane. This requirement leads to several relations between the elastic constants c_{ij}, the derivation of which is rather tedious and uninteresting.

In order to ascertain in what kind of materials polarised wave systems will occur we again consult the strain-energy functions of the various crystals.

It is easily shown that in the case of the triclinic, monoclinic and rhombic crystals equations (11) and (12) are for all values of φ dependent of ϑ; in these crystals a polarised system of the P and SV type is impossible. The three remaining classes will be discussed seperately.

I. In the most general case of the hexagonal and rhombohedral system (c_{14} and $c_{15} \neq 0$) we have according to (11):

$$\tan \psi = \frac{(c_{14} \cos 2\,\varphi - c_{15} \sin 2\,\varphi) + (c_{13}+c_{44}) \sin\,\varphi \cot \vartheta}{(c_{14} \sin 2\,\varphi + c_{15} \cos 2\,\varphi) + (c_{13}+c_{44}) \cos\,\varphi \cot \vartheta}$$

and this is independent of ϑ if

$$\frac{c_{14} \cos 2\,\varphi - c_{15} \sin 2\,\varphi}{c_{14} \sin 2\,\varphi + c_{15} \cos 2\,\varphi} = \frac{\sin\,\varphi}{\cos\,\varphi}$$

from which follows: $\tan 3\,\varphi = c_{14}/c_{15}$ and $\psi = \varphi$.

It appears that the other equation (11) as well as both equations (12) are also satisfied by these values of φ and ψ. Thus in the three directions φ differing 60° from each other, wave systems polarised with respect to the plane of incidence are possible in any crystal of these systems.

As for the rhombohedral classes III, IV and V the constant c_{14} disappears $\tan 3\,\varphi = 0$; then the three directions φ coincide with the hexagonal axes of the crystal.

It will be remembered that in these crystals a Rayleigh system cannot occur; with respect to the main subject of this paper the above is of no interest. Nevertheless we have included this investigation because it may be of interest to the experimental research based on the propagation of ultra-sonic waves in crystals (see page 41).

In the case of the still more special classes where c_{14} and c_{15} are both equal to zero (namely all hexagonal classes and the rhombohedral classes VI and VII) the interesting fact arises that φ is undeterminate. Hence polarised systems are possible in all directions φ and ϑ; these crystals are in this respect identical with transversely isotropic media. The results concerning the existence and the velocity of Rayleigh waves which we have derived in chapter II are applicable to this kind of crystals.

II. For tetragonal crystals the second equations of (11) and (12) both yield $\psi = \varphi$: the plane of polarisation coincides with the plane of incidence. Again the coefficients of the first equation (12) are all equal to zero and the first equation (11) becomes

$$\tan 2\,\varphi = \frac{(c_{12}+c_{66}) \sin 2\,\varphi + 2\,c_{16} \cos 2\,\varphi}{(c_{11}-c_{66}) \cos 2\,\varphi + 2\,c_{16} \sin 2\,\varphi}$$

or $\tan 4\,\varphi = \dfrac{2\,c_{16}}{{}^1\!/_2\,(c_{11}-c_{12}) - c_{66}}$.

Accordingly in 4 directions φ which differ 45° from each other (in agreement with the properties of symmetry of these crystals) polarised wave systems consisting of one incident and one reflected wave occur. In these directions Rayleigh systems consisting of only two plane waves are possible.

If $c_{16} = 0$, which obtains for the tetragonal classes I, II, V and VI, these directions coincide with the crystallographic tetragonal axes.

III. An inspection of the strain-energy function of the cubic crystals show that in this respect these crystals are similar to the tetragonal crystals with $c_{16} = 0$; the four directions coincide with two of the axes and with the bisectrices of the angle between these axes.

IV. TWO ISOTROPIC MEDIA

§ 1. A solid and a liquid medium

In this chapter we extend the theory to special wave-systems occurring in two homogeneous isotropic and perfectly elastic semi-infinite bodies which are in contact along the plane $z = 0$. We start this investigation with the case in which the rigidity of one of these media is infinitely small ($\mu = 0$).

The procedure followed here is quite analogous to that of chapter I: we calculate the direction of the stress on the interface exerted by one longitudinal or one transverse wave travelling in the solid medium; the Rayleigh (Stoneley) equation expresses the equality of these two directions. The only difference with the case of chapter I is that here these stresses are altered by the appearance of a wave in the liquid body

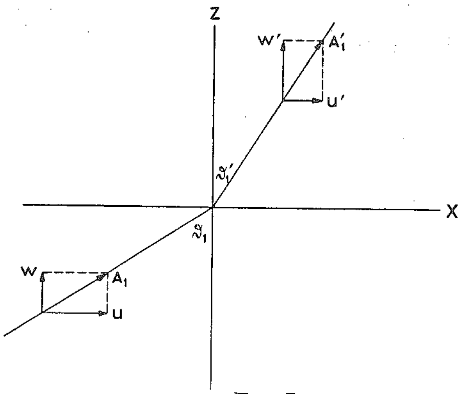

FIG. 7.

A longitudinal wave

$$A_1 \exp iv \left(t - \frac{x \sin \vartheta_1 + z \cos \vartheta_1}{c_1}\right)$$

travelling in the solid medium exerts on the plane $z = 0$ a stress given by

$$\left.\begin{aligned} T_{zx} &= -iv\varrho c_1 A_1 \,.\, 2 \sin^2\vartheta_2 \cot \vartheta_1 \\ &\text{and} \\ T_{zz} &= -iv\varrho c_1 A_1 \,.\, \cos 2\,\vartheta_2 \end{aligned}\right\} \quad (14)$$

where $\sin \vartheta_2 = \frac{c_2}{c_1} \sin \vartheta_1$,

c_1 = the longitudinal and c_2 = the transverse velocity. These expressions are easily derived from formulae (1) on page 12 by introducing ϑ_2. The phase-factor is omitted (as usual).

In the liquid medium a wave appears expressed by

$$A_1' \exp iv \left(t - \frac{x \sin \vartheta_1' + z \cos \vartheta_1'}{c'}\right)$$

which is connected with the movement in the solid body by the condition that the vertical motion only must be continuous. From this condition follows:

$$\frac{\sin \vartheta_1'}{c_1'} = \frac{\sin \vartheta_1}{c_1} \text{ and } A_1' \cos \vartheta_1' = A_1 \cos \vartheta_1.$$

This wave exerts on the interface a stress in the vertical direction equal to

$$T_{zz} = -iv\varrho' c_1' A_1' \text{ or } T_{zz} = -iv\varrho' c_1 A_1 \,.\, \frac{\cot \vartheta_1}{\cot \vartheta_1'}\,.$$

The result is that at $z = 0$ a stress-difference is caused, determined by

$$\left.\begin{aligned} \triangle T_{zx} &= -iv\varrho c_1 A_1 \,.\, 2\sin^2\vartheta_2 \cot\vartheta_1 \\ \triangle T_{zz} &= -iv\varrho c_1 A_1 \left(\cos 2\vartheta_2 - \frac{\varrho' \cot\vartheta_1}{\varrho \cot\vartheta_1'}\right) \end{aligned}\right\} \quad (15)$$

hence
$$\frac{\triangle T_{zx}}{\triangle T_{zz}} = \frac{2\sin^2\vartheta_2 \cot\vartheta_1}{\cos 2\vartheta_2 - \dfrac{\varrho' \cot\vartheta_1}{\varrho \cot\vartheta_1'}}.$$

In the same way we deal with the case of a transverse wave:

$$A_2 \exp iv \left(t - \frac{x \sin\vartheta_2 + z\cos\vartheta_2}{c_2}\right)$$

with
$$\left.\begin{aligned} T_{zx} &= -iv\varrho c_2 A_2 \,.\, \cos 2\vartheta_2 \\ T_{zz} &= +iv\varrho c_2 A_2 \,.\, 2\sin^2\vartheta_2 \cot\vartheta_2 \end{aligned}\right\} \quad (16)$$

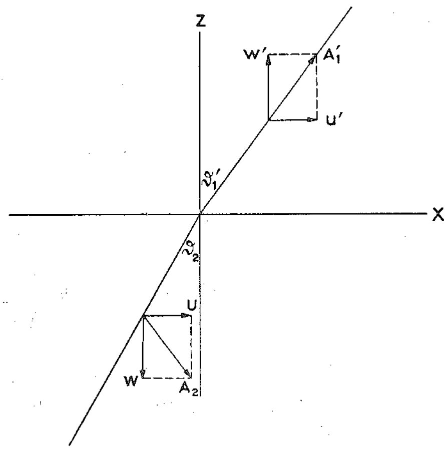

FIG. 8.

The accompanying wave in the liquid is propagated in the direction ϑ_1' given by $\dfrac{\sin\vartheta_1'}{\sin\vartheta_2} = \dfrac{c_1'}{c_2}$ with an amplitude A_1' satisfying $A_1' \cos\vartheta_1' = -A_2 \sin\vartheta_2$.

This wave exerts a stress $T_{zz} = iv\varrho' c_2 A_2 \,.\, \dfrac{1}{\cot\vartheta_1'}$.

The stress difference is then:

$$\left.\begin{aligned} \triangle T_{zx} &= -iv\varrho c_2 A_2 \,.\, \cos 2\vartheta_2 \\ \triangle T_{zz} &= +iv\varrho c_2 A_2 \left(2\sin^2\vartheta_2 \cot\vartheta_2 - \frac{\varrho'}{\varrho \cot\vartheta_1'}\right) \end{aligned}\right\} \quad (17)$$

which may be represented by a vector in the direction given by

$$\frac{\triangle T_{zx}}{\triangle T_{zz}} = -\frac{\cos 2\,\vartheta_2}{2\sin^2\vartheta_2 \cot\vartheta_2 - \dfrac{\varrho'}{\varrho \cot\vartheta_1'}}.$$

As the stress on the interface must be continuous the stress-differences have to disappear which is only possible if the two stress-differences (15) and (17) are in the same direction. This condition leads immediately to the equation

$$\frac{2\sin^2\vartheta_2 \cot\vartheta_1}{\cos 2\,\vartheta_2 - \dfrac{\varrho' \cot\vartheta_1}{\varrho \cot\vartheta_1'}} = -\frac{\cos 2\,\vartheta_2}{2\sin^2\vartheta_2 \cot\vartheta_2 - \dfrac{\varrho'}{\varrho \cot\vartheta_1'}}$$

or

$$\cos^2 2\,\vartheta_2 + 4\sin^4\vartheta_2 \cot\vartheta_1 \cot\vartheta_2 - \frac{\varrho' \cot\vartheta_1}{\varrho \cot\vartheta_1'} = 0 \tag{18}$$

This condition although necessary is obviously not sufficient; the stress on the interface disappears if moreover the amplitudes A_1 and A_2 satisfy the equation $(\triangle T_{zx})_1 = (\triangle T_{zx})_2$ or $c_1 A_1 \,.\, 2\sin^2\vartheta_2 \cot\vartheta_1 = c_2 A_2 \cos 2\,\vartheta_2$.

Equation (18) determines the directions ϑ of a three-wave system possible in two media one of which being liquid. Such a system will be a kind of Rayleigh wave if all waves decrease exponentially with increasing distance to the plane of contact.

§ 2. The existence of surface waves

In that case $\cot\vartheta_1$ and $\cot\vartheta_2$ are positive imaginary, and $\cot\vartheta_1'$ negative imaginary; with $\varepsilon = c_2/\sin\vartheta_2$ equation (18) becomes:

$$(2\,c_2^2-\varepsilon^2)^2 - 4\,\frac{c_2^3}{c_1}\sqrt{(c_2^2-\varepsilon^2)\,(c_1^2-\varepsilon^2)} + \frac{\varrho' c_1'}{\varrho c_1}\,\varepsilon^4 \sqrt{\frac{c_1^2-\varepsilon^2}{c_1'^2-\varepsilon^2}} = 0 \tag{19}$$

with $\varepsilon^2 < c_i^2$.

It is easily shown that such a root exists for every value of the material constants. For small values of ε^2 the left hand side of the equation is in first approximation equal to $-2\,c_2^2\varepsilon^2\,(1-c_2^2/c_1^2)$ which is negative as $c_1 > c_2$. Again if ε is equal to the smallest of the three velocities (which is either c_2 or c_1') the left hand side is positive. Consequently equation (19) always yields a root $\varepsilon^2 < c_i^2$.

For small values of the ratio between the acoustic resistances $\varrho' c_1'$ and ϱc_1 the transfer of the energy of waves from the solid to the liquid medium will be small too; then the propagation of waves in the solid body will be only slightly influenced by the liquid and therefore not very different from the propagation in a semi-infinite body bordering on the vacuum. In particular the velocity of the Rayleigh wave will then be almost equal to the velocity c_R of Rayleigh waves calculated in chapter I;

as this velocity is somewhat less than the velocity c_2 of the transverse waves it is to be expected that for small values of $\varrho' c_1'/\varrho_1 c_1$ the velocity of the surface wave travelling along the interface will not differ very much from c_2. However, the velocity discussed above is smaller than the velocities c_i, so that in the case where c_1' is much smaller than c_2 the velocity of the surface waves is also much smaller than c_2. This surface wave is therefore quite unlike the Rayleigh wave we expected to appear; apparently we must consider other roots of (19) in order to obtain this wave. Indeed it is evident that at small values of $\varrho' c_1'/\varrho c_1$ the velocity of ordinary Rayleigh waves satisfying the Rayleigh equation of the solid medium:

$$(2\, c_2^2 - \varepsilon^2)^2 - 4\, \frac{c_2^3}{c_1} \sqrt{(c_2^2 - \varepsilon^2)\,(c_1^2 - \varepsilon^2)} = 0$$

will approximately satisfy (19) too. As in the case under consideration $c_1' < c_2$ the third term of (19) is imaginary for $\varepsilon = c_R$ this root of (19) is complex with a real part $\approx c_R$.

The corresponding wave-system closely resembles the Rayleigh wave of chapter I, but the imaginary part of the root gives rise to a slight exponential decrease in the horizontal direction and a small sinusoidal variation in the vertical direction (CAGNIARD 1939).

§ 3. Two solid media

In the more general case where both media are solid we again follow the previous method; therefore we start with a wave system consisting of one longitudinal wave (A_1) in one of these media accompanied by a longitudinal (A_1') and a transverse wave (A_2') in the other medium. These waves are connected at $z = 0$ by the condition that the motion must be continuous; consequently:

$$A_1 \sin \vartheta_1 = A_1' \sin \vartheta_1' + A_2' \cos \vartheta_2'$$
$$A_1 \cos \vartheta_1 = A_1' \cos \vartheta_1' - A_2' \sin \vartheta_2'.$$

The solution of these equations is:

$$A_1' = \frac{c_1 A_1}{c_1' N} (1 + \cot \vartheta_1 \cot \vartheta_2')$$

$$A_2' = -\frac{c_1 A_1}{c_2' N} (\cot \vartheta_1 - \cot \vartheta_1')$$

with $N = 1 + \cot \vartheta_1' \cot \vartheta_2'$.

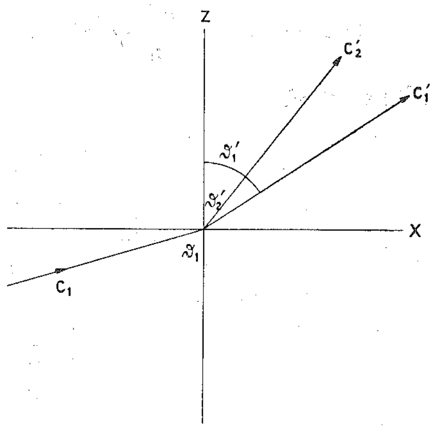

FIG 9.

Using the formulae of the stress-components of longitudinal and transverse waves (14) and (16) we obtain the following expressions for the stress-differences on the plane $z = 0$:

$$\triangle T_{zx} = -i\nu\,(\varrho c_1 A_1 . 2\sin^2\vartheta_2 \cot\vartheta_1 - \varrho' c_1' A_1' . 2\sin^2\vartheta_2' \cot\vartheta_1' - \varrho' c_2' A_2' . \cos 2\vartheta_2')$$
$$\triangle T_{zz} = -i\nu\,(\varrho c_1 A_1 . \cos 2\vartheta_2 - \varrho' c_1' A_1' . \cos 2\vartheta_2' + \varrho' c_2' A_2' . 2\sin^2\vartheta_2' \cot\vartheta_2').$$

By means of the above values of A_1' and A_2' the ratio between these quantities is easily reduced to

$$\frac{\triangle T_{zx}}{\triangle T_{zz}} = \frac{Ns\cot\vartheta_1 + \varrho'(\cot\vartheta_1 - \cot\vartheta_1')}{N(s+\varrho'-\varrho) + \varrho'\cot\vartheta_2'(\cot\vartheta_1 - \cot\vartheta_1')},$$

with $s = 2\varrho\sin^2\vartheta_2 - 2\varrho'\sin^2\vartheta_2'$.

In the same way we deal with the wave system consisting of one transverse wave (A_2) in one of the media. The amplitudes of the waves in the other medium are then:

$$A_1' = \frac{c_2 A_2}{c_1' N}(\cot\vartheta_2 - \cot\vartheta_2') \text{ and } A_2' = \frac{c_2 A_2}{c_2' N}(1+\cot\vartheta_2\cot\vartheta_1').$$

Substitution into the expressions for T_{xz} and T_{zz} leads finally to the ratio between the discontinuities of the stress-components at $z = 0$:

$$\frac{\triangle T_{zx}}{\triangle T_{zz}} = -\frac{N(s+\varrho'-\varrho) + \varrho'\cot\vartheta_1'(\cot\vartheta_2 - \cot\vartheta_2')}{Ns\cot\vartheta_2 + \varrho'(\cot\vartheta_2 - \cot\vartheta_2')}.$$

The continuity of the stress at $z = 0$ requires the disappearance of these discontinuities, which can only be effected by the above wave systems A_1, A_1', A_2' and A_2, A_1', A_2' if the two ratio's $\triangle T_{zx}/\triangle T_{zz}$ are equal to each other. This equation becomes after some reduction:

$$(s+\varrho'-\varrho)^2 + (s+\varrho')^2\cot\vartheta_1\cot\vartheta_2 + (s-\varrho)^2\cot\vartheta_1'\cot\vartheta_2'$$
$$+ s^2\cot\vartheta_1\cot\vartheta_2\cot\vartheta_1'\cot\vartheta_2' - \varrho\varrho'(\cot\vartheta_1\cot\vartheta_2' + \cot\vartheta_1'\cot\vartheta_2) = 0$$

which is called the Stoneley equation of the two media.

The corresponding wave system constitutes a surface wave if all waves decrease exponentially with increasing distance to the interface; hence $\cot\vartheta_1$ and $\cot\vartheta_2$ are positive imaginary, $\cot\vartheta_1'$ and $\cot\vartheta_2'$ negative imaginary.

§ 4. The existence of the Stoneley wave

As all waves decrease exponentially with increasing $|z|$ each plane wave and therefore the whole wave system travels parallel to the boundary with the same phase-velocity $\varepsilon = c_i/\sin\vartheta_i$.

This quantity fulfills the Stoneley equation:

$$\{2(\mu-\mu') + \varepsilon^2(\varrho'-\varrho)\}^2 - \{2(\mu-\mu') + \varepsilon^2\varrho'\}q_1 q_2 - \{2(\mu-\mu') - \varepsilon^2\varrho\}^2 q_1' q_2'$$
$$+ 4(\mu-\mu')^2 q_1 q_2 q_1' q_2' - \varrho\varrho'\varepsilon^4(q_1 q_2' + q_1' q_2) = 0$$

where $q_i = \sqrt{1 - \dfrac{\varepsilon^2}{c_i^2}}$ and $q_i' = \sqrt{1 - \dfrac{\varepsilon^2}{c_i'^2}}$ ($i = 1$ or 2) are real numbers.

A surface wave exists if this equation yields a real root of which the absolute value is smaller than the smallest of the 4 velocities c_i; we suppose that $c_2' < c_2$.

At values of ε^2 about equal to zero the left-hand side L of the equation may be written as

$$L = \left\{ (\mu - \mu') \left(\frac{1}{c_1'^2} + \frac{1}{c_2'^2} \right) + 2\varrho' \right\} \left\{ (\mu - \mu') \left(\frac{1}{c_1^2} + \frac{1}{c_2^2} \right) - 2\varrho \right\} \varepsilon^4 + \left\{ \dots \right\} \varepsilon^6 + \dots$$

The coefficient of ε^4 is

$$-\varrho\varrho' \left\{ \mu \left(\frac{1}{\mu'} + \frac{1}{\lambda' + 2\mu'} \right) + \frac{\lambda' + \mu'}{\lambda' + 2\mu'} \right\} \left\{ \mu' \left(\frac{1}{\mu} + \frac{1}{\lambda + 2\mu} \right) + \frac{\lambda + \mu}{\lambda + 2\mu} \right\},$$

which is negative as $\lambda > -{}^2/_3\,\mu$; hence L is negative for sufficiently small values of ε^2.

It follows that a root $< c_2'$ exists if L is positive for $\varepsilon = c_2'$; substitution of this value leads to the relation

$$\left\{ 2\frac{\mu}{\mu'} - \left(1 + \frac{\varrho}{\varrho'} \right) \right\}^2 - \left(2\frac{\mu}{\mu'} - 1 \right)^2 q_1 q_2 - \frac{\varrho}{\varrho'} q_2 q_1' > 0 \qquad (20)$$

with $$q_1 = \left(1 - \frac{c_2'^2}{c_1^2} \right)^{1/2}, \quad q_2 = \left(1 - \frac{c_2'^2}{c_2^2} \right)^{1/2}, \quad q_1' = \left(1 - \frac{c_2'^2}{c_1'^2} \right)^{1/2}.$$

We denote the ratios of the squares of the velocities as follows:

$$\frac{c_2^2}{c_1^2} = \alpha \left(= \frac{1}{2 + \lambda/\mu} \right), \quad \frac{c_2'^2}{c_1'^2} = \alpha' \left(= \frac{1}{2 + \lambda'/\mu'} \right) \text{ and } \frac{c_2'^2}{c_2^2} = \beta \left(= \frac{\mu'\varrho}{\mu\varrho'} \right);$$

the quantities q then become:

$$q_1 = (1 - \alpha\beta)^{1/2}, \quad q_2 = (1 - \beta)^{1/2}, \quad q_1' = (1 - \alpha')^{1/2},$$

and inequality (20) can be reduced to:

$$\left(\frac{\mu}{\mu'} \right) \{ (2 - \beta)^2 - 4\, q_1 q_2 \} - \frac{\mu}{\mu'} (4 - 2\beta - 4\, q_1 q_2 + \beta\, q_1' q_2) + (1 - q_1 q_2) > 0.$$

At given values of α and α' (which depend only on the ratios λ/μ and λ'/μ') the region in the $(\mu/\mu', \beta)$ plane where this inequality holds is bounded by the curve:

$$\left(\quad \right)^2 \{ (2 - \beta)^2 - 4\, q_1 q_2 \} - \frac{\mu}{\mu'} (4 - 2\beta - 4\, q_1 q_2 + \beta\, q_1' q_2) + (1 - q_1 q_2) = 0.$$

The general appearance of this curve is easily obtained by means of the following remarks:

1. for small values of β the equation becomes

$$4\left(\frac{\mu}{\mu'}\right)^2 (1-\alpha) + 2\left(\frac{\mu}{\mu'}\right)(1+2\,\alpha) - (1+\alpha) = 0$$

hence $\frac{\mu}{\mu'} = \frac{-(1+2\,\alpha) + \sqrt{5+4\,\alpha}}{4\,(1-\alpha)}$. This value varies from $\mu/\mu' = 0.309$ if $\alpha = 0$ (the case of an incompressible medium) to $\mu/\mu' = 0.319$ if $\alpha = {}^1/_3$ (a Poisson-medium with $\lambda = \mu$).

2. if we substitute $\beta = 1$ we obtain $(\mu/\mu' - 1)^2 = 0$, so that the line $\beta = 1$ is a tangent to the curve in $\mu/\mu' = 1$.

3. if for a certain value of β the coefficient of $(\mu/\mu')^2$ disappears this value determines an asymptote parallel to the μ/μ'-axis.

Comparing the equation

$$(2-\beta)^2 - 4\,q_1\,q_2 = 0 \text{ or } (2-\beta)^2 = 4\sqrt{1-\alpha\beta}\sqrt{1-\beta} \qquad (21)$$

with the Rayleigh equation:

$$\left(2-\frac{c_R^2}{c_2^2}\right)^2 = 4\sqrt{1-\frac{c_R^2}{c_1^2}}\sqrt{1-\frac{c_R^2}{c_2^2}}$$

where c_R = the Rayleigh velocity, we immediately see that the root of (21) is $\beta = c_R^2/c_2^2$.

In the case of an incompressible medium ($\lambda = \infty$) the asymptote is $\beta = 0.9126$ and in the case of a Poisson-medium ($\lambda = \mu$): $\beta = 0.8453$.

In fig. 10 the curve $L\,(\varepsilon = c_2') = 0$ has been drawn for the case of two Poisson-media; the equation of this curve is

$$\left\{(2-\beta)\frac{\mu}{\mu'} - 1\right\}^2 - \left(2\frac{\mu}{\mu'} - 1\right)^2 \sqrt{(1-{}^1/_3\,\beta)\,(1-\beta)} - \beta\,\frac{\mu}{\mu'}\sqrt{{}^2/_3\,(1-\beta)} = 0.$$

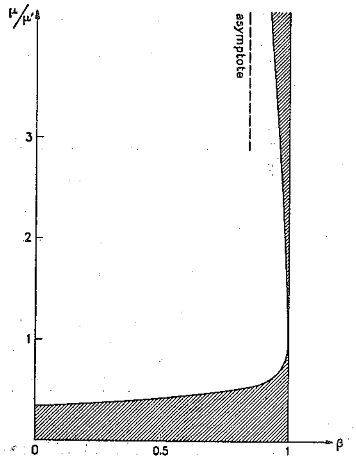

FIG. 10.

As $L > 0$ if $\mu'/\mu = 0$ the region where a root of the Stoneley equation exists is confined between the curve and the β-axis; moreover the assumption $c_2' < c_2$ limits this domain to the left-hand side of the line $\beta = 1$ (SCHOLTE, 1943). The region is represented by the shaded area of fig. 10.

V. PRACTICAL APPLICATIONS

§ 1. Seismology

Seismic data show that the propagation of elastic disturbances through the earth is approximately the same in all horizontal directions. Consequently when dealing with seismic waves we may consider the Earth as horizontally isotropic, so that we are entitled to use the theory of Rayleigh waves in an isotropic (chapter I) or in a transversely isotropic medium (chapter II).

1. As far as the polarisation of the Rayleigh waves is concerned it does not matter which of these two media we assume to exist as in both cases the movement takes place in the plane of incidence. This property has been used by HILLER (1950) in his method of determining the azimuth of the focus of an earthquake. It is evident that this direction is known if the movement of an observed Rayleigh wave has been reconstructed from the seismic records.

Again in both media a transverse wave (*SH*), which vibrates perpendicular to the plane of incidence, is possible in any direction. The identification of this purely horizontal movement in a seismogram is of some importance as the direction of this movement determines the plane of incidence. In this way HILLER obtains a second indication of the azimuth of the epicentre.

2. Other properties of the Rayleigh wave (and of all other waves too) as for instance its velocity depend on the elastic constants of the medium and will therefore be different for the two media mentioned above; measurements of these properties may be used in order to determine whether the body transversed by these waves is isotropic in all directions or merely in all horizontal directions.

The various layers of the crust of the Earth contain anisotropic crystalline material; however, the crystals are generally orientated at random so that the total effect over large areas will be that each layer may be considered as isotropic in all directions.

A noteworthy exception to this general state of affairs is formed by a layer of ice covering a lake or a part of the sea; apart from the uppermost layer the ice crystals are all orientated with the principal axis in the vertical direction.

In each horizontal plane the ice crystals appear to form groups of crystals with the hexagonal axes in the same direction; as the orientations of these clusters show a random distribution the overall effect will be that a sheet of ice behaves itself as transversely isotropic. Hence for waves lenghts small in comparison to the thickness of the sheet the velocity of the Rayleigh wave in ice is given by equation (6).

3. Returning to the commonly occurring case of a formation consisting of several layers of isotropic material we observe that such a structure may be considered as vertically anisotropic inasmuch as the time necessary for a seismic wave to travel a certain distance AB (fig. 11a) is a function of the angle ϑ.

It is evident that the resemblance of a stratified medium to a homogeneous trans-

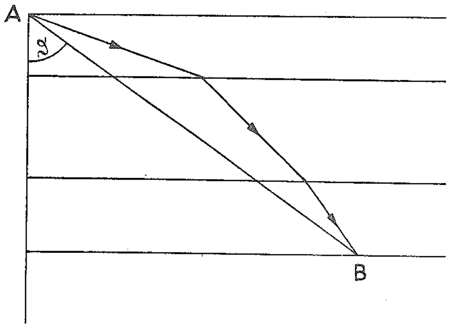

FIG. 11a.

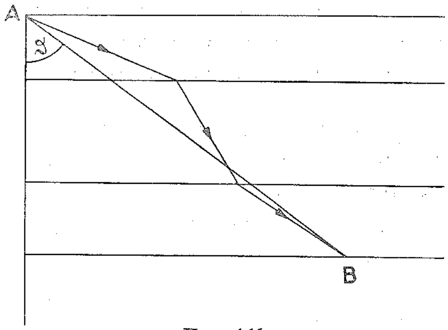

FIG. 11b.

versely isotropic medium is in general very slight as the actual path of propagation deviates strongly from the straight-line path followed in such a homogeneous medium. However, this deviation becomes smaller if the velocities of the waves in the layers are alternately great and small, as in fig. 11b; if moreover the thicknesses of the layers are small in comparison with the length of the waves the actual path may in first approximation be regarded as straight. In such a body a disturbance is propagated along (approximately) straight lines with a velocity which is a function of the direction ϑ; in other words this body is transversely isotropic. This kind of stratification is not unfrequently encountered in geophysical prospecting (UHRIG & VAN MELLE, 1955); in connection with this circumstance, POSTMA (1955) investigated the relation between the elastic constants of the layered structure and those of the equivalent homogeneous transversely isotropic medium.

As this paper is primarily concerned with the properties of Rayleigh waves, we use his results in order to calculate the velocity of these waves (equation 6), the values of the attenuation factors cot ϑ_1 and cot ϑ_2 (equation 4) and the ratio of the horizontal and vertical amplitudes (to be calculated from the condition that the stress is equal to zero at the boundary).

In the case of a formation consisting of alternating layers of limestone (thickness d_1) and sandstone (d_2) POSTMA obtained the following elastic constants of the substitute homogeneous anisotropic medium:

	ϱ	A	C	F	L	
$d_1/d_2 = 1/3$	3.4 gr/cm³	3.36	2.46	0.97	0.74	$\times\ 10^{11}$ dyne/cm²
$d_1/d_2 = 3$	2.6 gr/cm³	6.25	4.57	1.74	1.40	

The velocity of the Rayleigh waves is in the first medium 0.53 km/sec (and 0.70 km/sec in the second medium); the attenuation factors are 0.22 i and 1.37 i (0.19 i and 1.69 i). The ratio between horizontal and vertical amplitudes of these waves is in both cases about equal to 0.6.

This result is of some interest in connection with the discrepancy between the observed ratio between the horizontal and vertical amplitude, which is often greater than 1, and the theoretically calculated ratio, which is for isotropic media about 0.7. The suggestion is sometimes made that this discrepancy is due to some kind of heterogeneity in the neighbourhood of the seismographic station. The above result indicates that it is doubtful whether such a heterogeneity will be sufficient to alter the theoretical ratio in any decisive way.

4. In the theory dealt with in this paper it is assumed that the media are semi-infinite; therefore it is applicable only if the wavelength is small in comparison with the thickness of the medium.

The periods of the long waves observed in seismograms are about 15 sec so that the length of these earthquake-waves is of the order of 50 km; as the thickness of the crust of the Earth is about 40 km the above theory does not obtain for these waves.

It may be expected that in this case the eigenfrequencies of the elastic structure consisting of one or more layers will be predominant in the seismograms; the theory based on this conception has been worked out by several authors in order to deal with some particular problem. For instance: the appearance of a special kind of wave (T-phase), by EWING and his collaborators; the occurrence of a maximum amplitude of the long waves (Airy-phase), by the same authors (1950); the stratification of the crust, by STONELEY (1955); the existence of a low-velocity layer, by GUTENBERG (1954); the phenomenon of microseisms, by the present author (1943).

5. The occurrence of a surface wave travelling along the interface between two media is only of importance for an observer situated at a short distance from this surface, since the amplitude of these waves decreases rapidly with increasing distance to the plane of discontinuity. Thus Stoneley waves connected with the boundary between the crust and the mantle of the Earth are too small to be identified in the generally rather complicated coda of a seismogram.

This unfortunate circumstance does not arise in the case of surface waves propagated along the bottom of the ocean; the velocity ε of these waves is determined by equation (19):

$$(2c_2^2-\varepsilon^2)^2-4\frac{c_2^3}{c_1}\sqrt{c_2^2-\varepsilon^2}\sqrt{c_1^2-\varepsilon^2}+\frac{\varrho' c_1'}{\varrho c_1}\varepsilon^4\sqrt{\frac{c_1^2-\varepsilon^2}{c_1'^2-\varepsilon^2}}=0$$

where $c_1 = 6.5$ km/sec, $c_2 = 3.8$ km/sec (velocities in the silicic crust) $c_1' = 1.5$ km/sec (velocity of sound in water) and $\varrho'/\varrho = 0.36$.

The relevant roots of this equation are nearly 1.50 and $3.40+0.12\,i$.

If the seismic waves are generated by a disturbance in the solid medium, a Rayleigh wave with the velocity 3.40 km/sec will appear; this velocity is about equal to that of Rayleigh waves travelling along a free surface (3.42 km/sec) which is only slightly changed by the presence of the fluid layer. As the ratio between the acoustical resistances is rather small ($\varrho' c_1'/\varrho c_1 = 0.08$) the propagation in the solid medium is not greatly changed by the ocean.

A wave with the velocity ε, which is slightly smaller than the velocity of sound in water, will be observable if the primary disturbance takes place in the fluid medium, as happens in the case of a seismic disturbance of the ocean's bottom or at underwater explosions.

§ 2. Crystallography

In a recently developed technique the elastic constants of an anisotropic or crystalline body are determined by measuring the travel-time of ultrasonic waves along a path (*AB* in fig. 12) in these media (DE KLERK & MUSGRAVE 1952).

The waves are generated by a vibrating quartz crystal (*A*) in contact with one of the two parallel sides cut perpendicular to the principal axis of symmetry (*z*-axis) of the specimen to be examined. A similar crystal (*B*) adhering to the other side is used as a detector.

Usually these crystals are cut in a rectangular shape thus ensuring that the vibrations are polarised in the direction of a diagonal.

In general three waves generated at *A* travel in all directions with velocities given by equation (9). The travel time of only the first arrival in *B* is measured. If the orientation of the axes of the specimen is known it is possible to measure the angles ϑ and φ; entering these data and the velocity into equation (9) some information about the elastic constants is obtained.

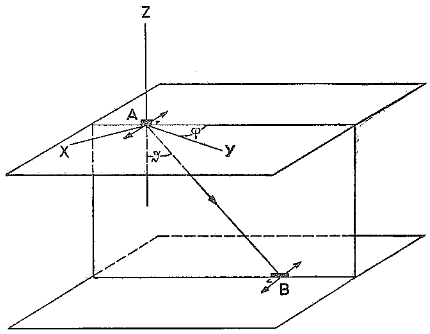

FIG. 12.

This equation is much more simple if polarised waves are used; as we show in chapter III, in several classes of crystals, in some directions waves can be propagated which vibrate transversely and parallel to the surface (*SH* waves). The velocity *c* of these waves obtained from the equations of motion (10) by putting $p' = r = 0$ and $\psi = \varphi$ (the auxiliary angle ψ used in these equations afterwards proved to be equal to the angle φ in fig. 12).
Hence *c* is determined by

$$\tfrac{1}{2}(a_{11}-a_{12}) \cos 2\varphi + a_{12} \sin 2\varphi - \tfrac{1}{2}(a_{11}+a_{22}) + \varrho\varepsilon^2 = 0$$

where a_{ij} are given by formulae (7) on page 12 and $\varepsilon = c/\sin\vartheta$.

As the direction of the waves must satisfy equations (11) and (13) with $\psi = \varphi$ we have

$$\tan 2\varphi = \frac{2\, a_{12}}{a_{11}-a_{22}};$$

it follows that

$$\varrho\varepsilon^2 = \tfrac{1}{2}(a_{11}+a_{22}) - \tfrac{1}{2}\{(a_{11}-a_{22})^2 + 4\, a_{12}{}^2\}^{1/2}. \tag{22}$$

These *SH* waves are generated only if the primary movement in *A* is horizontal and if moreover the angle φ between this movement and the *y*-axis satisfies equations (11) and (13). This axis is chosen perpendicular to a plane of symmetry of the crystal through the principal axis; when such a plane does not exist the *y*-axis is chosen along a digonal axis. In the case of cubic crystals one of the principal axes is taken as *y*-axis.

In order to establish the situation where *SH* waves are propagated towards the detector *B* both quartz crystals must be placed in such a way on the surfaces of the crystal that their vibrations are parallel to these surfaces; again the detector is turned perpendicular to the generator.

Rotating the crystal to be examined round its principal axis while the quartz crystals remain fixed the response of the detector will be zero if the angle φ fulfills equations (11) and (13). When now turning the detector over an angle of 90° a maximal response will be observed and the travel time of a pulse can be measured.

As in crystals of the hexagonal and rhombohedral, the tetragonal and the cubic classes *SH* waves are possible (in the remaining classes it is not possible to satisfy equations (11) and (13) simultaneously) this method is useful for many of the most frequently occurring crystals.

In the following these three classes will be considered seperately.

1. Hexagonal and rhombohedral crystals.

In this case the quantities a_{ij} appear to be:

$$a_{11} = \{c_{11}\cos^2\varphi + {}^1/_2\,(c_{11} - c_{12})\sin^2\varphi\} + 2\,(c_{14}\sin\varphi + c_{15}\cos\varphi)\cot\vartheta + c_{44}\cot^2\vartheta$$
$$a_{22} = \{c_{11}\sin^2\varphi + {}^1/_2\,(c_{11} - c_{12})\cos^2\varphi\} - 2\,(c_{14}\sin\varphi + c_{15}\cos\varphi)\cot\vartheta + c_{44}\cot^2\vartheta$$
$$a_{12} = {}^1/_2\,(c_{11} + c_{12})\sin\varphi\cos\varphi + 2\,(c_{14}\cos\varphi - c_{15}\sin\varphi)\cot\vartheta.$$

As shown on page 30 the angle φ is given by $\tan 3\,\varphi = c_{14}/c_{15}$.
The velocity equation (22) then becomes, after some reduction:

$$\varrho c^2 = c_{66}\sin^2\vartheta + c_{44}\cos^2\vartheta \pm \sqrt{c_{14}{}^2 + c_{15}{}^2}\,\sin 2\,\vartheta.$$

Measuring c in several directions ϑ the constants c_{44}, c_{66} and $\sqrt{c_{14}{}^2 + c_{15}{}^2}$ are obtained; moreover if the orientation of the axes of the crystal are known the determination of φ yields the value of c_{14}/c_{15}, so that c_{14} and c_{15} are known too.

In all hexagonal and in some rhombohedral crystals $c_{14} = c_{15} = 0$; as φ is then undeterminate the response of *B* is zero if *A* and *B* are perpendicular. Apparently no adjustment of the crystal is needed as *SH* waves are emitted in all directions.

2. Tetragonal crystals.

The quantities a_{ij} are:

$$a_{11} = (c_{11}\cos^2\varphi + 2\,c_{16}\sin\varphi\cos\varphi + c_{66}\sin^2\varphi) + c_{44}\cot^2\vartheta$$
$$a_{22} = (c_{11}\sin^2\varphi - 2\,c_{16}\sin\varphi\cos\varphi + c_{66}\cos^2\varphi) + c_{44}\cot^2\vartheta$$
$$a_{12} = c_{16}\cos 2\,\varphi + {}^1/_2\,(c_{12} + c_{66})\sin 2\,\varphi.$$

As $\tan 4\,\varphi = \dfrac{4\,c_{16}}{c_{11} - c_{12} - 2\,c_{66}}$ (see page 30) the velocity equation becomes:

$$\varrho c^2 = {}^1/_4\{(c_{11} - c_{12} + 2\,c_{66}) \pm \sqrt{(c_{11} - c_{12} - 2\,c_{66})^2 + 16\,c_{16}{}^2}\}\sin^2\vartheta + c_{44}\cos^2\vartheta.$$

Using these relations measurements of c and φ yield the value of c_{44}, c_{66}, c_{16} and c_{11}—c_{12}.

3. Cubic crystals.

In this case the formulae just obtained for tetragonal crystals are applicable with $c_{66} = c_{44}$ and $c_{16} = 0$; hence $\tan 4\,\varphi = 0$, or $\varphi = 0$ and $\varphi = \pi/4$. The corresponding velocities are given by

$\varrho c^2 = c_{44}$, if $\varphi = 0$,
$\varrho c^2 = {}^1/_2\,(c_{11}-c_{12}) \sin^2\vartheta + c_{44} \cos^2\vartheta$, if $\varphi = \pi/4$,

yielding the values of c_{44} and c_{11}—c_{12}.

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