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# CONSISTENCY OF APPROXIMATIONS IN DISCONTINUOUS FIELDS OF MOTION IN THE ATMOSPHERE WITH AN INTRODUCTION TO THE USE OF GENERALIZED FUNCTIONS OR DISTRIBUTIONS IN METEOROLOGY

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# GENERAL INTRODUCTION

Although the basic equations of dynamic meteorology are known already for years, their general solution is almost completely unknown. Drastic simplifications are necessary to obtain at least some information about the nature of the general solution. Roughly speaking, there are two types of simplifications which have been successfully applied. In the first type the complications due to the non-linearity of the equations are overcome by neglecting terms of minor importance. In the second attack one studies the exact solutions of the equations for simplified models. One may speak of mathematical and of physical simplification to denote their difference.

In the present publication a study is offered on the interplay between the method of mathematical approximation on the one hand and one of the most important physical models on the other hand *viz.* the Norwegian concept of a frontal surface. At its proper place a more specific and concrete description of this theme will be given. For the present it may be sufficient to remark that the main problem will be to develop and to apply a test for the correctness of approximations when frontal discontinuities are present in the atmosphere. The discussion, however, will not be limited to frontal surfaces only, for it will become clear that the results are valid for a larger class of atmospheric situations and have implications for all situations in which a region of strong baroclinity is present in the atmosphere. For the present study it is therefore hardly of importance whether frontal discontinuities in the original sense of the Norwegian school exist or not. The validity of the results found is even independent of any opinion which one may have in the present discussion on the revision of the concept of a front.

The study is divided into three parts. In the first part, the main mathematical tool will be described. It consists of the modern theory of distributions as developed by SCHWARTZ and others. For reasons which will be explained in full in the first part, this important new concept is not only almost unknown to meteorologists, but even to most scientists working in applied mathematics, notwithstanding the fact, that the main domain of application of the concept of distributions is to be found in applied mathematics. It will therefore be necessary to give an introduction to distribution theory. Since the author's task in this introduction is almost exclusively of a didactical nature, he did not try to find the shortest way to introduce distributions, but on the contrary he has thought it necessary to show that there are many ways leading to the same theory.

In the second part a critical revision of the basic equations of dynamic meteorology from the point of view of the theory of distributions has been given and the necessary collection of formulae has been derived. The third part is devoted to the main problem already formulated above.

PART I

# THEORY OF DISTRIBUTIONS

Distributions restore to mathematics the freedom that physicists have always desired.

IRVING KAPLANSKY

## 1. Introduction

The main mathematical tool which will be used in the present study is the theory of *distributions*. In the following an introduction to this new concept will be given, but before this can be done the reader must have some knowledge of the historical situation in which the problems which are solved by the distribution theory became a challenge to mathematicians. Otherwise he would not be able to appreciate the appraisal made by G. TEMPLE [74] [1]) in a Presidential Address to the London Mathematical Society "One of the great events in the contemporary history of mathematics is the invention of the theory of distributions by LAURENT SCHWARTZ".

The problem goes back to HEAVISIDE and is one of the many hard nuts which he gave mathematicians to crack. It is well known that in his book [33] and papers he repeatedly made use of a function which was later on called HEAVISIDE's *unit function* $U_a(x)$. It is defined by

$$(1.1) \qquad U_a(x) = \begin{cases} 0 \text{ for } x < a \\ 1 \text{ for } x > a \end{cases} \qquad \text{(see fig. 1)}$$

HEAVISIDE used other discontinuous functions also, but his unit function is typical and will be used further on as a good starting point for discussion. Although from the traditional point of view this was complete nonsense he differentiated his unit function many times, to the astonishment of all who had to admit that the final results of his calculations were quite correct. At first, however, the main problems taken from the work of HEAVISIDE which mathematicians tried to solve were his calculus of differential operators and his use of divergent series and the problem of the non-existent but useful differential quotients of the unit function did not obtain much attention.

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[1]) Numbers in square brackets refer to the list of references (page 85).

In the year 1925 many mathematicians became interested in the problems of the new quantum mechanics. At that time DIRAC [18, 19] began to use his

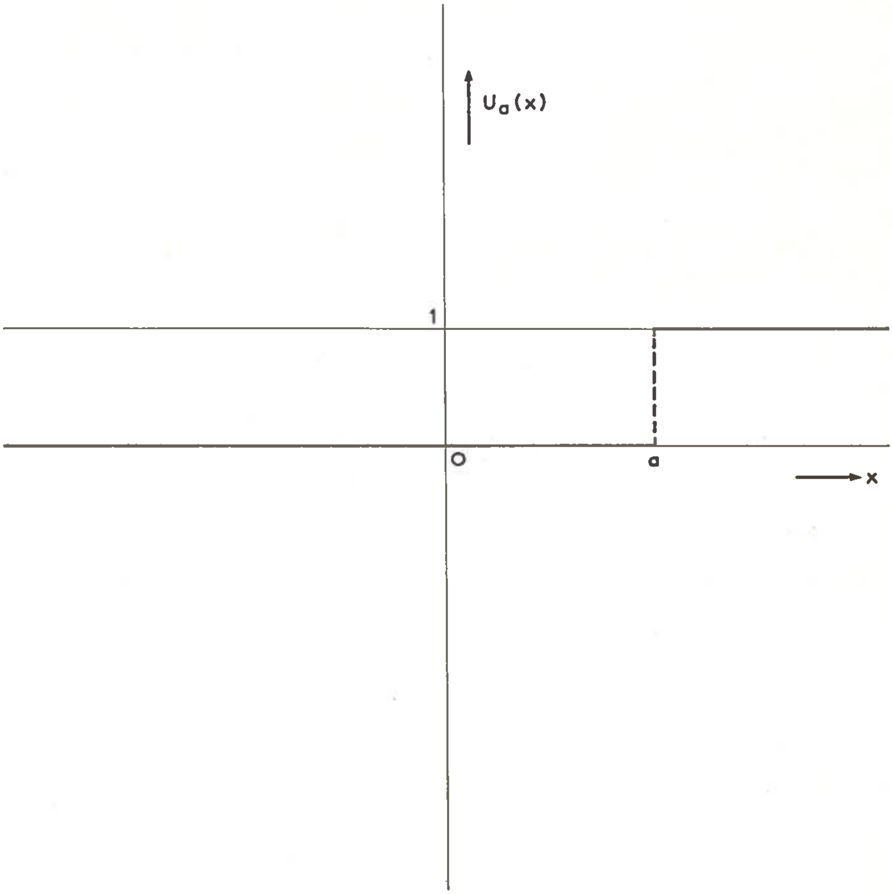


Figure 1.

famous *delta function*. It soon became clear that this function was the same as the first differential quotient of the unit function

$$\delta_a(x) = \frac{d\,U_a(x)}{dx}. \tag{1.2}$$

For the delta function was defined by its strange property that for "every" function $f(x)$ it is true that

$$(1.3) \qquad \int_{-\infty}^{+\infty} \delta_a(x) f(x)\, dx = f(a)$$

The relation between the two remarkable functions is "proved" by partial integration

$$(1.4) \qquad \int_{-\infty}^{+\infty} \delta_a(x) f(x)\, dx = \int_{-\infty}^{+\infty} f(x) \frac{d\,U_a(x)}{dx}\, dx =$$

$$= f(x)\, U_a(x) \Big/_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x)\, U_a(x)\, dx = 0 - \int_{a}^{+\infty} f'(x)\, dx =$$

$$= [-f(x)]_a^{+\infty} = f(a),$$

which is a "correct" reasoning if one supposes

$$(1.5) \qquad f(+\infty) = f(-\infty) = 0,$$

which on physical grounds often may be done.

So the attention of the mathematicians was for the second time drawn to the problem of the non-existent but useful differential quotients of discontinuous functions. A first solution was soon found, mainly by J. V. NEUMANN in his now classical theory of the spectral resolution of bounded linear operators in HILBERT spaces. This is not the place to describe this theory. In a certain sense the solution was no solution at all, as in V. NEUMANN's theory the astounding delta function was completely eliminated. Its basic idea was to rewrite the definition equation (1.3) for the delta function as

$$(1.6) \qquad \int_{-\infty}^{+\infty} \delta_a(x) f(x)\, dx = \int_{-\infty}^{+\infty} f(x)\, d\,U_a(x) = f(a),$$

in which the second integral is to be interpretated as a LEBESGUE-STIELTJES integral. So V. NEUMANN merely showed that it was unnecessary to use functions like the DIRAC function at all.

HEAVISIDE, however, did not only use the first derivative of his unit function but higher differential quotients as well and others followed him therein. For the use of higher derivatives V. NEUMANN's theory offered no solution. So the problem remained of the correct definition of the higher derivatives of the unit function, which was solved for the first time by SCHWARTZ in his theory of distributions. Finally one gets the following formulation for the basic purpose of the distribution theory:

*A theory in which in a certain sense differential-quotients of discontinuous functions exist.*

It is obvious that the derivatives of discontinuous functions cannot be ordinary functions, so that in one way or another a generalization of the concept of the function should be introduced.

At the end of this introduction three remarks must be made.

**1.1** After the distribution theory was developed by SCHWARTZ and others, it has been shown, that just as V. NEUMANN did with the delta function, the higher derivatives can be eliminated by a generalization of the STIELTJES integral. This was done by BURKILL [12]. However, the distribution theory seems to be an easier instrument than the quite complicated generalization of the STIELTJES integral which in practical problems is difficult to handle.

**1.2** The term distribution suggests a relation to the concept of the probability distribution. Indeed, a probability distribution is a distribution in the sense of SCHWARTZ, whose concept however is much more general. So there is an ambiguity in the term distribution, which for some scientists is a sufficient reason to use the term *generalized function*. Perhaps *distributor* – a term which can be translated in almost any language used in science – would be a better term.

**1.3** When in the next sections integrals occur, they usually should be interpreted as LEBESGUE integrals. Some of the theorems quoted are also valid for RIEMANN integrals but this will not be stated explicitely. There exists nowadays, short ways to introduce the LEBESGUE integral, which are quite in the spirit of the applied mathematician, as *e.g.* that which may be found in the books of RIESZ and SZ.-NAGY [63] and of ZAMANSKY [80].

## 2. Distributions as defined by Schwartz

**2.1** After an earlier attempt by SOBOLOV [70], the first complete theory of distributions was given by LAURENT SCHWARTZ in a two volume book [67]. SCHWARTZ, at that time professor at the University of Nancy, France, was largely influenced by the style of the French group of mathematical writers, who collectively publish their work under the pen-name BOURBAKI. So in his book the theory is developed on the basis of the theory of the locally convex linear topological vectorspaces. The topological setting of his theory was not predestinated to give it the desired familiarity under the applied mathematicians. One of the main tasks of all who have worked after SCHWARTZ on distributions, has been to eliminate as much topology as possible from it and this task has been performed with great success. Later on other theories, claimed to be elementary in some sense will be described, but a few words must be said on SCHWARTZ original theory, after which the reader may forget all about it and restrict himself to one of the later theories of his choice.

**2.2** Consider functions $\varphi(x)$ of the real variable $x$. SCHWARTZ's definition of the distribution $T$ is that it is a linear functional, that is a linear operation which assigns to $\varphi$ a number $a$:

(2.1) $$T[\varphi] = a.$$

It is obvious that not every function $\varphi$ can be subjected to an arbitrary operation $T$. Indeed if $T$ is a rather "wild" operation, then $\varphi$ should be a rather "tame" function. So it becomes necessary to specify the class of allowed operations and to restrict the class of functions $\varphi$.

First define the *support* of a function as the smallest closed set of the variable $x$, outside of which $\varphi(x)$ is identically zero. Consider now the collection $\mathfrak{D}$ of functions $\varphi(x)$ which have derivatives of all orders and have a bounded support. This class $\mathfrak{D}$ is called the *vector space of testfunctions.* By this wording it is said that if $\varphi_1(x)$ and $\varphi_2(x)$ are testfunctions then any linear combination of them with arbitrary real coefficients $a_1$ and $a_2$

(2.2) $$a_1\varphi_1(x) + a_2\varphi_2(x)$$

is also a testfunction.

In the space $\mathfrak{D}$ a definition of convergence is given as follows. It is said that an infinite sequence of testfunctions $\varphi_i(x)$ $(1 \leqq i \leqq \infty)$ converges to zero

(2.3) $$\lim_{i=\infty}{}^{*} \varphi_i(x) = 0$$

if (i) all $\varphi_i(x)$ have their support in the same bounded set, and (ii) the $\varphi_i(x)$ as well as their derivatives $\varphi_i^{(k)}(x)$ of all orders converge for every $k$ uniformly in $x$ in the ordinary sense to zero. This definition gives SCHWARTZ the opportunity to introduce the theory of locally convex linear topological vectorspaces.

A *distribution* $T$ is now defined as a linear functional

(2.4) $$T[\varphi] = a$$

which assigns to every testfunction a real number $a$ and which is supposed to be continuous in the following sense. If $\varphi_i$ is a sequence of testfunctions which according to the definition given above converges to zero then $T[\varphi_i]$ converges to zero.

The distribution is thus a continuous linear functional on $\mathfrak{D}$, which gives SCHWARTZ the opportunity to consider the whole collection of distributions as the dual space $\mathfrak{D}'$ of $\mathfrak{D}$. It must be observed that at the time at which SCHWARTZ wrote his book, only BANACH spaces were known, and that he had to develop the theory of locally convex linear topological vectorspaces as far as he needed it himself.

For every distribution a derivative $T'$ can be defined by

$$T'[\varphi] = -T[\varphi']. \tag{2.5}$$

Higher derivatives are given by

$$T^{(n)}[\varphi] = (-)^n T[\varphi^{(n)}]. \tag{2.6}$$

These definitions make sense, since by definition every testfunction possesses derivatives of all orders.

HEAVISIDE's unit function is now replaced by the unit distribution $U_a$ given by

$$U_a[\varphi] = \int_a^\infty \varphi(x)\,dx \tag{2.7}$$

and DIRAC's delta function is replaced by the distribution $\delta_a$ by means of

$$\delta_a = U'_a. \tag{2.8}$$

This is really what it should be since it follows from the definitions given above that

$$\delta_a[\varphi] = U'_a[\varphi] = -U_a[\varphi'] = -\int_a^\infty \varphi'(x)\,dx = \tag{2.9}$$

$$= [-\varphi(x)]_a^\infty = \varphi(a) - \varphi(\infty) = \varphi(a).$$

Notice that $\varphi(\infty) = 0$ for every testfunction $\varphi$ since by definition a test-function has a bounded support and is therefore equal to zero at infinity.

**2.3** Although the description given above has deliberately been restricted to the bare essentials, the reader will note that the theory is rather abstract and that what is called a distribution hardly looks like a generalization of a function. Furthermore the necessity to operate with testfunctions makes the application of distributions difficult. TEMPLE [74] has given the following example. Consider the wave-equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{2.10}$$

This equation should have the solution

$$\begin{cases} u = a \neq 0 & \text{for } x < t \\ u = 0 & \text{for } x > t, \end{cases} \tag{2.11}$$

representing a travelling shockwave. In SCHWARTZ's theory one is obliged to consider a distribution $T$, specified by

$$(2.12) \qquad T[\varphi] = \iint_{-\infty}^{+\infty} u(x, t)\, \varphi(x, t)\, dx\, dt$$

and to replace the differential equation for $u(x, t)$ (2.10) by the integral equation

$$(2.13) \qquad \iint_{-\infty}^{+\infty} u(x, t) \left\{ \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2} \right\} dxdt = 0,$$

which must be satisfied for all testfunctions $\varphi \in \mathbf{D}$.

If however one writes (2.11) as

$$(2.14) \qquad u = a\, U(t-x)$$

$U$ now being the original HEAVISIDE unit function then a straightforward calculation gives

$$(2.15) \qquad \frac{\partial^2 u}{\partial x^2} = a \frac{\partial^2}{\partial x^2} U(t-x) = a\delta'(t-x)$$

$$(2.16) \qquad \frac{\partial^2 u}{\partial t^2} = a \frac{\partial^2}{\partial t^2} U(t-x) = a\delta'(t-x),$$

so that (2.10) is satisfied.

The conclusion is that SCHWARTZ went too far away from the intuitive HEAVISIDE-DIRAC approach and this is not only caused by the abstract setting in which SCHWARTZ originally has presented his theory but also by the introduction of the testfunctions, which at first sight seem to bear no relation to physically relevant entities.

**2.4** Following most of the devices given by TEMPLE [75], LIGHTHILL in his book [46] succeeded in eliminating the more abstract notions from SCHWARTZ's approach and even found the way to use a more tractable kind of testfunctions [1]) called good functions by him. Since, however, the use of some type of testfunctions is not avoided by LIGHTHILL a description of his theory will not be given. It may be remarked, however, that LIGHTHILL's approach is very well adapted to the other theme of his book "*Fourier analysis*".

**2.5** Reference should be made to a booklet of HALPERIN [32], in which a sketch of SCHWARTZ's theory has been given with a minimum of topological

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[1]) Different classes of testfunctions may be used, which however may lead to slightly different types of distributions. The reader is referred to the literature for further details.

reasoning. A readable introduction to the original theory of SCHWARTZ, including an introduction to the topological background has been given by MARTINEAU and TREVES [48].

### 3. Mikusiński's weak limit scheme

Almost immediately after the first publications by SCHWARTZ on his new theory the Polish mathematician JAN MIKUSIŃSKI published a paper [54] in which he showed that the method used by SCHWARTZ to define his distributions was a special case of a very general scheme in which the concept of *weak limit* is fundamental.

Let three collections be given, a collection $F$ of elements $f$, a collection $\Phi$ of elements $\varphi$ and a collection $C$ of elements $c$. No restriction as to the nature of the collections will be made.

Suppose that for every element $f$ from $F$ and every $\varphi$ from $\Phi$ an element $c$ from $C$ is determined by

(3.1) $$f[\varphi] = c.$$

Let the collection $\Phi$ be *complete*, in the sense that if for all $\varphi$ from $\Phi$ one has

(3.2) $$f[\varphi] = g[\varphi]$$

it follows that

(3.3) $$f = g.$$

Let in the collection $C$ some kind of limit be defined, that is let a rule be laid down by which one tests whether for an infinite sequence $c_i$ of $C$

(3.4) $$\lim_{i=\infty} c_i = c$$

is true or not.

By definition a sequence $f_n$ converges weakly to $f$ (written $f_n \rightharpoonup f$) if for all $\varphi$ from $\Phi$ it is true that

(3.5) $$\lim_{n=\infty} f_n[\varphi] = f[\varphi].$$

Now, it may happen that the collection $F$ is not *complete with respect to the weak limit*. This phrase means that it may happen that a sequence $f_n[\varphi]$ converges for all $\varphi$ from $\Phi$, but that the collection $F$ does not possess an element $f$ so that (3.5) is valid. In this case one enlarges the collection $F$ by adding to it elements $f^*$ of the type

(3.6) $$f^* = \{f_1, f_2, \ldots . f_n, \ldots .\},$$

for which by definition $f^*[\varphi]$ is given by

(3.7) $$f^*[\varphi] = \lim_{n=\infty} f_n[\varphi].$$

By identifying two sequences $f_n$ $[\varphi]$ if the sequence of their differences converges to zero one obtains an enlarged collection $F^*$ which now is complete with respect to the weak limit. The reader should observe that this process of completing is familiar to him since it is an abstract description of a process which is much used, *e.g.* the completing of the system of rational numbers to the system of irrational numbers by fundamental sequences (CANTOR) follows the same line of thought.

Next make the following choices:

- $F$: The collection of all functions $f(x)$ which are LEBESGUE-integrable over every finite interval.
- $\Phi$: The collection of testfunctions $\mathfrak{D}$ as defined by SCHWARTZ.
- $C$: The collection of real numbers in which the usual limit will be used for (3.4).

Define $f[\varphi]$ as

$$f\,[\varphi] = \int_{-\infty}^{+\infty} f(x)\, \varphi\,(x)\, dx. \tag{3.8}$$

The collection $F$ of functions $f$ happens to be incomplete with respect to the weak limit corresponding with (3.4). MIKUSIŃSKI has shown that the elements which one should add to this collection to make it complete are exactly the distributions of SCHWARTZ.

Of course nothing is gained by this way of introducing distributions. The importance of MIKUSIŃSKI's general scheme lies in the possibility of choosing other specifications for the collections $F$, $\Phi$ and $C$. One may hope that by making a suitable choice for the specification of the collections $F$, $\Phi$ and $C$ one may arrive at a more simple theory of distributions. This is indeed what has been done by different authors. These theories are usually called elementary. The meaning of this term is rather specific. It means that the author has specified MIKUSIŃSKI's general scheme in some way or another, but has eliminated in his presentation of the theory all references to the general abstract scheme. So it looks as if no general theory is behind his presentation and as if he has succeeded in coming through by using exclusively methods of ordinary calculus. For a full understanding why it is possible that so many elementary approaches to distribution theory are possible the general scheme or even the more general concept of neutrices (see section **7.6**) should be kept in mind. In the next sections a few of these elementary theories will be described in the spirit of their inventors, that is without using MIKUSIŃSKI's scheme explicitely. An indication of how these theories fit in the general abstract scheme, however, will not be omitted.

## 4. Distributions defined by sequences

Different authors *e.g.* KOREVAAR [41] and MIKUSIŃSKI and SIKORSKI [56] have shown the possibility of defining distributions by sequences of ordinary functions. Since the difference between the approaches of KOREVAAR and MIKUSIŃSKI-SIKORSKI is not a matter of principle but is confined to details, only the latter theory will be described. First a fixed open interval $I$: $-\infty \leqq A < x < B \leqq \infty$ for the variable $x$ is defined.

A sequence $f_n(x)$ of functions continuous on $I$ is said to be *fundamental* if there exists a sequence $F_n(x)$ and an integer $k \geqq 0$ such that

(4.1) $$F_n{}^{(k)} = f_n(x) \text{ for all } n,$$

and

(4.2) the sequence $F_n(x)$ converges for $n \to \infty$ almost uniformly. The notion of *almost uniform convergence* is classical and may be defined by:

A sequence $F_n(x)$ converges on $I$ almost uniformly to $F(x)$, written

(4.3) $$F_n(x) \rightrightarrows F(x)$$

if it converges to $F(x)$ uniformly on each finite closed interval $I'$ contained in $I\,(I' \subset I);\ (I' \neq I)$.

Two fundamental sequences $f_n(x)$ and $g_n(x)$ are called *equivalent*, written

(4.4) $$\{f_n(x)\} \sim \{g_n(x)\}$$

if there exist sequences $F_n(x)$ and $G_n(x)$ and an integer $k \geqq 0$ such that

(4.5.1) $$F_n{}^{(k)}(x) = f_n(x) \quad \text{for all } n$$

(4.5.2) $$G_n{}^{(k)}(x) = g_n(x) \quad \text{for all } n$$

(4.5.3) $$F_n(x) \rightrightarrows F(x)$$

(4.5.4) $$G_n(x) \rightrightarrows G(x)$$

(4.5.5) $$F(x) = G(x).$$

It is easy to see that the relation of equivalence has the following properties

(4.6.1) $$\{f_n(x)\} \sim \{f_n(x)\}$$ (*reflexivity*)

(4.6.2) $$\text{If } \{f_n(x)\} \sim \{g_n(x)\} \text{ then } \{g_n(x)\} \sim \{f_n(x)\}$$ (*symmetry*)

(4.6.3) $$\text{If } \{f_n(x)\} \sim \{g_n(x)\},\ \{g_n(x)\} \sim \{h_n(x)\}, \text{ then } \{f_n(x)\} \sim \{h_n(x)\}$$ (*transitivity*)

By definition a *distribution* is a class of equivalent fundamental sequences.

Denote the class of all fundamental sequences equivalent to $\{f_n(x)\}$ by $[f_n(x)]$ then it follows from (4.6) that

(4.7.1) $\{f_n(x)\}$ belongs to $[f_n(x)]$;

(4.7.2) If $\{f_n(x)\} \sim \{g_n(x)\}$ then $[f_n(x)] = [g_n(x)]$;

(4.7.3) If it is not true that $\{f_n(x)\} \sim \{g_n(x)\}$ then the classes $[f_n(x)]$ and $[g_n(x)]$ have no common element.

Examples The sequence

$$f_n(x) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{nx^2}{2}}, \qquad I = (-1, 1) \tag{4.8}$$

is fundamental.

The sequence

$$g_n(x) = \begin{cases} 0 & \text{for } x < -1/n \\ n + n^2 x & \text{for } -1/n < x < 0 \\ n - n^2 x & \text{for } 0 < x < 1/n \\ 0 & \text{for } x > 1/n \end{cases} \qquad I = (-1, 1) \tag{4.9}$$

is fundamental. It is an easy exercise in classical analysis to prove that the sequences $f_n(x)$ and $g_n(x)$ are equivalent. Both define the same distribution *viz.* the DIRAC *delta distribution.*

By defining for a given continuous function $f(x)$, $f_n(x)$ by

$$f_n(x) = f(x) \quad \text{for all } n, \tag{4.10}$$

every continuous function can be interpreted as a distribution, so the concept of a distribution is clearly a generalization of that of a function.

It is proved in the further development of the theory that every distribution can be represented in the form $[p_n(x)]$ where $p_n(x)$ are polynomials. If a distribution is given in the form $[p_n(x)]$ where $p_n(x)$ are polynomials, the $m$ - th derivative of the distribution is defined by $[p_n^{(m)}(x)]$. From this it follows that each distribution has derivatives of arbitrary high order. Finally one proves the theorem that each distribution is the derivative in the distribution sense, of some order, of a continuous function.

After all this has been defined and proved, a slight generalization is necessary to obtain a set of distributions which is fully equivalent to that of SCHWARTZ, a generalization the sole purpose of which is to remove the restrictions implied by the initial choice of the open interval $I$. The intuitive base of the definition of distributions by sequences is clear. One knows – already since the time of HELMHOLTZ – that functions like (4.8) of (4.9) for large $n$ behave like the DIRAC delta function. One even knows, on the basis of a general theorem of LEBESGUE [45] (see also V. D. POL en BREMMER [61]) how to construct arbitrarily many functions of similar behaviour.

However, in all cases, the simple limit $n \to \infty$ of these functions does not exist. In the approach to distributions discussed in this section the problem is solved by not passing to the limit but by considering the sequences like (4.8) as new entities. Since there are sequences which would lead to the same limit if it would exist, some principle of identification is necessary. In modern mathematics this is always done by constructing an equivalence relation, having the property of reflexivity, symmetry and transitivity like (4.6) and by considering classes of equivalent entities as a new species of entities.

All this very well agrees with the general scheme of MIKUSIŃSKI as described in the previous section. Indeed, one has only to make the following specification in the general scheme:

Take for $F$: the collection of functions $f(x)$ continuous on $I$.

Take for $\Phi$: the collection of integers $k \geqq 0$.

Take for $C$: the collection of continuous functions with the limit concept as governed by the concept of almost uniform convergence on $I$.

Take for the relation (3.1)

(4.11) $$c(x) = f^{(-k)}(x)$$

in which $f^{(-k)}$ denotes the $k$-th iteration of the indefinite integral of $f(x)$.

Remarks:

1. The relations between the original definition of SCHWARTZ and the elementary theory of MIKUSIŃSKI and SIKORSKI are discussed in full by BOUIX [9].
2. A method for the introduction of distributions which lies in between the method of this section and the original method of SCHWARTZ has been constructed by RAVETZ [62]. The announced second paper of this author will perhaps never be published [private communication].

## 5. Distributions defined by derivatives

In the HEAVISIDE-DIRAC approach the delta distribution is from the beginning defined as a derivative of the unit function, neglecting the fact that it does not exist in the traditional sense. In distribution theory one has however the fundamental theorem, that, subjected to an at present inessential restriction, every distribution is (at least locally) the distribution derivative of some order of a continuous function. Now it is often possible in mathematics to construct a new theory by taking some fundamental theorem of an existing theory as the basic definition for the new theory. If this would be possible in distribution theory one would obtain a theory for distributions which approaches as much as possible the original ideas of DIRAC and HEAVISIDE. It has been shown by SIKORSKI that this can easily be done. An account of it may be found in MIKUSIŃSKI and SIKORSKI [56]. Consider pairs consisting of functions $F(x)$ continuous on $I$: $-\infty \leqq A < x < B \leqq \infty$ and integers $k \geqq 0$ : $\{F(x), k\}$.

Two pairs $\{F(x), k\}$ and $\{G(x), h\}$ are said to be *equivalent* if on $I$ either

$k \leqq h$ and $\dfrac{d^{h-k}}{dx^{h-k}} G(x) - F(x)$ exists and is a polynomial of degree $< k$

or

$h \leqq k$ and $\dfrac{d^{k-h}}{dx^{k-h}} F(x) - G(x)$ exists and is a polynomial of degree $< h$.

The conditions for equivalence are clearly less severe then the condition $F^{(k)} = G^{(h)}$ to which they are reducible if $F$ and $G$ are in the possession of the necessary differential quotients. It is easily seen that the equivalence relation is reflexive, symmetric and transitive. So one may consider classes of equivalent pairs which will be called *distributions*. It follows from the definitions given above that $\{F(x), 0\}$ is equivalent to $\{G(x), 0\}$ if and only if $F(x) = G(x)$. So the class of pairs equivalent to $\{F(x), 0\}$ contains exactly one element *viz.* $\{F(x), 0\}$ itself. It is therefore possible to identify the distribution $\{F(x), 0\}$ with the function $F(x)$ and vice versa.

The $m$-th derivative of a distribution $\{F(x), k\}$ is now defined by $\{F(x), k + m\}$. In particular the $m$-th derivative of $F(x)$ is the distribution $\{F(x), m\}$. It is easy to show that if $F(x)$ has already a continuous derivative in the classical sense $F'(x)$ that

(5.1) $$\{F'(x), 0\} = \{F(x), 1\}.$$

Example

Consider the continuous function $S_a(x)$ defined by

(5.2) $$S_a(x) = \begin{cases} 0 & \text{for } x < a \\ x-a & \text{for } x \geqq a. \end{cases}$$

One observes that the unit function is given by

(5.3) $$U_a(x) = \frac{d S_a(x)}{dx}$$

Since $U_a(x)$ is not a continuous function, it cannot be used to define the delta distribution. However, one may define this distribution by

(5.4) $$\delta_a(x) = \{ S_a(x), 2 \}.$$

A slight generalization is necessary to enlarge the set of distributions as defined in this section to obtain a set of distributions fully equivalent to the set of distributions as defined by SCHWARTZ.

The method described above is readily generalized to distributions in more variables and even to distributions in more general spaces as has been shown by SEBASTIÃO e SILVA [68]. The method has a long history which even goes back to the times before the invention of the distribution theory by SCHWARTZ, since already in 1932 BOCHNER [6] made use of it in the theory of Fourier integrals. For its relation with the general scheme of weak limits one may consult TEMPLE [74].

## 6. Distributions as pairs of functions

MIKUSIŃSKI has shown that there is still another way to introduce distributions. (MIKUSIŃSKI [55]).

Consider *semifunctions i.e.* functions which are identically zero for non-positive values of the independent variable. Between two semifunctions $f$ and $g$ one defines the convolution $f{*}g$ [German: *Faltung*; French: *Produit de composition*; Russian: *СВЕРТКА = Svertka*] as

(6.1) $$f{*}g = \left\{ \begin{array}{ll} \int_0^x f(x-\xi)\, g(\xi)\, d\xi = \int_0^x f(\xi)\, g(x-\xi) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{array} \right\} =$$

$$= U(x) \int_0^x f(x-\xi)\, g(\xi)\, d\xi = U(x) \int_0^x f(\xi)\, g(x-\xi)\, d\xi.$$

According to a theorem of TITCHMARSH the convolution $f{*}g$ is never identically zero for continuous or LEBESGUE integrable $f$ and $g$ unless at least one of the factors of the product is (almost) everywhere zero. In other words the system of semifunctions with $*$ as a product is a commutative ring without divisors of zero.

"Division", that is taking the inverse of the convolution is unique if the "quotient" exists. For if

(6.2) $$f{*}g_1 = f{*}g_2 = h(x)$$

then

$$f{*}(g_1 - g_2) = 0.$$

and according to TITCHMARSH's theorem $g_1 = g_2$.

However, in many cases the quotient does not exist. So one is obliged to introduce "fractions". This may be done in the following way.

A pair of semifunctions $\{f_1; g_1\}$ is said to be *equivalent* to a pair $\{f_2; g_2\}$ written

$$\{f_1; g_1\} \sim \{f_2; g_2\} \tag{6.3}$$

if

$$f_1 * g_2 = f_2 * g_1 \tag{6.4}$$

The equivalence relation defined above is reflexive, symmetric and transitive. So one may consider classes of equivalent pairs called *distributions*. The class of pairs equivalent to $F = \{f; g\}$ will be denoted by $[F] = [f; g]$.

Whereas in the preceding sections the definition of the elementary operations like addition with distributions gave no difficulties and was therefore not discussed, in the present method these definitions deserve some consideration.

### Scalar multiplication

If $c$ is a number and $[F]$ a distribution $[G] = c[F]$ will be defined by the rule: for all pairs $\{f; g\}$ belonging to $[F]$, $\{cf; g\}$ will belong to $[G]$.

### Addition

The addition

$$[F] + [G] = [H] \tag{6.5}$$

is defined by the rule:

if $\{f_1; g_1\}$ belongs to $[F]$ and $\{f_2; g_2\}$ belongs to $[G]$, then $\{f_1 * g_2 + f_2 * g_1; g_1 * g_2\}$ belongs to $[H]$.

### Multiplication

The convolution multiplication

$$[F] * [G] = [H] \tag{6.6}$$

is defined by the rule:

if $\{f_1; g_1\}$ belongs to $[F]$ and $\{f_2; g_2\}$ belongs to $[G]$, then $\{f_1 * f_2; g_1 * g_2\}$ belongs to $[H]$.

### Inverse

The inverse distribution $[F]^{-1} = H$ is defined by the rule:

if $\{f; g\}$ belongs to $[F]$, then $\{g; f\}$ belongs to $[H]$.

One readily sees that if one writes

$$[F] = [f; g] = \frac{f}{g} \tag{6.7}$$

one may do the arithmetical operations with distributions as with ordinary fractions. In particular it follows that the convolution division is always unique and possible provided that the denominator is not identically zero (except for $x \leq 0$). This implies the validity of the cancellation law:

From

(6.8) $$[F] * [G] = [F] * [H]$$

it follows that

(6.9) $$[G] = [H]$$

One now may identify an ordinary semifunction $f(x)$ with a distribution by the application

(6.10) $$f(x) \to [U*f;\ U],$$

$U$ being an abbreviation for the unitfunction $U_0(x)$.

At this point it is necessary to remark that constructions like $[f;\ 1]$ are impossible; the components of the pair – the numerator and the denominator of the fraction – should always be semifunctions. In (6.10) it is therefore impossible to divide numerator and denominator by $U$.

### Integration and differentiation

Let $f(x)$ be a semifunction. For the convolution with $U$ one has

(6.11) $$U(x) * f(x) = U(x) \int_0^x 1.f(\xi)\, d\xi = U(x) \int_0^x f(\xi)\, d\xi.$$

So one sees that the operator $U*$ applied to semifunctions is equivalent to integration (indefinite).

In particular

(6.12) $$U * U = xU(x).$$

Next if $f(x)$ possesses a semifunction as its classical differential quotient, one has

(6.13) $$U * f'(x) = U(x) \int_0^x f'(x)\, dx = f(x)\, U(x).$$

Translating this formula by (6.10) into the language of distributions one obtains

(6.14) $$[U * U * f';\ U] = [U * f;\ U].$$

By application of the cancellation law (6.8) it follows that

(6.15) $$[U * f';\ U] = [f;\ U]$$

and interpretation by (6.10) gives the rule: if a semifunction possesses as its ordinary differential quotient a semifunction, then the derivative may be found as the convolution quotient of the function and the unit function. However, not every semifunction possesses an ordinary derivative. But interpretated as a distribution, convolution division by the unit function is always possible. This may be considered as a generalization of the ordinary process of differentiation. So every semifunction possesses a derivative in the sense of distribution theory and even every distribution may be differentiated indefinitely.

By the rule (6.10) the unit function may be considered as the distribution

(6.16) $\quad [U * U;\, U].$

By differentiation one obtains the following representation for the DIRAC delta function or better the delta distribution

(6.17) $\quad \delta = [U * U;\, U * U] = [U;\, U].$

Originally the system of semifunctions was a ring without divisors of zero and without unit element. By the process outlined in this section the system has been imbedded in a larger system. Since in the enlarged system division is always possible, it has the algebraic structure of a *field* and must therefore possess a unit element. The unit element is exactly the delta distribution (6.17). For if $F = [f;\, g]$ is a distribution one has

(6.18) $\quad \delta * [F] = [U;\, U] * [f;\, g] = (\text{by } (6.6)) = [U * f;\, U * g] =$
$\quad = (\text{by the cancellation law } (6.8)) = [f;\, g] = [F]$

or

(6.19) $\quad \delta * [F] = [F].$

Next consider the derivative $\delta'$ of the delta distribution $\delta$ which by the rules given above may be written as

(6.20) $\quad \delta' = [U;\, U * U].$

It is easy to prove that for every distribution $[F] = [f;\, g]$

(6.21) $\quad \delta' * [F] = \dfrac{d}{dx}[F]$

in which the right hand side is to be interpreted as a differentiation in distribution sense. For

(6.22) $\quad \delta' * [F] = [U;\, U * U] * [f;\, g] = [U * f;\, U * U * g] =$
$\quad = [f;\, U * g] = \dfrac{d}{dx}[f;\, g]$

according to the rule for differentiation of distributions.

So one observes that the operator $\delta' *$ in distribution theory is the same thing as the operator $p = \frac{d}{dx}$ in HEAVISIDE's operator calculus. This is one of the many relations which exist between distribution calculus and the operator calculus or the theory of LAPLACE transforms. In fact in MIKUSIŃSKI's book [55] distribution theory is only mentioned incidentally, as its main theme is to develop a new simple algebraic approach to the operator calculus.

## 7. Remarks on different aspects of distribution theory

In this section a series of additional remarks on distribution theory are collected.

**7.1** It has been shown by EHRENPREIS [22] that it is possible to construct a distribution theory on locally compact spaces and by GÁL [28] that by introduction of uniform structures some difficult problems on the existence of solutions of equations with distributions as unknowns may be solved.

**7.2** Whereas convolution products of distributions give no difficulties, as may be inferred from (6.6), ordinary products of distributions do not always exist. Even the square of the delta distribution does not exist. On the base of a very abstract theory KÖNIG [39, 40] succeeded in generalizing the distribution concept to the effect that some of the difficulties are removed but even then the square of the delta distribution cannot be defined.

**7.3** A quite different approach has been found by SCHMIEDEN and LAUGWITZ [66, 43]. In their theory almost nothing has to be changed in the concept of the function. They consider a new type of independent variables, which are no longer numbers but infinite sequences of numbers. In this theory the delta function exists as an ordinary function of the new type of variables and therefore its square exists also. However one has to pay for this freedom. In this theory not one unique delta distribution exists but infinitely many. In fact every sequence like (4.8) of (4.9) defines its own delta function. The authors maintain that in applied mathematics one would profit by the multiplicity of delta functions but until now they failed to prove their thesis. It must, however, be admitted, that their theory is extremely elementary, using hardly anything which goes beyond ordinary calculus.

**7.4** Although there exist many introductions to distribution theory for one variable, their is still a need for an introductory text on distributions in more variables. Formulae like

(7.4.1) $$\delta(x)\,\delta(y) = \frac{\delta\left(\sqrt{x^2+y^2}\right)}{\pi\;\sqrt{x^2+y^2}} = \frac{\delta(r)}{\pi r}$$

have been known for long times and have been proved by the HEAVISIDE-DIRAC technique. They are of fundamental importance for the applications. In text-books on distribution theory they are however hardly discussed at all, except by SCHWARTZ [67] and by LAVOINE [44].

**7.5** Consider the LAPLACE transform as defined by

$$L(f) = \int_0^\infty e^{-st} f(t)dt, \tag{7.5.11}$$

which gives for a delta function concentrated at $x = a$, $\delta_a(x)$:

$$L(\delta_a) = \int_0^\infty e^{-st}\, \delta_a(t)dt = e^{-sa}. \tag{7.5.2}$$

Now one may ask the following question. If one starts with some class of functions in which the delta function is still not defined, would it be possible to define distributions so that (7.5.2) is an almost trivial consequence of the definition? The program has been investigated by WESTON, who in his first paper [78] succeeded in constructing distributions in an elementary way by using a weak limit technique. In his second paper [79] he gives more of the abstract background of his method and discusses also the relations between his theory and the HEAVISIDE operator calculus.

**7.6** Recently VAN DER CORPUT [15, 16, 17] has given a first sketch of a new theory called by him *Neutrix calculus*. This theory has a very broad scope and is of a remarkable generality. A general kind of distributions are defined in it, of which the distributions of SCHWARTZ are only a very special example. It may be that the further developments of the neutrix calculus become of great importance in applied mathematics. Even in this theory the ordinary product of distributions gives difficulties, so one may safely assume that in near future one will not have a theory at hand in which the square of the delta distribution exists.

**7.7** In the second and third part of this study the application of distribution theory is confined to dynamic meteorology. Readers interested in other applications may consult *e.g.* the papers of BREMMER [10] (electromagnetic waves), DORFNER [20] (supersonic gasdynamics), TAYLOR [73] (classical electrodynamics), BOUIX [8] (MAXWELL and HELMHOLTZ equations).

**7.8** In the survey given in the preceding sections of distribution theory no attention has been given to Russian literature. Reference should be made to a four volume book by GEL'FAND and ŠILOV [30] of which English and

German translations of the first volume have been announced. Furthermore there exist translations of papers by GEL'FAND and ŠILOV [29], KOSTYUČENKO and ŠILOV [42] and BOROK [7] in which distribution theory is applied to some problems of pure mathematics. From this one may infer that Russian mathematicians pay due attention to distribution theory and that their methods are some amalgamation of those of SCHWARTZ [67] and LIGHTHILL-TEMPLE [46, 75].

## 8. The practice of distribution calculus

**8.1** As has been shown, there are many ways to introduce distributions *i.e.* to generalize the concept of a function in order to give sense to differential quotients which in the traditional sense do not exist at all. All methods of introducing lead to the same final conclusion *viz.* that one may use the HEAVISIDE-DIRAC technique without falling into troubles. This means that one may safely forget all what has been said about the foundations of distribution theory. All what is needed is the knowledge that there is *some* – in fact there are even more – theory which guarantees the correctness of the intuitive HEAVISIDE-DIRAC technique. Using the intuitive technique a search will be made in order to detect what kind of physically meaningful entities correspond with the standard types of distributions, generalized functions, singular functions or whatever they may be called: $\delta(x)$, $\delta'(x)$ etc.

The discussion will be made using a simple example taken from electrostatics. Consider an electric potential $V(x)$ defined by

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \dfrac{xV_0}{a} & \text{for} \quad 0 < x < a \\ V_0 & \text{for} \quad x > a. \end{cases} \tag{8.1}$$

(See fig. 2)

The potential $V$ can easily be realized by taking two conducting halfspaces $x < 0$ and $x > a$ and bringing them to the potentials 0 and $V_0$.

With the aid of the unit function one may summarize (8.1) into a single formula

$$V(x) = \frac{xV_0}{a} U(x) + \left(V_0 - \frac{xV_0}{a}\right) U(x-a). \tag{8.2}$$

If $V$ were a differentiable function one could find the electrical field strength $E$ by differentiation.

$$E = -\frac{dV}{dx} \tag{8.3}$$

The use of distributions gives sense to (8.3) also for non-differentiable $V(x)$ and one finds

$$E = -\left[\frac{V_0}{a} U(x) - \frac{V_0}{a} U(x-a) + \frac{xV_0}{a}\delta(x) + V_0\left(1 - \frac{x}{a}\right) \delta(x-a)\right], \tag{8.4}$$

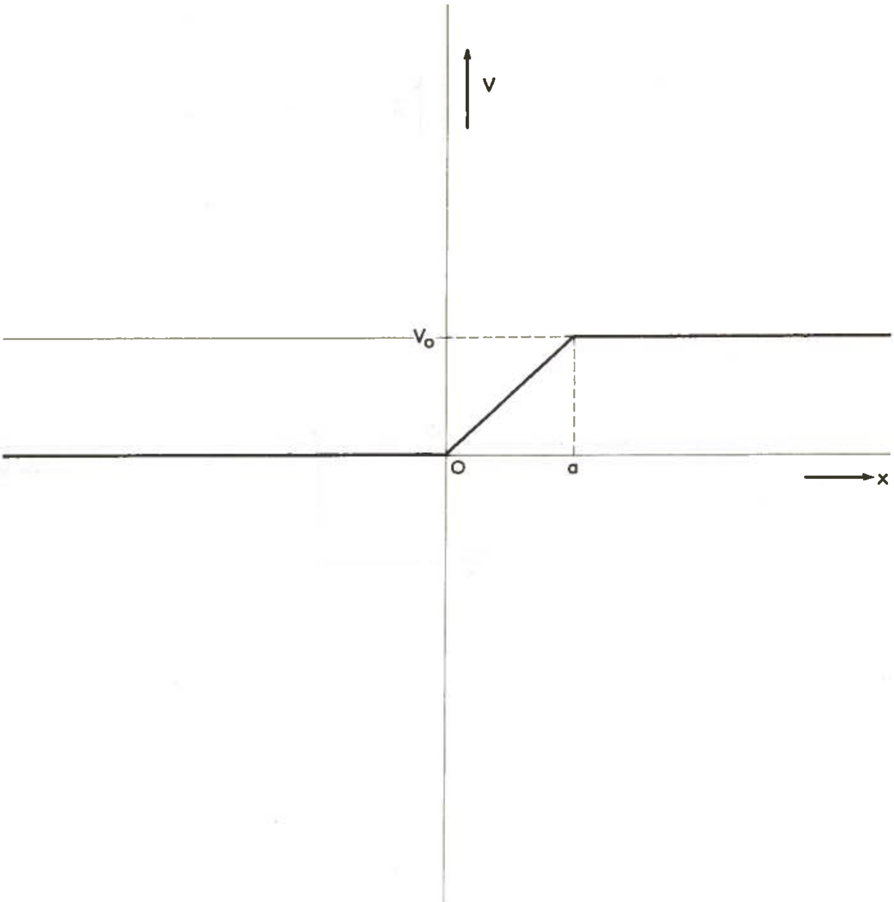


Figure 2.

which due to the relation

(8.5) $f(x)\,\delta(x) = f(0)\,\delta(x)$

may be simplified as

(8.6) $$E = -\left[\frac{V_0}{a}U(x) - \frac{V_0}{a}U(x-a)\right] = \frac{V_0}{a}\left[U(x-a) - U(x)\right] =$$

$$= \begin{cases} 0 & \text{for } x < 0 \\ -\dfrac{V_0}{a} & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$ (see fig. 3)

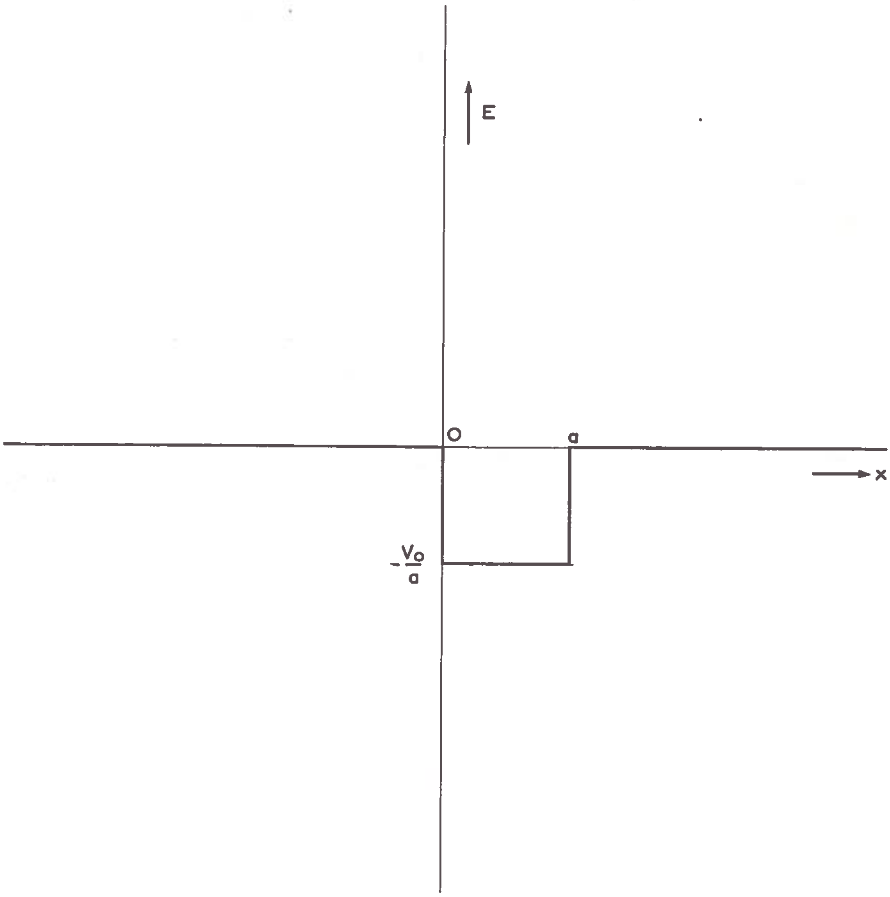


Figure 3.

One observes that the electrical field exists only outside the conductors and is homogeneous in the empty part of the space exactly as it should be according to known theory.

If $E(x)$ were a differentiable function one could find the space charge density $\varrho(x)$ by differentiation

$$\varrho(x) = \frac{dE}{dx} = -\frac{d^2V}{dx^2} \tag{8.7}$$

The use of distributions gives sense to (8.7) also for non-differentiable $E(x)$ and one finds

$$\varrho(x) = \frac{V_0}{a}\,[\delta(x-a) - \delta(x)]. \tag{8.8}$$

Now, one knows that in the case discussed the space charge density is identically zero and that there should be surface charges of opposite sign on the boundaries of the conductors. The strength of the surface charges is known to be proportional to the discontinuity of $E(x)$ at the boundaries. They should be equal to $-V_0/a$ at $x = 0$ and $V_0/a$ at $x = a$.The strengths appear to be exactly the coefficients of the delta functions in (8.8). So one obtains an interpretation for a singular density like

(8.9) $\varrho(x) = \beta\,\delta(x - x_0)$

which simply means a point charge of strength $\beta$ at the point $x = x_0$. Thus the intuitive content of the delta function is that of what is usually called a *point singularity of unit strength.*

Now proceed to the construction of singularities of the dipole type. To do this, place a point singularity of strength $(-e)$ at $x = x_0$ and one of strength


Figure 4.

$+e$ at $x = x_0 + a$. (See fig. 4). According to what has been said above, one may describe this by a singular density given by

(8.10) $\varrho(x) = e\,\delta(x_0 + a - x) - e\,\delta(x_0 - x).$

The next step is to let the charges approach each other by making $a \to 0$ at the same time increasing the strengths such that $m$

(8.11) $m = ea$

remains constant. One then obtains

(8.12) $$\lim \varrho(x) = \lim_{a=0} \left[ m \frac{\delta(x_0 - x + a) - \delta(x_0 - x)}{a} \right] =$$

$$= m \lim_{a=0} \frac{\delta(x_0 - x + a) - \delta(x_0 - x)}{a} = m\,\delta'(x - x_0).$$

One observes that a singularity of the dipole type with moment $m$ placed at $x = x_0$ is represented by the singular density

(8.13) $\varrho(x) = m\,\delta'(x - x_0)$

**8.2** Now both $\delta(x)$ and $\delta'(x)$ have been correlated to physically significant concepts. As remarked already distribution theory has not been able to give a satisfactory definition of the square of the delta function. This problem will be studied now. Consider the family of functions $\varphi_\varepsilon(x)$ (See fig. 5).

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\left[U\left(x+\frac{\varepsilon}{2}\right) - U\left(x-\frac{\varepsilon}{2}\right)\right] = \begin{cases} 0 & \text{for } x < -\varepsilon/2 \\ 1/\varepsilon & \text{for } -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2} \\ 0 & \text{for } x > \varepsilon/2 \end{cases} \quad (8.14)$$

$(\varepsilon > 0)$

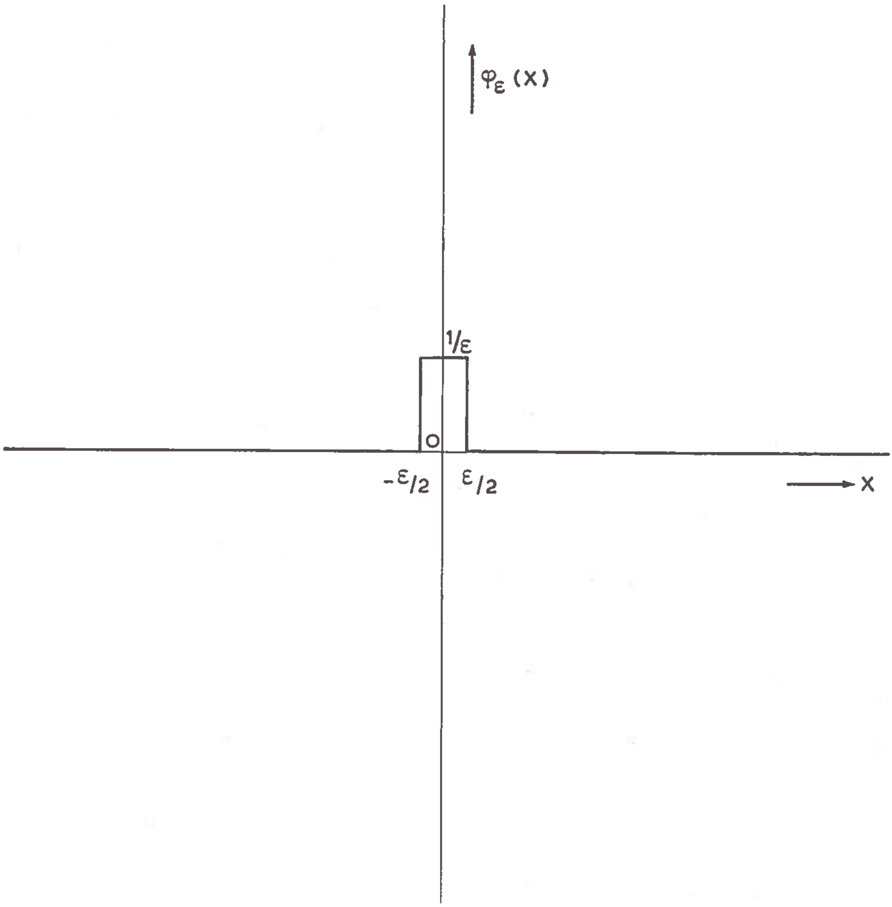


Figure 5.

One has

$$\lim_{\varepsilon = 0} \varphi_\varepsilon(x) = \frac{dU}{dx} = \delta(x). \quad (8.15)$$

This follows immediately from (8.14) by definition, but one obtains further insight by using a test function $f(x)$, and by considering

(8.16) $$c_\varepsilon = \int_{-\infty}^{+\infty} \varphi_\varepsilon(x) f(x)\, dx = \frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} f(x)\, dx.$$

If $f(x)$ satisfies some mild conditions one may apply the mean value theorem of integral calculus to the last integral. According to this theorem there exists a real number $\xi$,

(8.17) $$-\frac{\varepsilon}{2} \leqq \xi \leqq \frac{\varepsilon}{2}$$

such that

(8.18) $$c_\varepsilon = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\varepsilon/2} f(x)\, dx = \frac{1}{\varepsilon} f(\xi)\, \varepsilon = f(\xi).$$

Taking the limit for $\varepsilon \to 0$ one obtains

(8.19) $$\lim_{\varepsilon=0} \int_{-\infty}^{+\infty} \varphi_\varepsilon(x)\, f(x)\, dx = \lim_{\xi=0} f(\xi) = f(0).$$

So one sees that for small values of $\varepsilon$, $\varphi_\varepsilon(x)$ may be considered as an approximation of $\delta(x)$. It is therefore worth while to try to define $\delta^2(x)$ by means of this approximation *e.g.* by writing tentatively

(8.20) $$\delta^2(x) = \lim_{\varepsilon=0} \varphi^2{}_\varepsilon(x).$$

Doing this one obtains

(8.21) $$\int_{-\infty}^{+\infty} f(x)\, \delta^2(x)\, dx = \lim_{\varepsilon=0} \int_{-\infty}^{+\infty} f(x)\, \varphi^2{}_\varepsilon(x)\, dx = \lim_{\varepsilon=0} \frac{1}{\varepsilon^2} \int_{-\frac{\varepsilon}{2}}^{\varepsilon/2} f(x)\, dx =$$

$$= \lim_{\varepsilon=0} \frac{1}{\varepsilon^2} f(\xi)\, \varepsilon = \lim_{\varepsilon=0} \frac{f(\xi)}{\varepsilon} = \frac{f(0)}{0}$$

This makes no sense if $f(0) \neq 0$, and if $f(0)$ happens to be zero by chance, the result may sometimes perhaps be interpreted as $f'(0)$. But generally speaking one does not know how to interpret integrals of the type (8.21). The same conclusion follows from the formula

(8.22) $$\int_{-\infty}^{+\infty} f(x)\, \delta^2(x)\, dx = \int_{-\infty}^{+\infty} [f(x)\, \delta(x)]\, \delta(x)\, dx = f(0)\, \delta(0).$$

Now, one knows that $\delta(x)$ has an intuitive significant meaning. The graph of $\delta(x)$ can be imagined as a small and tall pike around $x = 0$ whose area is unit. The discussion above shows that the graph of $\delta^2(x)$ should be something like a small and tall pike around $x = 0$ with infinite area. There seems to be no physically significant concept associated with this picture. So, not only does $\delta^2(x)$ not exist in any distribution theory but also its intuitive content – if it would exist – has no correlate in physical thinking. Much emphasis has been laid upon this problem in this section because in part three (Applications) the argument will be used that theories leading to the square of the delta function are wrong. One should observe that this argument remains valid even if in the future development of distribution theory somebody will construct a generalization in which $\delta^2(x)$ has a mathematical meaning, since no physical meaning can be found.

**8.3** Before passing over to the next part of this study a terminological remark must be made with respect to the term *continuous*. This term offers no ambiguity in pure mathematics. Different equivalent definitions of the term continuous are known for which one may consult *e.g.* KELLEY [37].

However in applied mathematics the term continuous is used more losely. Often one calls a function continuous when what is really meant is that the function is indefinitely differentiable. Sometimes a function is called continuous if it possesses differential quotients of those orders in which the author happens to be interested. So sometimes a function like $V(x)$ defined by (8.1) would be termed discontinuous although according to every mathematician it is clearly continuous.

HADAMARD [31] has tried to save the situation by introducing terms like "*discontinuity of the first, second order etc.*". He would say that $V(x)$ from (8.1) possesses a discontinuity of the first order because it is the first derivative of this function which according to the strict definition is discontinuous. His method works as long as one confines oneself to functions. But HADAMARD also used terms like "*points of discontinuity of the first order*" or "*surfaces of discontinuity of the second order*". Since however in practical applications one has to deal with more than one function (*e.g.* pressure, temperature, humidity, *etc.*) which do not necessarily show the same behaviour at the point or surface of discontinuity the method becomes rather cumbersome because one is obliged to denote which function one considers as critical for determining the order of discontinuity.

The conclusion is that the term continuous function should be only used in the strict mathematical sense. One observes that the whole problem is a product of the tradition in physics and engineering not to distinguish between continuity and differentiability and that just because distribution theory is developed to make every continuous function differentiable one should stick to an exact use of correct terminology.

PART II

# THE EQUATIONS OF DYNAMIC METEOROLOGY

## 9. Surfaces of discontinuity

**9.1** Since the introduction of the frontal theory by the Norwegian school of meteorologists, surfaces of discontinuity play an important role in dynamic meteorology. However, it has become clear that a front is not always a surface of discontinuity in the strict sense. Some authors, from which BLEEKER [3, 4, 5] must be mentioned as the leading authority, even hold that the whole concept of a front needs a fundamental revision. For the present study the different opinions about fronts and surfaces of discontinuity are hardly of importance. Although the concept of discontinuity will be used this does not mean that the results are without value if discontinuities do not exist in the atmosphere. Nobody denies the existence of layers in the atmosphere with pronounced strong baroclinity. It is possible to interpret the whole theory of discontinuities as an approximative theory for layers of strong baroclinity. Still better the surface of discontinuity is to be considered as an exaggeration of a baroclinic layer, used in order to obtain a clear picture of the role played by baroclinity. So even if one denies the existence of discontinuities in the strict sense it may be worth while to study them. This is the more true for the present study since the conclusions which will be drawn are preponderant of a qualitative nature so that they can easily be generalized to baroclinic layers.

**9.2** Consider a surface of discontinuity (SD). In the part of the space through which it moves one considers the time by which the SD passes the point $(x, y, z)$.

$$t = S(x, y, z). \tag{9.1}$$

By total or substantial derivation of this equation to the time one gets

$$1 = \boldsymbol{v} \cdot \nabla S \tag{9.2}$$

in which $\boldsymbol{v}$ denotes the windvector. One should note that (9.2) makes only sense in the points of the SD. It is however possible to write (9.2) in another way in order to obtain an equation valid in the whole space by multiplication with a delta function

$$(1 - \boldsymbol{v} \cdot \nabla S)\, \delta(t - S) = 0. \tag{9.3}$$

For $t \neq S$ the delta function is zero and (9.3) is an identity.

Integrating (9.3) over the whole time one obtains

$$(9.4)\qquad \int_{-\infty}^{+\infty} (1 - \boldsymbol{v} \cdot \nabla S)\,\delta(t-S)\,dt = (1 - \boldsymbol{v} \cdot \nabla S)_{t=S} = 0,$$

which is equivalent to (9.2) restricted to the points of the SD.
By the same method, one may write every equation like

$$(9.5)\qquad F = G \quad \text{(on SD)}$$

in the form valid for the whole space

$$(9.6)\qquad (F-G)\,\delta(t-S) = 0.$$

The physical interpretation of (9.2) or (9.4) offers no difficulties. It simply means that the SD moves with a velocity which is equal to the normal component of $\boldsymbol{v}$ to the SD.

## 10. Dynamic boundary condition

Let $P_1 = P_1(x, y, z, t)$ be the pressure at one side of a SD, and $P_2 = P_2(x, y, z, t)$ be the pressure at the other side. For the pressure $P = P(x, y, z, t)$ in an arbitrary point of the space one may write

$$(10.1)\qquad P = P_1\,U(t-S) + P_2\,U(S-t).$$

Since $(t-S)$ has a different sign on either side of the SD exactly one of the unit functions is equal to zero, the other one being equal to unity. It is classic to state that the pressure should be continuous at the SD. This may be expressed by the formula

$$(10.2)\qquad (P_1-P_2)\,\delta(t-S) = 0 \qquad \textit{(dynamic boundary condition)}.$$

By differentiation of (10.1) one obtains

$$(10.3.1)\qquad \frac{\partial P}{\partial t} = \frac{\partial P_1}{\partial t}\,U(t-S) + \frac{\partial P_2}{\partial t}\,U(S-t) + (P_1-P_2)\,\delta(t-S) \qquad ^{1)}$$

and

$$(10.3.2)\qquad \nabla P = \nabla P_1\,U(t-S) + \nabla P_2\,U(S-t) + (P_2-P_1)\,\delta(t-S)\,\nabla S,$$

which on account of (10.2) may be simplified to

$$(10.4.1)\qquad \frac{\partial P}{\partial t} = \frac{\partial P_1}{\partial t}\,U(t-S) + \frac{\partial P_2}{\partial t}\,U(S-t)$$

and

$$(10.4.2)\qquad \nabla P = \nabla P_1\,U(t-S) + \nabla P_2\,U(S-t).$$

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1) Use is made of the fact that the delta function is an even function. Proof will follow in section **14.1**.

These equations express the wellknown fact that the pressure gradient ($-\nabla P$) and the pressure tendency $\partial P/\partial t$ are discontinuous at the SD but do not show a singular behaviour.

## 11. Kinematic boundary condition

Let $\boldsymbol{v}_1 = \boldsymbol{v}_1(x, y, z, t)$ be the wind on one side of a SD, and $\boldsymbol{v}_2 = \boldsymbol{v}_2(x, y, z, t)$ the wind on the other side. For the wind $\boldsymbol{v} = \boldsymbol{v}(x, y, z, t)$ in an arbitrary point of the space one may write

(11.1) $$\boldsymbol{v} = \boldsymbol{v}_1 U(t-S) + \boldsymbol{v}_2 U(S-t),$$

the argument being the same as in the preceding section. It is classic to state that the normal component of $\boldsymbol{v}$ on the SD should be continuous. This may be expressed by the formula

(11.2) $$(\boldsymbol{v}_1-\boldsymbol{v}_2) \cdot \nabla S\, \delta(t-S) = 0 \qquad \textit{(kinematic boundary condition)}.$$

Taking the divergence of $\boldsymbol{v}$ one obtains according to (11.1)

(11.3) $$\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}_1\, U(t-S) + \operatorname{div} \boldsymbol{v}_2\, U(S-t) + (\boldsymbol{v}_2-\boldsymbol{v}_1) \cdot \nabla S\, \delta(t-S),$$

which due to the kinematic boundary condition (11.2) may be simplified to

(11.4) $$\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}_1\, U(t-S) + \operatorname{div} \boldsymbol{v}_2\, U(S-t).$$

This equation expresses the fact that the wind divergence is discontinuous on the SD but does not show a singular behaviour, a fact which already is far more less known than the corresponding theorem regarding the pressure gradient and the pressure tendency (10.4.1) and (10.4.2). However, it must be remarked, that it is only the wind divergence which is free of singularities, but that the mass divergence has a singular term as follows from

(11.5) $$\operatorname{div}(\varrho \boldsymbol{v}) = -\frac{\partial \varrho}{\partial t} = -\frac{\partial \varrho_1}{\partial t} U(t-S) - \frac{\partial \varrho_2}{\partial t} U(S-t) + \\ + (\varrho_2-\varrho_1)\, \delta(t-S)$$

or

(11.6) $$\operatorname{div}(\varrho \boldsymbol{v}) = \operatorname{div}(\varrho_1 \boldsymbol{v}_1)\, U(t-S) + \operatorname{div}(\varrho_2 \boldsymbol{v}_2)\, U(S-t) + \\ + (\varrho_2 \boldsymbol{v}_2 - \varrho_1 \boldsymbol{v}_1) \cdot \nabla S\, \delta(t-S).$$

(Cf. section **12** for the use of the equation of continuity).

By differentiation of (11.1) to the time one obtains

(11.7) $$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial \boldsymbol{v}_1}{\partial t} U(t-S) + \frac{\partial \boldsymbol{v}_2}{\partial t} U(S-t) + (\boldsymbol{v}_1-\boldsymbol{v}_2)\, \delta(t-S)$$

and in this equation the last term on the right hand side cannot be put equal to zero on account of some boundary condition. So the local wind variation with the time is not only discontinuous but even has a singularity. The same is true for the (relative) vorticity $\vec{\zeta}$[1])

$$\vec{\zeta} = \mathrm{rot}\, \boldsymbol{v} = \mathrm{rot}\, \boldsymbol{v}_1\, U(t-S) + \mathrm{rot}\, \boldsymbol{v}_2\, U(S-t) + + (\boldsymbol{v}_1-\boldsymbol{v}_2) \wedge \nabla S\, \delta(t-S). \tag{11.8}$$

These singularities are of great importance for the dynamics of SD's as will be seen in later sections.

## 12. Equation of continuity

One usually derives the so-called equation of continuity by the following argumentation.
The formula

$$\iiint \frac{\partial \varrho}{\partial t}\, dV + \iint \varrho \boldsymbol{v} \cdot \boldsymbol{dO} = 0, \tag{12.1}$$

expresses that the variation of mass contained in a fixed but arbitrary volume of the space is caused by the transport of matter through its boundaries. The application of GAUSS's theorem then gives:

$$\iiint \left\{ \frac{\partial \varrho}{\partial t} + \mathrm{div}\, (\varrho \boldsymbol{v}) \right\} dV = 0. \tag{12.2}$$

Since the volume considered is wholly arbitrary it must be true that

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}\, (\varrho \boldsymbol{v}) = 0 \tag{12.3}$$

or

$$\frac{d\varrho}{dt} + \varrho\, \mathrm{div}\, \boldsymbol{v} = 0. \tag{12.4}$$

The mathematical basis of the reasoning given above is not without some points which may give opportunity to criticism. However, it should be kept in mind that the proof of every fundamental equation in physics is more or less not conclusive and that heuristic elements never can be eliminated from it. Indeed, it would perhaps be better to consider equations like (12.3) or (12.4) as axioms

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[1]) For typographical reasons vectors are denoted by bold-faced latin letters or by greek letters with an arrow.

rather than as equations deductible from other basic principles. For the real proof for the correctness of equations of this type does not lie in the fact that they can be deduced from other equations but in the possibility of building an acceptable theory with these equations as starting points. So nothing can be said against the assumption of validity of the equation (12.3) for the case in which the variables are not ordinary functions but distributions. Next write (12.3) in the form

$$\frac{\partial \varrho}{\partial t} + \varrho \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \varrho = 0 \tag{12.5}$$

and use

$$\frac{\partial \varrho}{\partial t} = \frac{\partial \varrho_1}{\partial t} U(t-S) + \frac{\partial \varrho_2}{\partial t} U(S-t) + (\varrho_1-\varrho_2)\, \delta(t-S) \tag{12.6}$$

and

$$\nabla \varrho = \nabla \varrho_1\, U(t-S) + \nabla \varrho_2\, U(S-t) + (\varrho_2-\varrho_1)\, \nabla S\, \delta(t-S) \tag{12.7}$$

to obtain

$$\begin{aligned} &\left\{\frac{\partial \varrho_1}{\partial t} + \varrho \operatorname{div} \boldsymbol{v}_1 + \boldsymbol{v} \cdot \nabla \varrho_1\right\} U(t-S) + \\ &+ \left\{\frac{\partial \varrho_2}{\partial t} + \varrho \operatorname{div} \boldsymbol{v}_2 + \boldsymbol{v} \cdot \nabla \varrho_2\right\} U(S-t) + \\ &+ \left\{(\varrho_1-\varrho_2) + (\varrho_2-\varrho_1)\, \boldsymbol{v} \cdot \nabla S\right\} \delta(t-S) = 0. \end{aligned} \tag{12.8}$$

On account of the condition (9.3) the term containing the delta function is identically equal to zero.

Furthermore

$$\begin{aligned} \varrho\, U(t-S) &= \varrho_1\, U(t-S), \\ \varrho\, U(S-t) &= \varrho_2\, U(S-t), \\ \boldsymbol{v}\, U(t-S) &= \boldsymbol{v}_1\, U(t-S), \\ \boldsymbol{v}\, U(S-t) &= \boldsymbol{v}_2\, U(S-t), \end{aligned} \tag{12.9}$$

so that (12.7) may be simplified to

$$\left\{\frac{\partial \varrho_1}{\partial t} + \operatorname{div} (\varrho_1 \boldsymbol{v}_1)\right\} U(t-S) + \left\{\frac{\partial \varrho_2}{\partial t} + \operatorname{div} (\varrho_2 \boldsymbol{v}_2)\right\} U(S-t) = 0. \tag{12.10}$$

A more direct method for deriving this equation will be described later on (section **14.2**).

## 13. Equation of motion

The usual derivation of the equation of motion is as follows. Let $\boldsymbol{F}$ be the force per unit of mass, then in an inertial coordinate system one has

$$(13.1) \qquad \frac{d}{dt}(\varrho \boldsymbol{v} \delta V) = \boldsymbol{F} \varrho \delta V$$

in which $\delta V$ is the volume of a small but arbitrary portion of matter. From this, it follows that in a coordinate system attached to the rotating earth the equation of motion is given by

$$(13.2) \qquad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2 \vec{\Omega} \wedge \boldsymbol{v} + \nabla \Phi + \frac{1}{\varrho} \nabla P = 0$$

in the traditional notation (cf. PETTERSSEN [58; vol. I]). As in the preceding section there is no reason why this equation should not be valid in the case that the variables are not ordinary functions but distributions. On account of

$$\begin{matrix}(13.3) = \\ (11.5)\end{matrix} \qquad \frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial \boldsymbol{v}_1}{\partial t} U(S-t) + \frac{\partial \boldsymbol{v}_2}{\partial t} U(S-t) + (\boldsymbol{v}_1 - \boldsymbol{v}_2)\, \delta(t-S)$$

$$\begin{aligned}(13.4) \qquad \boldsymbol{v} \cdot \nabla \boldsymbol{v} &= \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1\, U(t-S) + \boldsymbol{v}_2 \cdot \nabla \boldsymbol{v}_2\, U(S-t) + \\ &+ (\boldsymbol{v}_2 - \boldsymbol{v}_1)\, \boldsymbol{v} \cdot \nabla S\, \delta(t-S) = \\ &= \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1\, U(t-S) + \boldsymbol{v}_2 \cdot \nabla \boldsymbol{v}_2\, U(S-t) + \\ &+ (\boldsymbol{v}_2 - \boldsymbol{v}_1)\, \delta(t-S) \qquad \text{(See (9.3))}\end{aligned}$$

and

$$\begin{matrix}(13.5) = \\ (10.4.2)\end{matrix} \qquad \nabla P = \nabla P_1\, U(t-S) + \nabla P_2\, U(S-t)$$

it is found that

$$\begin{aligned}(13.6) \qquad &\left[\frac{\partial \boldsymbol{v}_1}{\partial t} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1 + 2 \vec{\Omega} \wedge \boldsymbol{v}_1 + \nabla \Phi + \frac{1}{\varrho_1} \nabla P_1\right] U(t-S) + \\ + &\left[\frac{\partial \boldsymbol{v}_2}{\partial t} + \boldsymbol{v}_2 \cdot \nabla \boldsymbol{v}_2 + 2 \vec{\Omega} \wedge \boldsymbol{v}_2 + \nabla \Phi + \frac{1}{\varrho_2} \nabla P_2\right] U(S-t) = 0.\end{aligned}$$

This is the required equation of motion. In order to specify the motion completely a last equation is necessary, describing the thermodynamical character of the motion. In most cases of practical importance, it is appropriate to assume the motion to be adiabatic. Using the potential temperature $\Theta$,

$$(13.7) \qquad \Theta = T \left(\frac{P_0}{P}\right)^{\varkappa},$$

in which

$$P_0 = \text{arbitrary but fixed reference pressure}$$

$$\varkappa = \frac{R}{c_p} = \frac{\text{gas constant for dry air}}{\text{specific heat at constant pressure}}.$$

The conservatism of the potential temperature is expressed by

$$\frac{d\Theta}{dt} = 0. \tag{13.8}$$

It will be seen later on that from (13.8) and

$$\Theta = \Theta_1\, U(t-S) + \Theta_2\, U(S-t) \tag{13.9}$$

it follows immediately that

$$\frac{d\Theta_1}{dt}\, U(t-S) + \frac{d\Theta_2}{dt}\, U(S-t) = \frac{d\Theta}{dt}. \tag{13.10}$$

So finally one obtains as the complete system of equations which describes atmospheric motions the equations:

| | | |
|---|---|---|
| (i) | Equation for the movement of SD | (9.3) |
| (ii) | Dynamic boundary condition | (10.2) |
| (iii) | Kinematic boundary condition | (11.2) |
| (iv) | Equation of continuity | (12.10) |
| (v) | Equation of motion | (13.6) |
| (vi) | Thermal equation | (13.8) |

## 14. Some general theorems

In this section some general theorems are proved which are useful in that their application saves time and labour in calculations.

**14.1** The delta function is an even function, *i.e.*

$$\delta(x) = \delta(-x). \tag{14.1}$$

For the proof it is sufficient to remark that instead of

$$\delta(x) = \frac{dU}{dx} \tag{14.2}$$

one may also write

$$\delta(x) = \frac{d}{dx}\left[-\frac{1}{2} + U(x)\right]. \tag{14.3}$$

Now the function under the differentiation operator is an odd function so $\delta(x)$ is an even function. In general one finds

$$\delta^{(n)}(x) = (-)^n\, \delta^{(n)}(-x). \tag{14.4}$$

**14.2** Next it will be proved that

$$\frac{d}{dt}\, U(t-S) = \frac{d}{dt}\, U(S-t) = 0. \tag{14.5}$$

The proof is easy and follows from (9.3) and

$$\frac{d}{dt}\, U(t-S) = \{\, 1 - \boldsymbol{v} \cdot \nabla S \,\}\, \delta(t-S) = 0. \tag{14.6}$$

The intuitive content of (14.5) is obvious. $U(t-S)$ is a function which is everywhere constant except at the SD where it shows a discontinuity. Considered from the point of view of an observer which moves with the motion, nothing changes in the situation with time. The formula (14.5) is of paramount importance because it expresses that the operator $\frac{d}{dt}$ applied to a non-singular function leads to a result in which no singularities occur. So the substantial derivative of a non-singular entity is non-singular itself. The use of this theorem would have simplified the derivation of the equations (12.10) and (13.6) and proves the correctness of (13.10) as an immediate consequence of (13.9). For the same reason it is true that

$$\frac{d}{dt}\, \delta(t-S) = 0. \tag{14.7}$$

**14.3** As an application of these theorems the structure of so-called balance equations [1]) will be studied now. Consider an arbitrary entity $F$ of the form:

$$F = F_1\, U(t-S) + F_2\, U(S-t). \tag{14.8}$$

Let for $F$ a balance equation be valid

$$\frac{\partial}{\partial t}\, (\varrho F) + \operatorname{div}\, (\varrho F \boldsymbol{v}) = F^Q \tag{14.9}$$

The three terms in this equation will be called

$$F^L = \frac{\partial}{\partial t}\, (\varrho F) \qquad = \text{local variation}, \tag{14.10.1}$$

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[1]) Here the term balance equation is used in the traditional and physical sense and not in the more specific sense used in studies on numerical weather prediction.

(14.10.2) $F^A = \operatorname{div}(\varrho F \boldsymbol{v}) \quad = \text{advection},$

(14.10.3) $F^Q \quad = \text{local production}.$

So (14.9) expresses the fact the local variation $F^L$ is compensated by the sum of what is transported away by advection $F^A$ and the "consumption" ($-F^Q$).

Now one finds

$$(14.11) \qquad F^L = \frac{\partial}{\partial t}(\varrho F) = \frac{\partial}{\partial t}[\varrho_1 F_1\, U(t-S) + \varrho_2 F_2\, U(S-t)] =$$

$$= \frac{\partial}{\partial t}(\varrho_1 F_1)\, U(t-S) + \frac{\partial}{\partial t}(\varrho_2 F_2)\, U(S-t) +$$

$$+ (\varrho_1 F_1 - \varrho_2 F_2)\, \delta(t-S) =$$

$$= F_1^L\, U(t-S) + F_2^L\, U(S-t) + (\varrho_1 F_1 - \varrho_2 F_2)\, \delta(t-S).$$

So that the local variation is seen to have a singular term. Further, one obtains

$$(14.12) \qquad F^A = F_1^A\, U(t-S) + F_2^A\, U(S-t) + (\varrho_2 F_2 \boldsymbol{v}_2 - \varrho_1 F_1 \boldsymbol{v}_1) \cdot \nabla S\, \delta(t-S),$$

so that the advection contains also a singular term. Adding these results one, however, obtains

$$(14.13) \qquad F^L + F^A = (F_1^L + F_1^A)\, U(t-S) + (F_2^L + F_2^A)\, U(S-t) +$$

$$+ [(\varrho_2 F_2 \boldsymbol{v}_2 - \varrho_1 F_1 \boldsymbol{v}_1) \cdot \nabla S + \varrho_1 F_1 - \varrho_2 F_2]\, \delta(t-S),$$

in which the singular terms happens to be identically zero on account of the kinematic boundary condition. The final conclusion is that for quantities of the form (14.8) the left hand side of the balance equation cannot have a singular term. Then also the right hand side of (14.9) cannot have a singular term which may be expressed by saying that no singular production exists. The proof of this theorem may be simplified by using the rule (14.5) by which it follows from (14.8) that

$$(14.14) \qquad \frac{dF}{dt} = \frac{dF_1}{dt}\, U(t-S) + \frac{dF_2}{dt}\, U(S-t),$$

so that

$$(14.15) \qquad \varrho \frac{dF}{dt} = \varrho_1 \frac{dF_1}{dt}\, U(t-S) + \varrho_2 \frac{dF_2}{dt}\, U(S-t) =$$

$$= \left[\frac{\partial}{\partial t}(\varrho_1 F_1) + \operatorname{div}(\varrho_1 \boldsymbol{v}_1 F_1)\right] U(t-S) +$$

$$+ \left[\frac{\partial}{\partial t}(\varrho_2 F_2) + \operatorname{div}(\varrho_2 \boldsymbol{v}_2 F_2)\right] U(S-t) =$$

$$= (F_1^L + F_1^A)\, U(t-S) + (F_2^L + F_2^A)\, U(S-t).$$

The systematic use of the general theorems proved in this section, greatly facilitates calculations which otherwise become unnecessary complicated. More specifically it follows from (14.5) that it is advantageous to employ as much as possible the substantial differential operator rather than the local one, which always introduces singularities of higher order into the calculations.

## 15. The formula of Margules

At this point the systematic development of the calculus of discontinuities will be interrupted and an example will be given of what can be done with the calculus as developed so far. The example will be that of the well-known formula of MARGULES [47] for the slope of the SD and its relation to the geostrophic wind.

Starting with the dynamic boundary condition

(15.1) = (10.2) $\quad (P_1 - P_2)\,\delta(t-S) = 0$

one obtains

(15.2) $\quad (\nabla P_1 - \nabla P_2)\,\delta(t-S) + (P_2 - P_1)\,\nabla S\,\delta'(t-S) = 0.$

By taking the outproduct of (15.2) with $\nabla S$ one obtains

(15.3) $\quad \nabla S \wedge (\nabla P_1 - \nabla P_2)\,\delta(t-S) = 0$

and by multiplication with the unitvector in the vertical direction $\boldsymbol{k}$:

(15.4) $\quad \boldsymbol{k} \wedge \{\nabla S \wedge (\nabla P_1 - \nabla P_2)\}\,\delta(t-S) = 0.$

Write (15.4) in the developed form

(15.5) $\quad [\nabla S\{\boldsymbol{k}\cdot(\nabla P_1 - \nabla P_2)\} - (\nabla P_1 - \nabla P_2)\,\boldsymbol{k}\cdot\nabla S]\,\delta(t-S) = 0.$

Next let $\boldsymbol{n}$ be the unitvector normal to the SD so that

(15.6) $\quad \nabla S = \boldsymbol{n}\,|\nabla S|$

then it follows that

(15.7) $\quad [\boldsymbol{n}\{\boldsymbol{k}\cdot(\nabla P_1 - \nabla P_2)\} - (\nabla P_1 - \nabla P_2)\,\boldsymbol{k}\cdot\boldsymbol{n}]\,\delta(t-S) = 0$

Since it is known, that SD's are nearly horizontal it is allowed to make the approximation

(15.8) $\quad \boldsymbol{k}\cdot\boldsymbol{n} \approx 1$

and thus

(15.9) $$\boldsymbol{n}\,\delta(t-S) = \frac{\nabla P_1 - \nabla P_2}{\boldsymbol{k}\cdot(\nabla P_1 - \nabla P_2)}\,\delta(t-S),$$

which gives the slope of the SD, the formula usually being written in the form

(15.10) $$\operatorname{tg}\alpha = \frac{\dfrac{\partial P_1}{\partial x} - \dfrac{\partial P_2}{\partial x}}{\dfrac{\partial P_1}{\partial z} - \dfrac{\partial P_2}{\partial z}} = -\frac{\dfrac{\partial P_1}{\partial x} - \dfrac{\partial P_2}{\partial x}}{g(\varrho_1 - \varrho_2)}.$$

For the numerator of the right hand side of (15.9) one may substitute the geostrophic approximation

(15.11) $\nabla P_1 - \nabla P_2 \approx f(\varrho_1 \boldsymbol{v}_{g1} - \varrho_2 \boldsymbol{v}_{g2}) \wedge \boldsymbol{k}$

in which

(15.12) $f = 2\, \boldsymbol{k} \cdot \vec{\Omega} = 2\, |\vec{\Omega}|\, \sin \varphi$

($f$ = Coriolisparameter; $\varphi$ = geographical latitude
$\boldsymbol{v}_{g1}; \boldsymbol{v}_{g2}$ = geostrophic wind).

whereas in the denominator one may use the hydrostatic equation

(15.13) $\dfrac{\partial P}{\partial z} = -g\varrho.$

By this, one obtains from (15.9) for the slope of the SD the equation

(15.14) $\boldsymbol{n}\, \delta(t-S) = \dfrac{f}{g} \cdot \dfrac{\varrho_2 \boldsymbol{v}_{g2} - \varrho_1 \boldsymbol{v}_{g1}}{\varrho_2 - \varrho_1}\, \delta(t-S)$

in which the traditional formula is readily recognized. One observes, that the derivation of the formula of MARGULES is not more complicated than the derivation given in textbooks on dynamic meteorology. Of course, at this point in the developement of the calculus it is premature to expect new results.

## 16. The vorticity equation

At present it is almost generally accepted that in dynamic meteorology vorticity is perhaps a more important quantity than velocity. Reference may be made to the fact that some calculation schemes used in numerical weather prediction are based on the vorticity equation rather than on the equations of motion. Although the importance of the concept of vorticity was only recently appreciated, it must be remembered that the concept of vorticity has been studied already long ago. TRUESDELL [77] has made an extensive study of the history of our knowledge on vorticity and came to the conclusion that some of the novelties of the last years are merely rediscoveries of old truth. Already as long ago as 1870–1875 BELTRAMI's [1] knowledge of the laws governing the vorticity was fairly complete. To give only one example: it seems not generally known that PETTERSSEN's well-known decomposition of a linear field is completely contained in BELTRAMI's work together with the refinements studied by HINKELMANN [35]. For further details on the history of the knowledge of vorticity the reader may consult TRUESDELL's book [77].

The following formula has already been given

(16.1) = (11.8) $\vec{\zeta} = \mathrm{rot}\, \boldsymbol{v} = \mathrm{rot}\, \boldsymbol{v}_1\, U(t-S) + \mathrm{rot}\, \boldsymbol{v}_2\, U(S-t) + (\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \nabla S\, \delta(t-S),$

which will be written as

(16.2) $$\vec{\zeta} = \vec{\zeta_1}\, U(t-S) + \vec{\zeta_2}\, U(S-t) + \vec{\zeta_3}\, \delta(t-S),$$

with the abbreviation

(16.3) $$\vec{\zeta_3}\, \delta(t-S) = (\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \nabla S\, \delta(t-S).$$

With regard to the singular term (16.3) of (16.1) and (16.2) it must be observed that on account of the kinematic boundary condition (11.2) the components of the wind normal to the SD are equal so that in (16.3) only the tangential components enter. This means that $\vec{\zeta_3}$ is a vector lying in the tangential plane and is in that plane normal and proportional to the windshear. Using the rules of section **14.3** one obtains from (16.2)

(16.4) $$\frac{d\vec{\zeta}}{dt} = \frac{d\vec{\zeta_1}}{dt}\, U(t-S) + \frac{d\vec{\zeta_2}}{dt}\, U(S-t) + \frac{d\vec{\zeta_3}}{dt}\, \delta(t-S).$$

Only the last term of the right hand side of this equation will be discussed, the others giving no new points of view. One obtains

(16.5) $$\frac{d\vec{\zeta_3}}{dt}\, \delta(t-S) = \frac{d}{dt}\Big[(\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \nabla S\Big]\, \delta(t-S) =$$

$$= \left[\left(\frac{d\boldsymbol{v}_1}{dt} - \frac{d\boldsymbol{v}_2}{dt}\right) \wedge \nabla S + (\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \frac{d}{dt}(\nabla S)\right] \delta(t-S) =$$

$$= \left[\left(\frac{d\boldsymbol{v}_1}{dt} - \frac{d\boldsymbol{v}_2}{dt}\right) \wedge \nabla S + (\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \boldsymbol{v} \cdot \nabla\nabla S\right] \delta(t-S).$$

So the substantial variation of the singular part of the vorticity consists of two terms. The first term is proportional to the tangential component of the shear in the SD of the acceleration. The second term is a contribution resulting from the advective variation of the slope of the SD. So much for the kinematics of vorticity. Proceeding to the dynamics it is advantageous to introduce the absolute vorticity $\boldsymbol{Q}$ defined by

(16.6) $$\boldsymbol{Q} = \vec{\zeta} + 2\vec{\Omega},$$

for which the so-called vorticity equation is valid:

(16.7) $$\frac{d\boldsymbol{Q}}{dt} - \boldsymbol{Q} \cdot \nabla \boldsymbol{v} + \boldsymbol{Q}\, \mathrm{div}\, \boldsymbol{v} + \nabla \frac{1}{\varrho} \wedge \nabla P = 0.$$

Instead of (16.4) one may use

(16.8) $$\frac{d\boldsymbol{Q}}{dt} = \frac{d\boldsymbol{Q}_1}{dt}\, U(t-S) + \frac{d\boldsymbol{Q}_2}{dt}\, U(S-t) + \frac{d\boldsymbol{Q}_3}{dt}\, \delta(t-S),$$

with

$$(16.9)\qquad Q_i = \vec{\zeta}_i + 2\vec{\Omega} \qquad\qquad (i = 1, 2, 3)$$

Further one obtains for the various terms of (16.7)

$$(16.10)\qquad -Q \cdot \nabla v = -Q \cdot \nabla [v_1\, U(t-S) + v_2\, U(S-t)] = \\ = -[Q_1 \cdot \nabla v_1\, U(t-S) + Q_2 \cdot \nabla v_2\, U(S-t) + \\ + (v_2 - v_1)\, Q \cdot \nabla S\, \delta(t-S)],$$

$$(16.11)\qquad Q \operatorname{div} v = (\text{see } 11.4) = Q_1 \operatorname{div} v_1\, U(t-S) + Q_2 \operatorname{div} v_2\, U(S-t) + \\ + Q_3 \operatorname{div} v\, \delta(t-S),$$

$$(16.12)\qquad \frac{1}{\varrho} \nabla P = \frac{1}{\varrho_1} \nabla P_1\, U(t-S) + \frac{1}{\varrho_2} \nabla P_2\, U(S-t)$$

and

$$(16.13)\qquad \nabla \frac{1}{\varrho} \wedge \nabla P = \operatorname{rot}\left(\frac{1}{\varrho} \nabla P\right) = \nabla \frac{1}{\varrho_1} \wedge \nabla P_1\, U(t-S) + \\ + \nabla \frac{1}{\varrho_2} \wedge \nabla P_2\, U(S-t) + \left(\frac{\nabla P_1}{\varrho_1} - \frac{\nabla P_2}{\varrho_2}\right) \wedge \nabla S\, \delta(t-S).$$

So finally one obtains for the vorticity equation the formula

$$(16.14)\qquad \left[\frac{dQ_1}{dt} - Q_1 \cdot \nabla v_1 + Q_1 \operatorname{div} v_1 + \nabla \frac{1}{\varrho_1} \wedge \nabla P_1\right] U(t-S) + \\ + \left[\frac{dQ_2}{dt} - Q_2 \cdot \nabla v_2 + Q_2 \operatorname{div} v_2 + \nabla \frac{1}{\varrho_2} \wedge \nabla P_2\right] U(S-t) + \\ + \left[\frac{dQ_3}{dt} + (v_1 - v_2)\, Q \cdot \nabla S + Q_3 \operatorname{div} v + \right. \\ \left. + \left(\frac{\nabla P_1}{\varrho_1} - \frac{\nabla P_2}{\varrho_2}\right) \wedge \nabla S\right] \delta(t-S) = 0.$$

One observes that the substantial variation of the singular term of the vorticity in the SD consists of three parts. The first contribution

$$(16.15)\qquad (v_1 - v_2)\, Q \cdot \nabla S$$

deals with the difference of the dragging of the normal component of the vorticity along the SD by the various winds in the air masses separated by the SD, whereas the third contribution

$$(16.16)\qquad \left(\frac{\nabla P_1}{\varrho_1} - \frac{\nabla P_2}{\varrho_2}\right) \wedge \nabla S$$

is proportional to the difference of the tangential components of the specific pressure gradient forces in the SD, or proportional to the tangential shear of the specific pressure gradient forces. The interpretation on the middle term

$$(16.17) \qquad Q_3 \operatorname{div} \boldsymbol{v}$$

readily follows from the remarks made on the interpretation of $\vec{\zeta}_3$ (see 16.3).

## 17. Introduction of pressure as an independent variable

Using the ordinary Cartesian coordinate system it is difficult to take into account the different role played in meteorology by the vertical direction as compared to the horizontal directions. In the last decade of years, it has become popular to overcome this difficulty by introducing instead of the vertical height $z$, the pressure $P$ as an independent variable. In this $(x, y, P)$-coordinate system the geopotential $\Phi$ becomes a dependent variable. For the application to continuous fields one may consult ELIASSEN [23]. This new technique gets its full strength only if combined with some approximations. Apart from the fact, that one considers the new coordinate system as an orthogonal system, which implies the neglect of the so-called metrical accelerations, one usually neglects the vertical acceleration. It is assumed that the reader is fully acquainted with the philosophy behind the use of the $(x, y, P)$-system in the case of continuous fields from which these and other approximations receive their justification. Instead of the vertical velocity one uses the quantity $\omega$ defined by

$$(17.1) \qquad \omega = \frac{dP}{dt}.$$

Due to lack of a better term $\omega$ is usually also called "vertical velocity". It should, however, be kept in mind that $\omega > 0$ means a descending motion, and thus corresponds to $\boldsymbol{v}_z < 0$.

The windcomponent in the surfaces $P =$ const. will be indicated by $\boldsymbol{u}$, and it will be termed "the isobaric component of the wind".

Use will be made of the isobaric component $\triangledown$ of $\nabla$ given by

$$(17.2) \qquad \triangledown = \left\{ \left(\frac{\partial}{\partial x}\right)_P ; \left(\frac{\partial}{\partial y}\right)_P ; 0 \right\}$$

the subscripts $P$ indicating that the partial differentiations should be made for constant $P$. With these notations the operator of the substantial differentiation becomes

$$(17.3) \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \triangledown + \omega \frac{\partial}{\partial P}.$$

It will be convenient to introduce a special notation for the isobaric divergence. Here the notation *dip* will be used, which is simpler than notations like $div_H$ *etc.* found in literature. So for the isobaric divergence of a vectorfield $\boldsymbol{a}$ here will be written

$$\text{(17.4)} \qquad \text{dip}\,\boldsymbol{a} = \nabla \cdot \boldsymbol{a} = \left(\frac{\partial a_x}{\partial x}\right)_P + \left(\frac{\partial a_y}{\partial y}\right)_P.$$

For the time $t$ at which a SD passes through the point $(x, y, P)$ of the space will be written

$$\text{(17.5)} \qquad t = \sigma = \sigma(x, y, P).$$

By substantial derivation one obtains

$$\text{(17.6)} \qquad \left(1 - \boldsymbol{u} \cdot \nabla\sigma - \omega \frac{\partial \sigma}{\partial P}\right) \delta(t - \sigma) = 0. \qquad \text{(compare (9.3))}.$$

The equation of motion now reads

$$\text{(17.7)} \qquad \left[\frac{d\boldsymbol{u}_1}{dt} + f\boldsymbol{k} \wedge \boldsymbol{u}_1 + \nabla\Phi_1\right] U(t - S) + \\ + \left[\frac{d\boldsymbol{u}_2}{dt} + f\boldsymbol{k} \wedge \boldsymbol{u}_2 + \nabla\Phi_2\right] U(S - t) = 0.$$

Some remarks on this equation should be made.

(i) One observes that (17.7) is a two-component equation. Since vertical accelerations are neglected no equation for a vertical component should enter into (17.7).

(ii) The Coriolis-term has been simplified, in accordance with the usual practice.

(iii) The geopotential $\Phi$ has become a dependent variable.

The equation of continuity now reads

$$\text{(17.8)} \qquad \text{dip}\,\boldsymbol{u} + \frac{\partial \omega}{\partial P} = 0,$$

or explicitly

$$\text{(17.9)} \qquad \left[\text{dip}\,\boldsymbol{u}_1 + \frac{\partial \omega_1}{\partial P}\right] U(t - \sigma) + \left[\text{dip}\,\boldsymbol{u}_2 + \frac{\partial \omega_2}{\partial P}\right] U(\sigma - t) + \\ + \left[(\boldsymbol{u}_2 - \boldsymbol{u}_1) \cdot \nabla\sigma + (\omega_2 - \omega_1)\frac{\partial \sigma}{\partial P}\right] \delta(t - \sigma) = 0.$$

Application of the kinematic boundary condition in the form

$$\text{(17.10)} \qquad \left[(\boldsymbol{u}_2 - \boldsymbol{u}_1) \cdot \nabla\sigma + (\omega_2 - \omega_1)\frac{\partial \sigma}{\partial P}\right] \delta(t - \sigma) = 0$$

gives instead of (17.9)

$$\left[\operatorname{dip} \boldsymbol{u}_1 + \frac{\partial \omega_1}{\partial P}\right] U(t-\sigma) + \left[\operatorname{dip} \boldsymbol{u}_2 + \frac{\partial \omega_2}{\partial P}\right] U(\sigma-t) = 0. \tag{17.11}$$

It has already been seen earlier that the kinematic boundary condition is equivalent to the condition that the equation of continuity has no singular terms, both expressing the conservation of matter. From the remark (i) at (17.7) it follows that one equation has been lost. It is necessary to fill the gap which is done by the so-called hydrostatic equation,

$$\frac{1}{\varrho} + \frac{\partial \Phi}{\partial P} = 0, \tag{17.12}$$

or explicitly

$$\left[\frac{1}{\varrho_1} + \frac{\partial \Phi_1}{\partial P}\right] U(t-\sigma) + \left[\frac{1}{\varrho_2} + \frac{\partial \Phi_2}{\partial P}\right] U(\sigma-t) + \tag{17.13}$$

$$+ (\Phi_2 - \Phi_1) \frac{\partial \sigma}{\partial P} \delta(t-\sigma) = 0.$$

However, by application of the dynamic boundary condition in the form

$$(\Phi_1 - \Phi_2)\, \delta(t-\sigma) = 0. \tag{17.14}$$

(17.13) may be simplified to

$$\left[\frac{1}{\varrho_1} + \frac{\partial \Phi_1}{\partial P}\right] U(t-\sigma) + \left[\frac{1}{\varrho_2} + \frac{\partial \Phi_2}{\partial P}\right] U(\sigma-t) = 0. \tag{17.15}$$

Sometimes it will be necessary to take into account the variation of the Coriolis parameter $f$ with the geographic latitude. This is usually done by introducing the ROSSBY parameter $\beta$ given by

$$\beta = \frac{df}{dy}. \tag{17.16}$$

This, however, involves an orientation of the coordinate system by rotating it around its vertical axes so that the $y$-direction is orientated to the north. This is against the spirit of the $(x, y, P)$-calculus and this syncretism may be avoided by introducing a vector $\vec{\beta}$ given by

$$\vec{\beta} = \nabla f. \tag{17.17}$$

Sometimes, the variation of $f$ is neglected. For equations valid under this approximation the sign $\overset{*}{=}$ will be used instead of the usual sign of equality. So

$$\vec{\beta} \overset{*}{=} 0. \tag{17.18}$$

To complete the formalism the isobaric Laplace-operator $\triangle$ is introduced by

$$(17.19)\qquad \triangle = \nabla \cdot \nabla = \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}_P .$$

## 18. The geostrophic wind

By definition the (isobaric) geostrophic wind $\boldsymbol{u}_g$ is given by

$$(18.1)\qquad \boldsymbol{u}_g = \frac{1}{f} \boldsymbol{k} \wedge \nabla \Phi$$

or, with (17.14) taken into account

$$(18.2)\qquad \boldsymbol{u}_{g1}\, U(t-\sigma) + \boldsymbol{u}_{g2}\, U(\sigma-t) = \frac{1}{f} \boldsymbol{k} \wedge \left[ \nabla \Phi_1\, U(t-\sigma) + \nabla \Phi_2\, U(\sigma-t) \right].$$

From this it follows that

$$(18.3)\qquad \frac{\partial \boldsymbol{u}_g}{\partial P} = \frac{1}{f} \boldsymbol{k} \wedge \frac{\partial(\nabla \Phi)}{\partial P} = \frac{1}{f} \boldsymbol{k} \wedge \nabla \frac{\partial \Phi}{\partial P} = (\text{see } 17.12) = -\frac{1}{f} \boldsymbol{k} \wedge \nabla \frac{1}{\varrho}.$$

Using

$$(18.4)\qquad \frac{1}{\varrho} = \frac{RT}{P}$$

and

$$(18.5)\qquad \nabla P = 0$$

one obtains

$$(18.6)\qquad \nabla \frac{1}{\varrho} = \frac{R}{P} \nabla T = \frac{\nabla T}{\varrho T},$$

so that

$$(18.7)\qquad \frac{\partial \boldsymbol{u}_g}{\partial P} = -\frac{\boldsymbol{k} \wedge \nabla T}{f \varrho T} = -\frac{R}{f} \frac{\boldsymbol{k} \wedge \nabla T}{P}.$$

This is the well-known equation for the *thermal wind* in differential form [1]).

Now one has

$$(18.8)\qquad \frac{\partial \boldsymbol{u}_g}{\partial P} = \frac{\partial \boldsymbol{u}_{g1}}{\partial P} U(t-\sigma) + \frac{\partial \boldsymbol{u}_{g2}}{\partial P} U(\sigma-t) + (\boldsymbol{u}_{g2} - \boldsymbol{u}_{g1}) \frac{\partial \sigma}{\partial P} \delta(t-\sigma)$$

---

[1]) In the author's opinion the thermal wind should be defined as the difference between two *geostrophic* winds at different heights above the same place. The difference between *observed* winds should be only identified with the thermal wind as an approximation, which is not always fully justified, and should be called vertical windshear.

and

(18.9) $$\nabla T = \nabla T_1\, U(t-\sigma) + \nabla T_2\, U(\sigma-t) + (T_2-T_1)\, \nabla\sigma\, \delta(t-\sigma)$$

from which it follows that

(18.10) $$\left\{ \frac{\partial \boldsymbol{u}_{g1}}{\partial P} + \frac{\boldsymbol{k} \wedge \nabla T_1}{f \varrho_1 T_1} \right\} U(t-\sigma) + \left\{ \frac{\partial \boldsymbol{u}_{g2}}{\partial P} + \frac{\boldsymbol{k} \wedge \nabla T_2}{f \varrho_2 T_2} \right\} U(\sigma-t) +$$
$$+ \left\{ (\boldsymbol{u}_{g2}-\boldsymbol{u}_{g1}) \frac{\partial \sigma}{\partial P} + \frac{T_2-T_1}{f \varrho T} \boldsymbol{k} \wedge \nabla\sigma \right\} \delta(t-\sigma) = 0.$$

Now, analysing the equations for the thermal wind (18.8) and (18.10) one observes in (18.8) a singular term which is proportional to the windshear in the SD which in (18.10) is seen to be counterbalanced by a term proportional to the temperature difference between the air masses.

Let further the geostrophic isobaric winddivergence $D_g$ be defined by

(18.11) $$D_g = \text{dip}\, \boldsymbol{u}_g.$$

Then

(18.12) $$D_g = \text{dip}\left\{ \frac{1}{f} \boldsymbol{k} \wedge \nabla\Phi \right\} = \nabla \frac{1}{f} \cdot (\boldsymbol{k} \wedge \nabla\Phi) + \frac{1}{f} \text{dip}\, (\boldsymbol{k} \wedge \nabla\Phi) =$$
$$= -\frac{1}{f_2} \vec{\beta} \cdot (\boldsymbol{k} \wedge \nabla\Phi) + \frac{1}{f} \{ \nabla\Phi\, \text{rot}\, \boldsymbol{k} - \boldsymbol{k}\, \text{rot}\, \nabla\Phi \} =$$
$$= -\frac{1}{f^2} \vec{\beta} \cdot (\boldsymbol{k} \wedge \nabla\Phi).$$

Neglecting the variation of the Coriolisparameter this gives:

(18.13) $$D_g \stackrel{*}{=} 0.$$

This result is the basis for the well-known theorem called JEFFREYS' paradox. Since

(18.14) $$\text{dip}\, \boldsymbol{u}_g = D_{g1}\, U(t-\sigma) + D_{g2}\, U(\sigma-t)$$

on account of the kinematic boundary condition it follows also that

(18.15) $$D_{g1}\, U(t-\sigma) + D_{g2}\, U(t-\sigma) \stackrel{*}{=} 0.$$

and the solution of JEFFREYS' paradox cannot be found in the introduction of discontinuities, as one would perhaps believe remembering ERTEL's so-called theory of singular advection, which will be discussed later on in more detail.

Next, consider the geostrophic isobaric vorticity $\vec{\zeta}_g$ defined by

(18.16) $$\vec{\zeta}_g = \nabla \wedge \boldsymbol{u}_g.$$

One readily obtains

$$(18.17) \qquad \vec{\zeta}_g = \nabla \wedge \{ \frac{1}{f} \boldsymbol{k} \wedge \nabla \Phi \} = \nabla \frac{1}{f} \wedge (\boldsymbol{k} \wedge \nabla \Phi) +$$

$$+ \frac{1}{f} \nabla \wedge (\boldsymbol{k} \wedge \nabla \Phi) = - \frac{1}{f^2} \boldsymbol{k} (\vec{\beta} \cdot \nabla \Phi) + \frac{1}{f} \boldsymbol{k} \triangle \Phi$$

using the obvious relation

$$(18.18) \qquad \boldsymbol{k} \cdot \vec{\beta} = 0$$

and well-known theorems from vector calculus. In (18.17) the term containing $\vec{\beta}$, cannot have a singularity due to the dynamic boundary condition. So, in order to study the singularities, nothing is lost by making the approximation

$$(18.19) = (17.17) \qquad \vec{\beta} \overset{*}{=} 0.$$

Then

$$(18.20) \qquad \vec{\zeta}_g \overset{*}{=} \frac{1}{f} \boldsymbol{k} \triangle \Phi \overset{*}{=} \frac{\boldsymbol{k}}{f} \operatorname{dip} \left[ \nabla \Phi_1 \, U(t-\sigma) + \nabla \Phi_2 \, U(\sigma - t) \right] \overset{*}{=}$$

$$\overset{*}{=} \frac{\boldsymbol{k}}{f} \Big[ \triangle \Phi_1 \, U(t-\sigma) + \triangle \Phi_2 \, U(\sigma - t) +$$

$$+ (\nabla \Phi_2 - \nabla \Phi_1) \cdot \nabla \sigma \, \delta(t-\sigma) \Big].$$

For the singular part of $\vec{\zeta}_g$ one finds, using the definition of the geostrophic wind (18.1)

$$(18.21) \qquad (\vec{\zeta}_g)_{sing} \overset{*}{=} \frac{\boldsymbol{k}}{f} \left[ -f \{ \boldsymbol{k} \wedge (\boldsymbol{u}_{g1} - \boldsymbol{u}_{g2}) \} \cdot \nabla \sigma \right] \delta(t-\sigma) \overset{*}{=}$$

$$\overset{*}{=} \boldsymbol{k} \cdot \left[ \boldsymbol{k} \wedge (\boldsymbol{u}_{g2} - \boldsymbol{u}_{g1}) \cdot \nabla \sigma \, \delta \right] (t-\sigma).$$

The following comment on this equation may be made.

(i) Since the term with $\vec{\beta}$ in (18.17) cannot become singular, the sign $\overset{*}{=}$ in (18.21) may be replaced by the sign for ordinary equality.

(ii) The singular part of the geostrophic isobaric vorticity is completely determined by the windshear in the SD.

(iii) The geostrophic wind is used in (18.21) as a substitute for $\nabla \Phi$. So the geostrophic *approximation* does not enter into this equation.

## 19. The vorticity equation with pressure as an independent variable

With the notations

(19.1) $D = \text{dip}\, \boldsymbol{u} = \nabla \cdot \boldsymbol{u}$ = isobaric divergence,

(19.2) $\vec{\zeta}_p = \nabla \wedge \boldsymbol{u}$ = relative isobaric vorticity,

(19.3) $\zeta_p = \boldsymbol{k} \cdot \vec{\zeta}_p$ = vertical component of the relative isobaric vorticity,

(19.4) $\eta = \zeta_p + f$ = vertical component of the absolute isobaric vorticity,

one may write the vorticity equation as

(19.5) $$\frac{d\eta}{dt} + \eta\, D + \boldsymbol{k} \cdot \left(\nabla \omega \wedge \frac{\partial \boldsymbol{u}}{\partial p}\right) = 0.$$

From

(19.6) $$\zeta_p = \boldsymbol{k} \cdot (\nabla \wedge \boldsymbol{u}) = \zeta_{p1}\, U(t-\sigma) + \zeta_{p2}\, U(\sigma-t) + \\ + \boldsymbol{k} \cdot \nabla \sigma \wedge (\boldsymbol{u}_2 - \boldsymbol{u}_1)\, \delta(t-\sigma)$$

and the abbreviation

(19.7) $$\zeta_{p3} = \boldsymbol{k} \cdot \nabla \sigma \wedge (\boldsymbol{u}_2 - \boldsymbol{u}_1)$$

one gets using the rules from section **14.3**

(19.8) $$\frac{d\eta}{dt} = \frac{d\eta_1}{dt}\, U(t-\sigma) + \frac{d\eta_2}{dt}\, U(\sigma-t) + \frac{d\eta_3}{dt}\, \delta(t-\sigma).$$

Further

(19.9) $$D = \text{dip}\, \boldsymbol{u} = D_1\, U(t-\sigma) + D_2\, U(\sigma-t) + (\boldsymbol{u}_2 - \boldsymbol{u}_1) \cdot \nabla \sigma\, \delta(t-\sigma),$$

(19.10) $$\eta D = \eta_1\, D_1\, U(t-\sigma) + \eta_2\, D_2\, U(\sigma-t) + \\ + \left[ D\boldsymbol{k} \cdot \nabla\sigma \wedge (\boldsymbol{u}_2 - \boldsymbol{u}_1) + \eta\, (\boldsymbol{u}_2 - \boldsymbol{u}_1) \cdot \nabla\sigma \right] \delta(t-\sigma) + \\ + \left[ \boldsymbol{k} \cdot \nabla\sigma \wedge (\boldsymbol{u}_2 - \boldsymbol{u}_1) \right] (\boldsymbol{u}_2 - \boldsymbol{u}_1) \cdot \nabla\sigma\, \delta^2(t-\sigma),$$

(19.11) $$\nabla\omega = \nabla\omega_1\, U(t-\sigma) + \nabla\omega_2\, U(\sigma-t) + (\omega_2 - \omega_1)\, \nabla\sigma\, \delta(t-\sigma),$$

(19.12) $$\frac{\partial \boldsymbol{u}}{\partial P} = \frac{\partial \boldsymbol{u}_1}{\partial P}\, U(t-\sigma) + \frac{\partial \boldsymbol{u}_2}{\partial P}\, U(\sigma-t) + (\boldsymbol{u}_2 - \boldsymbol{u}_1) \frac{\partial \sigma}{\partial P}\, \delta(t-\sigma),$$

and

$$(19.13)\qquad \boldsymbol{k}\cdot\nabla\omega\wedge\frac{\partial\boldsymbol{u}}{\partial P}=\boldsymbol{k}\cdot\left[\nabla\omega_1\wedge\frac{\partial\boldsymbol{u}_1}{\partial P}U(t-\sigma)+\nabla\omega_2\wedge\frac{\partial\boldsymbol{u}_2}{\partial P}U(\sigma-t)\right]+$$

$$+\boldsymbol{k}\cdot\left[(\omega_2-\omega_1)\nabla\sigma\wedge\frac{\partial\boldsymbol{u}}{\partial P}+\right.$$

$$\left.+\nabla\omega\wedge(\boldsymbol{u}_2-\boldsymbol{u}_1)\frac{\partial\sigma}{\partial P}\right]\delta(t-\sigma)+$$

$$+\boldsymbol{k}\cdot\nabla\sigma\wedge(\boldsymbol{u}_2-\boldsymbol{u}_1)(\omega_2-\omega_1)\frac{\partial\sigma}{\partial P}\delta^2(t-\sigma).$$

Taking terms together one obtains the vorticity equation in the form

$$(19.14)\qquad \left[\frac{d\eta_1}{dt}+\eta_1 D_1+\boldsymbol{k}\cdot\nabla\omega_1\wedge\frac{\partial\boldsymbol{u}_1}{\partial P}\right]U(t-\sigma)+$$

$$+\left[\frac{d\eta_2}{dt}+\eta_2 D_2+\boldsymbol{k}\cdot\nabla\omega_2\wedge\frac{\partial\boldsymbol{u}_2}{\partial P}\right]U(\sigma-t)+$$

$$+\left[\frac{d\eta_3}{dt}+D\,\boldsymbol{k}\cdot\nabla\sigma\wedge(\boldsymbol{u}_2-\boldsymbol{u}_1)+\eta\,(\boldsymbol{u}_2-\boldsymbol{u}_1)\cdot\nabla\sigma+\right.$$

$$\left.+(\omega_2-\omega_1)\nabla\sigma\wedge\frac{\partial\boldsymbol{u}}{\partial P}+\nabla\omega\wedge(\boldsymbol{u}_2-\boldsymbol{u}_1)\frac{\partial\sigma}{\partial P}\right]\delta(t-\sigma)=0.$$

One observes, that the terms with $\delta^2$ have dropped from the final equation, thanks to the kinematic boundary condition, otherwise the result would have been without any value. The interpretation of (19.14) is similar to that of (16.14) except for some terminological transformations. One should further note the fact that the introduction of pressure as an independent variable did not considerably simplify the vorticity equation. By using the pressure as an independent variable one takes into account that the vertical direction, due to the hydrostatic equation, occupies an exceptional position in dynamic meteorology. However, in the case of discontinuities present, there is a second direction of importance *viz.* that of the normal on the SD, so in this case the vertical direction looses much of its exceptional position.

## 20. Lagrangian Coordinates

**20.1** The reader, having seen in the preceding section that the introduction of pressure as an independent variable has in the case of discontinuous fields not the advantages known from applications to continuous fields, may recall a statement of ERTEL [25]: "Die LAGRANGEsche Form der hydrodynamischen Gleichungen hat der EULERschen Form gegenuber den Vorteil, dass sie den Bedürfnissen der "Luftkörper"-Meteorologie ganz besonders angepasst ist".

It cannot be denied that in spite of ERTEL's statement, the use of Lagrangian coordinates never has been become popular, even for purposes which specifically belong to the domain of air mass meteorology. Indeed, their use is even so impopular, that every author introducing Lagrangian coordinates feels himself obliged to do this only after writing a long introduction in which he develops a new notation which he believes to be more tractable than existing notations and which he also believes will make an end to the impopularity. The present author also could not avoid an introduction and a new notation, but is rather pessimistic with respect to the question of popularity. However, since in one of the problems studied in part III (Applications) the use of Lagrangian coordinates seems to contribute essentially to the full understanding, the subject of Lagrangian coordinates could not be avoided. Furthermore, the present part, in which the formal side of the calculus related to atmospheric discontinuities is studied, could hardly be considered complete if the Lagrange techniques were left out of consideration. So far for the introduction; now the notations follow.

**20.2** Let $\delta_{ij}$ be the KRONECKER delta

(20.1) $$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \quad (1 \leq i \leq 3) \text{ [1]} \\ 0 & \text{for } i \neq j \quad (1 \leq j \leq 3) \end{cases}$$

and $\varepsilon_{ijk}$ the signed permutation operator

(20.2) $$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } (i, j, k) \text{ even permutation of } (1, 2, 3) \\ -1 & \text{for } (i, j, k) \text{ odd permutation of } (1, 2, 3) \\ 0 & \text{for } (i, j, k) \text{ no permutation of } (1, 2, 3). \end{cases}$$

The author is aware of the fact that with the aid of the tensor calculus another interpretation of $\varepsilon_{ijk}$ is possible. However, for the purpose of this section it is sufficient to consider $\varepsilon_{ijk}$ as a signed permutation operator.

Well known are the rules

(20.3) $$\varepsilon_{ijk}\, \varepsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl},$$

(20.4) $$\varepsilon_{ijk}\, \varepsilon_{jkl} = 2\delta_{il},$$

(20.5) $$\varepsilon_{ijk}\, \varepsilon_{ijk} = 3!$$

Here, and in the following the usual summation convention is used.

For a determinant of 3 rows and 3 columns with elements $A_{ij}$ one may write with the ε-notation

(20.6) $$\det(A_{ij}) = \frac{1}{3!}\, \varepsilon_{ijk}\, \varepsilon_{lmn}\, A_{il}\, A_{jm}\, A_{kn}.$$

---

[1]) Without further notification all indices like *h*, *i*, *j*, *k*, *l*, *m*, *n* *etc.* are supposed to run from 1 to 3.

For the Lagrange or particle coordinates, which usually are written as $(a, b, c)$ here the notation $a_i = (a_1, a_2, a_3)$ will be used. For the geometrical coordinates or the radiusvector of a particle in motion $\boldsymbol{x}$ will be used. So

(20.7) $$\boldsymbol{x} = \boldsymbol{x}(a_i; t).$$

There exist two types of connecting quantities *viz.* $\nabla a_i$ and

(20.8) $$\boldsymbol{b}_i = \frac{\partial \boldsymbol{x}}{\partial a_i}.$$

Between these quantities there are two relations

(20.9) $$\boldsymbol{b}_i \cdot \nabla a_j = \delta_{ij}$$

and

(20.10) $$\boldsymbol{b}_i \nabla a_i = \mathrm{I},$$

in which I denotes the unit tensor, unit dyadic or unit matrix in the $\boldsymbol{x}$-space

(20.11) $$\mathrm{I} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}.$$

Of great importance are the functional determinants or Jacobians

(20.12) $$J = \frac{\partial (x, y, z)}{\partial (a_1, a_2, a_3)} = \frac{1}{6} \varepsilon_{ijk} (\boldsymbol{b}_i \wedge \boldsymbol{b}_j) \cdot \boldsymbol{b}_k$$

(20.13) $$\frac{1}{J} = \frac{\partial (a_1, a_2, a_3)}{\partial (x, y, z)} = \frac{1}{6} \varepsilon_{ijk} (\nabla a_i \wedge \nabla a_j) \cdot \nabla a_k$$

The notation $J$ (Jacobian) is used instead of the customary $\Theta$, in order to have $\Theta$ available for the potential temperature. If one interprets (20.9) as a linear equation for $\nabla a_j$ and (20.10) as an equation for $\boldsymbol{b}_i$ then by CRAMER's rule one obtains for the solutions

(20.14) $$2 \nabla a_i = \frac{1}{J} \varepsilon_{ijk} \boldsymbol{b}_j \wedge \boldsymbol{b}_k$$

and

(20.15) $$2 \boldsymbol{b}_i = J \varepsilon_{ijk} \nabla a_j \wedge \nabla a_k.$$

In an easily understood symbolism one may write

(20.16) $$\frac{\partial}{\partial a_i} = \frac{\partial \boldsymbol{x}}{\partial a_i} \cdot \nabla = \boldsymbol{b}_i \cdot \nabla$$

and

(20.17) $$\nabla = \nabla a_i \cdot \frac{\partial}{\partial a_i}.$$

which one may apply to prove relations of which typical examples are given by

(20.18) $$\operatorname{div} \boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial a_i} \cdot \nabla a_1$$

and

(20.19) $$\frac{\partial \varphi}{\partial a_i} = \boldsymbol{b}_i \cdot \nabla \varphi_i.$$

**20.3** Already EULER knew that there is a remarkable relation between the equation of continuity and the Jacobian $J$. In its Eulerian form the equation of continuity is written as

(20.20) $$\frac{d\varrho}{dt} + \varrho \operatorname{div} \boldsymbol{v} = 0.$$

Translating this equation into the Lagrangian form one obtains

(20.21) $$\frac{1}{\varrho}\frac{\partial \varrho}{dt} + \operatorname{div} \frac{\partial \boldsymbol{x}}{dt} = 0$$

which may be transformed as follows

(20.22) $$\frac{1}{\varrho}\frac{\partial \varrho}{\partial t} + \nabla a_i \cdot \frac{\partial^2 \boldsymbol{x}}{\partial a_i\, \partial t} = 0,$$

(20.23) $$\frac{1}{\varrho}\frac{\partial \varrho}{\partial t} + \frac{1}{2J}\, \varepsilon_{ijk}\, \boldsymbol{b}_i \wedge \boldsymbol{b}_k \cdot \frac{\partial \boldsymbol{b}_i}{\partial t} = 0,$$

(20.24) $$\frac{1}{\varrho}\frac{\partial \varrho}{\partial t} + \frac{1}{6J}\, \varepsilon_{ijk} \frac{\partial}{\partial t}\left[\boldsymbol{b}_i \wedge \boldsymbol{b}_k \cdot \boldsymbol{b}_i\right] = 0,$$

(20.25) $$\frac{1}{\varrho}\frac{\partial \varrho}{\partial t} + \frac{1}{J}\frac{\partial J}{\partial t} = 0$$

or

(20.26) $$J\varrho = \text{const.} = J_0\, \varrho_0,$$

in which $J_0$ and $\varrho_0$ denote the values of $J$ and $\varrho$ at some arbitrary but fixed time say $t = t_0$. That (20.24) is indeed a consequence of (20.23) is proved by

(20.27) $$\varepsilon_{ijk} \frac{\partial}{\partial t}\left[\boldsymbol{b}_j \wedge \boldsymbol{b}_k \cdot \boldsymbol{b}_i\right] =$$
$$= \varepsilon_{ijk}\left[\frac{\partial \boldsymbol{b}_j}{\partial t} \wedge \boldsymbol{b}_k \cdot \boldsymbol{b}_i + \boldsymbol{b}_j \wedge \frac{\partial \boldsymbol{b}_k}{\partial t} \cdot \boldsymbol{b}_i + \boldsymbol{b}_j \wedge \boldsymbol{b}_k \cdot \frac{\partial \boldsymbol{b}_i}{\partial t}\right] =$$
$$= 3\, \varepsilon_{ijk}\, \boldsymbol{b}_j \wedge \boldsymbol{b}_k \cdot \frac{\partial \boldsymbol{b}_i}{\partial t}.$$

Sometimes one chooses the coordinates $\boldsymbol{a}_i$ in such a manner that for the time $t = t_0$, $a_i = x_i$ in order to make $J_0 = 1$. For general purposes this is not to be advocated. Making $J_0 = 1$ deprives the equations of their symmetrical structure and makes dimensional checking more troublesome. The equation of motion in Lagrangian form follows from

$$\text{(20.28)} \qquad \frac{d\boldsymbol{v}}{dt} + 2\vec{\Omega} \wedge \boldsymbol{v} + \nabla\Phi + \frac{1}{\varrho}\nabla P = 0$$

by the well-known manipulations

$$\text{(20.29)} \qquad \frac{\partial^2 \boldsymbol{x}}{\partial t^2} + 2\vec{\Omega} \wedge \frac{\partial \boldsymbol{x}}{\partial t} + \nabla\Phi + \frac{1}{\varrho}\nabla P = 0,$$

$$\text{(20.30)} \qquad \frac{\partial^2 \boldsymbol{x}}{\partial t^2} + 2\vec{\Omega} \wedge \frac{\partial \boldsymbol{x}}{\partial t} + \nabla a_i \frac{\partial \Phi}{\partial a_i} + \frac{1}{\varrho}\nabla a_i \frac{\partial P}{\partial a_i} = 0,$$

$$\text{(20.31)} \qquad \boldsymbol{b}_i \cdot \left\{\frac{\partial^2 \boldsymbol{x}}{\partial t^2} + 2\vec{\Omega} \wedge \frac{\partial \boldsymbol{x}}{\partial t}\right\} + \frac{\partial \Phi}{\partial a_i} + \frac{1}{\varrho}\frac{\partial P}{\partial a_i} = 0.$$

To these equations one should add the thermal equation which in Lagrangian coordinates takes the simple form

$$\text{(20.32)} \qquad \frac{\partial \Theta}{\partial t} = 0,$$

$\Theta$ being as usually the potential temperature.

**20.4** In textbooks of dynamic meteorology the vorticity equation is as a rule not given in the Lagrangian form. There is even a wide spread belief amongst meteorologists that an equation of this type does not exist at all. TRUESDELL in his monograph [77] to which reference already was made, has clearly demonstrated that also in this matter BELTRAMI [1] was in the possession of all essential knowledge. All the present author had to do, was to translate BELTRAMI's results into the notation adopted in this study, and to introduce a rotating coordinate system.

Let $\boldsymbol{V}$ be the absolute velocity

$$\text{(20.33)} \qquad \boldsymbol{V} = \boldsymbol{v} + \vec{\Omega} \wedge \boldsymbol{x}$$

and $\boldsymbol{Q}$ the absolute vorticity

$$\text{(20.34)} \qquad \boldsymbol{Q} = \text{rot}\, \boldsymbol{V}$$

then it follows from (20.17)

$$Q = \nabla \wedge V = \nabla a_i \frac{\partial}{\partial a_i} \wedge (v + \vec{\Omega} \wedge x) = \tag{20.35}$$

$$= \frac{1}{2J} \varepsilon_{ijk} (b_j \wedge b_k) \wedge \frac{\partial}{\partial a_i} \left(\frac{\partial x}{\partial t} + \vec{\Omega} \wedge x\right) =$$

$$= \frac{1}{2J} \varepsilon_{ijk} (b_j \wedge b_k) \wedge \left(\frac{\partial b_i}{\partial t} + \vec{\Omega} \wedge b_i\right) =$$

$$= \frac{1}{2J} \varepsilon_{ijk} \left[ b_k \left\{ b_j \cdot \left(\frac{\partial b_i}{\partial t} + \vec{\Omega} \wedge b_i\right) - b_j \cdot \left(\frac{\partial b_i}{\partial t} + \vec{\Omega} \wedge b_i\right) \right\} \right],$$

or

$$Q = \frac{1}{J} \varepsilon_{ijk} b_k \left[ b_j \cdot \left(\frac{\partial b_i}{\partial t} + \vec{\Omega} \wedge b_i\right) \right]. \tag{20.36}$$

Now, if one defines the quantity $Z_h$ by

$$Z_h = \varepsilon_{hij} b_i \cdot \left(\frac{\partial b_j}{\partial t} + \vec{\Omega} \wedge b_j\right) = \varepsilon_{hij} b_i \cdot \frac{\partial}{\partial a_i} (v + \vec{\Omega} \wedge x) \tag{20.37}$$

then one obtains

$$J Q = b_h Z_h. \tag{20.38}$$

The introduction of the quantity $Z_h$ solves the first part of the problem which consists in the construction of an entity which in the Lagrange technique may play the same role as $Q$ in the Eulerian system. One observes in the definition (20.37) of $Z_h$ that this quantity is related to the difference between the velocities of two material points which in the $a_i$-space differ from each other by an infinitisimal rotation, just as one would expect for a vorticity to do. The second part of the problem consists in the construction of a differential equation for $Z_h$ which must describe its behaviour in time. To do this, one differentiates the equation of motion (20.31)

$$\frac{\partial^2 x}{\partial a_i \partial a_j} \cdot \left\{ \frac{\partial^2 x}{\partial t^2} + 2\vec{\Omega} \wedge \frac{\partial x}{\partial t} \right\} + \frac{\partial x}{\partial a_i} \cdot \left\{ \frac{\partial^3 x}{\partial a_j \partial t^2} + 2\vec{\Omega} \wedge \frac{\partial^2 x}{\partial a_j \partial t} \right\} + \tag{20.39}$$

$$+ \frac{\partial^2 \Phi}{\partial a_i \partial a_j} + \frac{\partial}{\partial a_j} \left(\frac{1}{\varrho}\right) \frac{\partial P}{\partial a_i} + \frac{1}{\varrho} \frac{\partial^2 P}{\partial a_j \partial a_i} = 0.$$

If one multiplies this equation with $\varepsilon_{hij}$, then all terms of (20.39) which are symmetric in the indices $i$ and $j$ drop out since for any symmetrical quantity $c_{ij} = c_{ji}$ one has

$$\varepsilon_{hij} c_{ij} = \varepsilon_{hij} c_{ji} = - \varepsilon_{hji} c_{ji} = - \varepsilon_{hij} c_{ij} = 0. \tag{20.40}$$

So, after multiplication of (20.39) with $\varepsilon_{hij}$ one gets

$$\varepsilon_{hij} \boldsymbol{b}_i \cdot \left\{ \frac{\partial^2 \boldsymbol{b}_j}{\partial t^2} + 2 \vec{\Omega} \wedge \frac{\partial \boldsymbol{b}_j}{\partial t} \right\} + \varepsilon_{hij} \frac{\partial}{\partial a_i} \left( \frac{1}{\varrho} \right) \frac{\partial P}{\partial a_j} = 0. \tag{20.41}$$

Now, it follows from the definition of $Z_h$ (20.37)

$$\frac{\partial Z_h}{\partial t} = \varepsilon_{hij} \boldsymbol{b}_i \cdot \left\{ \frac{\partial^2 \boldsymbol{b}_j}{\partial t^2} + 2 \vec{\Omega} \wedge \frac{\partial \boldsymbol{b}_j}{\partial t} \right\} \tag{20.42}$$

so that finally one obtains the vorticity equation in the form

$$\frac{\partial Z_h}{\partial t} + \varepsilon_{hij} \frac{\partial}{\partial a_i} \left( \frac{1}{\varrho} \right) \frac{\partial P}{\partial a_j} = 0. \tag{20.43}$$

So the meteorologists who believe that in Lagrangian coordinates a vorticity equation could not be formulated are wrong. As a matter of fact the vorticity equation has even taken a simpler form in Lagrangian coordinaties than the more familiar equation in Eulerian coordinates. One should not forget to observe that the variation in time of $Z_h$ is completely determined by the number of isosteric-isobaric solenoides in the $a_i$-space.

**20.5** With the introduction of the quantity $Z_h$ the formal apparatus for the manipulation with Lagrangian coordinates is completed, and now a moment of reflection seems to be appropriate. One observes that the notations developed in this section enable us to work with Lagrangian coordinates without using explicitely determinants and minors of determinants which make the application of other systems of notation so cumbersome and tedious. However, it must be admitted that even this does not make Lagrangian coordinates an easy instrument of investigation. The main difficulty seems to be the fact that calculations which may be termed straightforward in Eulerian coordinates cannot be imitated in Lagrangian coordinates without some additional detours. However, one may expect that some advantages of the Lagrangian coordinates will manifest themselves, if one takes discontinuous fields into consideration, as, according to ERTEL, these coordinates in particular should be adjusted to the demands of air mass meteorology. Indeed, it will be seen in the next part of this section that the introduction of discontinuities into the calculations is an easy job in the Lagrangian technique.

**20.6** The main simplification, which gives the use of LAGRANGIAN coordinates in the case of discontinuous fields over the use of Eulerian coordinates, stems from the simple form by which the SD can be described in Lagrangian coordinates. Since the SD consists at all times of the same air particles the SD may be described by an equation of the type

$$\Psi(a_i) = 0, \tag{20.44}$$

in which the time does not enter. As an immediate consequence of the absence of $t$ in this description it follows that now there is no need for an equation like

$$(20.45) = (9.3) \qquad (1 - \boldsymbol{v} \cdot \nabla S)\, \delta(t - S) = 0.$$

The kinematic boundary condition takes the form

$$(20.46) \qquad (\boldsymbol{v}_2 - \boldsymbol{v}_1) \cdot \nabla \Psi\, \delta(\Psi) = 0$$

or, on account of (20.17)

$$(20.47) \qquad \varepsilon_{ijk} \left( \frac{\partial \boldsymbol{x}_2}{\partial t} - \frac{\partial \boldsymbol{x}_1}{\partial t} \right) \cdot (\boldsymbol{b}_i \wedge \boldsymbol{b}_k) \frac{\partial \Psi}{\partial a_i}\, \delta(\Psi) = 0$$

whereas the dynamic boundary condition is expressed by

$$(20.48) \qquad (P_1 - P_2)\, \delta(\Psi) = 0.$$

The further development of the calculus will be postponed untill it becomes necessary for the applications. For the present it may be sufficient to observe that the equations related to the SD are indeed somewhat simpler than in the Eulerian form. One may, however, expect that these simplifications are as a rule not important enough to compensate for the general complications involved by the use of the Lagrangian coordinates. As a matter of fact in the next part (Applications) of this study, only one example will be found in which the use of Lagrangian coordinates seems to contribute essentially in a simple manner to the understanding of the problem. Attention may be drawn to the remarkable fact that this happens to be a problem pertaining exactly to the only equation which is usually not studied at all with the aid of the Lagrangian technique *viz.* the vorticity equation.

With the description of the use of Lagrangian coordinates in cases in which the variables are not ordinary functions but distributions, the development of the formal apparatus for handling discontinuous fields is completed.

PART III

# APPLICATIONS

## 21. Introduction

In this part a beginning is made with the application of the calculus developed in the preceding sections to specific problems. Although it is next to impossible to give all applications, the examples given are carefully selected so that they represent typical cases. In the first example (sect. **22**) the atmospheric energy balances are studied. The next section (**23**) is devoted to the construction of a criterion for the correctness of approximations already announced in the General Introduction. As a first application of this criterion the correctness is studied of various approximations of which the so-called isallobaric wind is typical (sect. **24**). In this section also an old controversial point – the theory of singular advection – is discussed, on which some new observations can be made on the basis of the present theory. The criterion will be further applied to discuss the basic equations for the technique of differential analysis (sect. **25**) and to the use of the ROSSBY approximation of the potential vorticity (sect. **26**). One of the most important discoveries made by the use of the calculus of distributions deals with the incorrectness of the ROSSBY approximation. The next section (**26**) is devoted to the study of the relation between potential vorticity and dynamic instability in which some results are found which hitherto seem to have been unknown even in the case of continuous fields. The last example (sect. **27**) of the application of the criterion for the correctness of approximations deals with the tendency equation. In the last but one section (**28**) new light is thrown on an old question pertaining to the problem why it is impossible that in a fluid whose motion is governed by the NAVIER-STOKES equation discontinuities exists, whereas in the concluding section (**29**) some remarks are made on the difficulties encountered in numerical weather prediction.

If one takes into account that the concept of potential vorticity and its ROSSBY approximation take important places in idealized models of which some are in daily use one observes, that with the aid of the calculus of discontinuities remarks on almost every subject discussed in contemporary dynamic meteorology can be made, of which some seem to have important consequences and hardly could have been made without the use of this calculus.

## 22. Energy balances

Starting with the equation of motion in the form

(22.1) $$\frac{d\boldsymbol{v}}{dt} + 2\vec{\Omega} \wedge \boldsymbol{v} + \nabla\Phi + \frac{1}{\varrho}\nabla P = 0$$

one obtains by scalar multiplication with $\boldsymbol{v}$ for the specific kinetic energy:

(22.2) $$K = \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}$$

the equation

(22.3) $$\frac{dK}{dt} + \boldsymbol{v}\cdot\nabla\Phi + \frac{1}{\varrho}\boldsymbol{v}\cdot\nabla P = 0,$$

which on account of the well-known formula

(22.4) $$\varrho\frac{dF}{dt} = \frac{\partial}{\partial t}(\varrho F) + \operatorname{div}(\varrho F\boldsymbol{v})$$

valid for any quantity $F$, may also be written as

(22.5) $$\frac{\partial}{\partial t}(\varrho K) + \operatorname{div}(\varrho K\boldsymbol{v}) + \varrho\,\boldsymbol{v}\cdot\nabla\Phi + \boldsymbol{v}\cdot\nabla P = 0.$$

For the specific potential energy $\Phi$ one has:

(22.6) $$\frac{d\Phi}{dt} = \boldsymbol{v}\cdot\nabla\Phi$$

or

(22.7) $$\frac{\partial}{\partial t}(\varrho\,\Phi) + \operatorname{div}(\varrho\,\Phi\,\boldsymbol{v}) - \varrho\,\boldsymbol{v}\cdot\nabla\,\Phi = 0.$$

Adding these equations one obtains the balance equation for the specific mechanical energy in the form:

(22.8) $$\frac{\partial}{\partial t}[\varrho\,(K+\Phi)] + \operatorname{div}[\varrho\,(K+\Phi)\,\boldsymbol{v}] + \boldsymbol{v}\cdot\nabla P = 0.$$

By the first law of thermodynamics one obtains for the specific internal energy $E$ the balance equation:

(22.9) $$\frac{dE}{dt} = -P\frac{d}{dt}\left(\frac{1}{\varrho}\right) = \frac{P}{\varrho^2}\frac{d\varrho}{dt} = -\frac{P}{\varrho}\operatorname{div}\boldsymbol{v},$$

or

(22.10) $$\frac{\partial}{\partial t}(\varrho\,E) + \operatorname{div}(\varrho\,E\,\boldsymbol{v}) + P\operatorname{div}\boldsymbol{v} = 0.$$

Strictly speaking, this is not true. Since the system under discussion *viz*. the atmosphere is an open system, there strictly speaking does not exist a first law of thermodynamics as has been shown by TOLHOEK and DE GROOT [76]. However, as all motions are supposed to be adiabatic, in this special case (22.9) may still be used.

Finally one obtains

(22.11) $$\frac{\partial}{\partial t}\,[\varrho\,(K+\Phi+E)+\operatorname{div}\,[\varrho\,(K+\Phi+E)\,\boldsymbol{v}]+\operatorname{div}\,(P\boldsymbol{v})=0.$$

Introduction of the specific enthalpy $H$, given by

(22.12) $$\varrho\,E+P=\varrho\,H$$

makes it possible to rewrite (22.11) as

(22.13) $$\frac{\partial}{\partial t}\,[\varrho\,(K+\Phi+E)]+\operatorname{div}\,[\varrho\,(K+\Phi+H)\,\boldsymbol{v}]=\text{o}.$$

If one uses instead of (22.9) the corresponding equation

(22.14) $$\frac{dH}{dt}=\frac{1}{\varrho}\frac{dP}{dt}$$

then one finds as an equation equivalent to (22.11)

(22.15) $$\frac{\partial}{\partial t}\,[\varrho\,(K+\Phi+H)]+\operatorname{div}\,[\varrho\,(K+\Phi+H)\,\boldsymbol{v}]=\frac{\partial P}{\partial t}$$

All these balance equations have the form

(22.16) $$F^L+F^A=F^Q$$

with

(22.17) $$F^L=\frac{\partial}{\partial t}\,(\varrho\,F),$$

and

(22.18) $$F^A=\operatorname{div}\,(\varrho\,F\,\boldsymbol{v}).$$

So the theorem proved in section **14.3** may be used in the following form: if in any of the balance equations the quantity $F$ has the structure

(22.19) = (14.8) $$F=F_1\,U(t—S)+F_2\,U(S—t)$$

without a term containing a delta function, then the local production $F^Q$ cannot show a singular behaviour. It is easily seen that none of the quantities $K$, $\Phi$, $E$ or $H$ shows a singular behaviour so it follows from the general theorem that none of the local production terms $K^Q$, $\Phi^Q$, $E^Q$ or $H^Q$ may posses a singularity, which means that for the kinetic energy, the potential energy, the internal energy and the enthalpy it may be stated that no singular production exists.

## 23. A general principle for discussing correctness of approximations

In the following sections some of the most frequently used approximations will be studied in order to detect whether they can be used in cases in which discontinuities are present or not. Since in the discussion the same type of argument always will be followed, the logical structure of the reasoning may be described separately.

Consider any quantity $F$ and let it be known that it has the structure

$$F = F_1\, U(t-S) + F_2\, U(S-t) + F_3\, \delta(t-S). \tag{23.1}$$

Let $F^*$ be an approximation of $F$ having the structure

$$F^* = F_1^*\, U(t-S) + F_2^*\, U(S-t) + F_3^*\, \delta(t-S). \tag{23.2}$$

For a good approximation it is necessary that

$$F_1 \approx F_1^*; \qquad F_2 \approx F_2^*; \qquad F_3 \approx F_3^* \tag{23.3}$$

The argument to be used consists in the principle that it cannot be tolerated that one of the following situations occur:

$$\left\{\begin{array}{lll} (A) & F_3 = 0; & F_3^* \neq 0 \\ \text{or} & & \\ (B) & F_3 \neq 0; & F_3^* = 0. \end{array}\right. \tag{23.4}$$

In the first case the approximation possesses a singularity which is not present in the unapproximated quantity whereas in the second case the approximation fails to show the singular behaviour of the unapproximated quantity. In both cases the approximation shows from the physical point of view a significant difference in behaviour with respect to the unapproximated quantity.

Already in the General Introduction to the present study it has been emphasized that the implications of the discussion are not restricted to situations in which discontinuities occur but are also valid in the more general situations in which layers of pronounced baroclinity are present. The principle formulated above is the main example of a result which is of more general importance in this sense. It should be recalled that a discontinuity may be considered as originating from a continuous field by some kind of limit process. If in the limit of discontinuity *e.g.* case $A$ is found to exist this means that in the limit $F$ remains finite whereas $F^*$ becomes infinite. However, this is only possible if before the passing to the limit $F^*$ has already become extremely large as compared to $F$. More generally if two quantities show a different type of singularity in the limit then they must show a different type of behaviour already before the limit is reached. So any difference in the type of singularity between two quantities in the case of a frontal surface reveals a physically different type of behaviour of these quantities in a layer of pronounced baroclinity. Approximations which belong to one of the cases (***A***)or (***B***) of (23.4) are therefore of no value inside baroclinic layers or frontal zones. All this is quite in the spirit of HEAVISIDE's dictum "What is proved for a discontinuity is proved for any sort of variation". (HEAVISIDE [33], Vol. III, sec. 452).

In the next sections examples will follow. The success of this method for discussing approximations rests on the fact that although exact solutions of the equations of dynamic meteorology are practically unknown, it is possible to determine exactly which type of singularity meteorological elements possess on discontinuities for almost every element of interest.

## 24. Isallobaric winds

**24.1** In order to find approximative solutions of the equation of motion

$$(24.1) = (17.7) \qquad \frac{d\boldsymbol{u}}{dt} + f\boldsymbol{k} \wedge \boldsymbol{u} + \nabla\Phi = 0$$

one usually follows a method first described by HESSELBERG [34] and made popular by H. PHILIPPS [59]. By vectorial multiplication of (24.1) by ***k*** one obtains

$$(24.2) \qquad \boldsymbol{k} \wedge \frac{d\boldsymbol{u}}{dt} - f\boldsymbol{u} + \boldsymbol{k} \wedge \nabla\Phi = \left(\boldsymbol{k} \wedge \frac{d}{dt} - f\right)\boldsymbol{u} + \boldsymbol{k} \wedge \nabla\Phi = 0,$$

from which the series development follows:

$$(24.3) \qquad \boldsymbol{u} = \frac{1}{f - \boldsymbol{k} \wedge \dfrac{d}{dt}} \cdot \boldsymbol{k} \wedge \nabla\Phi = \frac{1}{f} \sum_{n=1}^{\infty} \frac{1}{f^n} \left(\boldsymbol{k} \wedge \frac{d}{dt}\right)^n (\boldsymbol{k} \wedge \nabla\Phi).$$

The first term of this series appears to be the geostrophic wind

$$(24.4) \qquad \boldsymbol{u}_g = \frac{1}{f} \boldsymbol{k} \wedge \nabla\Phi.$$

The second term, representing the ageostrophic component of the wind, may be written as

$$(24.5) \qquad \boldsymbol{u}_a = \frac{1}{f^2} \left(\boldsymbol{k} \wedge \frac{d}{dt}\right) (\boldsymbol{k} \wedge \nabla\, \Phi) = \frac{1}{f^2} \frac{d}{dt} [\boldsymbol{k} \wedge (\boldsymbol{k} \wedge \nabla\, \Phi)] =$$

$$= -\frac{1}{f^2} \frac{d}{dt} (\nabla\Phi) = -\frac{1}{f^2} \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla + \omega \frac{\partial}{\partial P}\right) \nabla\, \Phi.$$

Usually this term is written as the sum of two terms $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ with

(24.6) $$\boldsymbol{u}_a = \boldsymbol{u}_i + \boldsymbol{u}_j,$$

(24.7) $$\boldsymbol{u}_i = -\frac{1}{f^2}\frac{\partial}{\partial t}\nabla\Phi,$$

and

(24.8) $$\boldsymbol{u}_j = -\frac{1}{f^2}\left(\boldsymbol{u}\cdot\nabla + \omega\frac{\partial}{\partial P}\right)\nabla\Phi.$$

The term $\boldsymbol{u}_i$ is usually referred to as the *isallobaric wind* and was first described by BRUNT and DOUGLAS [11]; the term $\boldsymbol{u}_j$ will be called here the *anisallobaric wind.* Exact determination of the anisallobaric wind is impossible because the unknown occurs on both sides of the equation (24.8). However, it is known that one may apply a kind of iteration process by substituting in the right-hand side of (24.8) for $\boldsymbol{u}$ the geostrophic approximation. So one obtains an approximation $\boldsymbol{u}_j^*$ for the anisallobaric wind given by

(24.9) $$\boldsymbol{u}_j^* = -\frac{1}{f^2}\boldsymbol{u}_g\cdot\nabla\nabla\Phi.$$

As usually, when considering geostrophic approximations, the vertical velocity $\omega$ has been assumed to be zero.

**24.2** In the next it will be shown, that this treatment of the second term in the HESSELBERG-PHILIPPS series development *i.e.* of the ageostrophic component of the wind must be used with caution if one wishes to avoid inconsistencies of the kind described in the preceding section.

From

(24.10) $$\Phi = \Phi_1\,U(t-\sigma) + \Phi_2\,U(\sigma-t),$$

one gets using the dynamic boundary condition

(24.11) $$\nabla\Phi = \nabla\Phi_1\,U(t-\sigma) + \nabla\Phi_2\,U(\sigma-t) + (\Phi_2-\Phi_1)\nabla\sigma\,\delta(t-\sigma) = \\ = \nabla\Phi_1\,U(t-\sigma) + \nabla\Phi_2\,U(\sigma-t)$$

So the isallobaric wind is given by

(24.12) $$\boldsymbol{u}_i = -\frac{1}{f^2}\left[\frac{\partial}{\partial t}(\nabla\Phi_1)\,U(t-\sigma) + \frac{\partial}{\partial t}(\nabla\Phi_2)\,U(\sigma-t) + \right. \\ \left. + (\nabla\Phi_1-\nabla\Phi_2)\,\delta(t-\sigma)\right]$$

and the anisallobaric wind by

$$(24.13)\qquad \boldsymbol{u}_j = -\frac{1}{f^2}\left\{(\boldsymbol{u}_1 \cdot \nabla)\,\nabla\Phi_1 + \omega_1 \frac{\partial\nabla\Phi_1}{\partial P}\right\} U(t-\sigma) -$$
$$-\frac{1}{f^2}\left\{(\boldsymbol{u}_2 \cdot \nabla)\,\nabla\Phi_2 + \omega_2 \frac{\partial\nabla\Phi_2}{\partial P}\right\} U(\sigma-t) -$$
$$-\frac{1}{f^2}(\nabla\Phi_2-\nabla\Phi_1)\left\{\boldsymbol{u} \cdot \nabla\sigma + \omega\frac{\partial\sigma}{\partial P}\right\} \delta(t-\sigma).$$

It is essential for the present discussion that in the derivation of equation (24.13) the geostrophic approximation is not used so that (24.13) gives an exact description of the behaviour of the anisallobaric wind. So, it has been proved exactly, using no approximation whatever, that both the isallobaric wind and the anisallobaric wind become singular on a SD. However if one adds these winds together according to (24.6) one finds that these singularities cancel, as:

$$(24.14)\qquad -\frac{1}{f^2}(\nabla\Phi_1-\nabla\Phi_2)\left[-1 + \boldsymbol{u} \cdot \nabla\sigma + \omega\frac{\partial\sigma}{\partial P}\right] \delta(t-\sigma) = 0$$

on account of (17.6).

So the ageostrophic component of the wind, being the second term in the series (24.3) is non singular, but if one considers this term as the sum of two parts $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$, these parts show singularities of equal magnitude but of different sign. This may expressed by saying that the quantities $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ are not independent but are strongly negatively coupled. It is quite clear that any reasoning which is based solely on the use of the BRUNT-DOUGLAS isallobaric wind is therefore fundamentally unsound. It is known that in the subjective forecasting techniques problems of development of pressure systems are often discussed by arguments in which the isallobaric wind plays a preponderant role. It follows from the present discussion that this is a dangerous technique for exactly when the isallobaric wind becomes large – as is the case in baroclinic zones – the anisallobaric wind becomes nearly as much as large but with opposite sign.

It might be observed, that the theorem which states that the second term of the series (24.9) is non-singular may be proved in a more direct way. For according to (24.11) $\nabla\Phi$ is non-singular, and it is known that the operator $\frac{d}{dt}$ introduces no new singularities as was proved in section **14.2**. However, the detour made in the derivation of the theorem was necessary in order to show the singular behaviour of its constituents $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$. Before discussing further details it may be repeated, that in the derivation of these results no approximation whatever has been made.

**24.3** For practical applications, however, one cannot use $\boldsymbol{u}_j$ which cannot be calculated exactly, but one is compelled to use the approximation $\boldsymbol{u}_j^*$ (24.9). It is easy to show that in using this approximation one obtains a singular term in the second term of the series (24.3) given by

$$(24.15) \qquad (\boldsymbol{u}_a)_{\text{sing}} = -\frac{1}{f^2}(\nabla\Phi_1 - \nabla\Phi_2)\,[1 - \boldsymbol{u}_g \cdot \nabla\sigma]\,\delta(t-\sigma).$$

So this approximation is an example of case (*A*) of the preceding section. The approximation introduces a singularity on the SD whereas it just has been proved exactly that the quantity discussed could not show a singular behaviour. In order to obtain a correct approximation one therefore should put

$$(24.16) \qquad (1 - \boldsymbol{u}_g \cdot \nabla\sigma)\,\delta(t-\sigma) = 0.$$

At first sight this looks like a rather new ad hoc hypothesis. However, comparing (24.16) with the exact equation

$$(24.17) = (17.6) \qquad \left(1 - \boldsymbol{u} \cdot \nabla\sigma - \omega\frac{\partial\sigma}{\partial P}\right)\delta(t-\sigma) = 0,$$

the physical background of (24.16) is readily discovered. Equation (24.17) expresses the fact that the SD moves with the component of the wind normal to the SD. It is clear that if one uses approximations of the wind field the velocity of the SD must be adapted to this approximation. The analysis given above clearly shows how this must be done in order not to distrurb consistency. The substitution which leads to (24.9) may be also described by saying that in (24.5) the operator $\frac{d}{dt}$ is approximated geostrophically. The final conclusion may then be given in the following wording. If one wishes to approximate the operator $\frac{d}{dt}$ in (24.5) geostrophically one should also use the geostrophic approximation (24.16) of (24.17) in order to avoid the appearance of singularities in the approximation which according to the exact theory are not present in the unapproximated wind.

If the correct method of approximation is used one finds the following expression for the second term in the HESSELBERG-PHILIPPS series

$$(24.18) \qquad \boldsymbol{u}_a^* = -\frac{1}{f^2}\left[\frac{\partial}{\partial t} + \boldsymbol{u}_g \cdot \nabla\right]\Phi =$$
$$= -\frac{1}{f^2}\left[\nabla\frac{\partial\Phi_1}{\partial t} + \boldsymbol{u}_{g1} \cdot \nabla\nabla\Phi_1\right]U(t-\sigma) -$$
$$-\frac{1}{f^2}\left[\nabla\frac{\partial\Phi_2}{\partial t} + \boldsymbol{u}_{g2} \cdot \nabla\nabla\Phi_2\right]U(\sigma-t)$$

under the condition (24.16).

This approximation is consistent provided that one does not try to split up the expression (24.18) in two components, an isallobaric and an anisallobaric component.

**24.4** Looking back on the preceding discussion the following observations seem to be appropriate:

(i) It is of paramount importance to investigate the independence of additive components. Often, and especially in subjective forecasting techniques, an effect is attributed to one component, without it being certain that the effect is not compensated by an other component which might be negatively coupled or correlated with the first one. In some cases as *e.g.* in the case of the HESSELBERG-PHILIPPS series development for the wind, the independence or rather the non-independence of additive components may be investigated by studying their behaviour on surfaces of discontinuity. The example given shows that coupling may become very strong in frontal zones or baroclinic layers.

(ii) In using approximations of the wind, the displacement of surfaces of discontinuity should be adapted to the approximation used. It is easy to see that one should approximate in the same manner all other advective processes.

(iii) One of the most discussed theories in recent history of dynamic meteorology which shows a violation of the rule given in (ii) has been the theory of the so-called singular advection. It is clear that the equation (24.17) is substantially equivalent to the kinematic boundary condition. So (ii) is equivalent to the rule that if one approximates the wind one must do it either without violating the kinematic boundary condition or use an approximated kinematic boundary condition adapted to the wind approximation used. It has been proved by SCHMIDT [65] that this exactly was the point in which the theory of the singular advection was basically wrong. If its protagonists had used the distribution technique it never would have come to a discussion as one could witness before SCHMIDT's paper [65] was published, because this technique gives the investigator tools to avoid inconsistant approximations in an easier way than otherwise could possibly have been found.

(iv) In conclusion of this section a remark may be made pertaining to the basic equations used in numerical weather prediction. Taking the pieces (24.4) and (24.18) together one obtains for the sum of the first two terms of the series (24.3) the expression

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a = \frac{1}{f}\boldsymbol{k} \wedge \nabla\Phi - \frac{1}{f^2}\left\{\frac{\partial}{\partial t} + \boldsymbol{u}_g \cdot \nabla\right\}\Phi. \tag{24.19}$$

Now it is known that the geostrophically approximated barotropic vorticity equation

$$(24.20) \qquad \frac{\partial \zeta_g}{\partial t} + \boldsymbol{u}_g \cdot \nabla \zeta_g + \vec{\beta} \cdot \boldsymbol{u}_g = 0$$

may be obtained by taking the divergence of (24.19) and assuming

$$(24.21) \qquad \text{dip}\, \boldsymbol{u} = 0,$$

as has been shown by HOLLMANN and REUTER [36]. This means that from the present point of view the (approximated) barotropic vorticity equation may be considered as sound as long as the assumption of nondivergence (24.21) is not violated.

Now

$$(24.22) \qquad \text{dip}\, \boldsymbol{u} = \text{dip}\, [\boldsymbol{u}_1\, U(t-\sigma) + \boldsymbol{u}_2\, U(\sigma-t)] =$$
$$= \text{dip}\, \boldsymbol{u}_1\, U(t-\sigma) + \text{dip}\, \boldsymbol{u}_2\, U(\sigma-t) + (\boldsymbol{u}_2-\boldsymbol{u}_1) \cdot \nabla\sigma\, \delta(t-\sigma)$$

which reveals, that this assumption cannot be valid at surfaces of discontinuity or more generally inside layers of pronounced baroclinity. So the application of (24.20) for purposes of numerical weather prediction is really confined to barotropic situations as one would expect it to do.

## 25. Differential analysis

In this section the basic equations of a technique usually called *differential analysis* will be discussed from the point of view of the criterion of consistency as constructed in section 23. Consider a layer $dP$ between two surfaces of constant pressure $P_{(1)}$ and $P_{(2)}$. In the traditional expositions of the technique of differential analysis one usually considers a finite layer (*cf.* e.g. SUTCLIFFE and FORSDYKE [72]). However, by considering an infinitisimal layer integration along the vertical and the use of some process of averaging may be avoided and the presentation may be simplified without loss of generality.

The geopotential difference or "thickness" $d\Phi$ is given by

$$(25.1) \qquad d\Phi = d\Phi_1\, U(t-\sigma) + d\Phi_2\, U(\sigma-t)$$

taking the dynamic boundary condition into account.

A second equation for $d\Phi$ follows from the hydrostatic equation (17.12)

$$(25.2) \qquad d\Phi = \frac{1}{\varrho}\, dP = RTd\,(\ln P) = R\,[T_1\, U(t-\sigma) + T_2\, U(\sigma-t)]\, d\,(\ln P).$$

By differentiation to the time one obtains a tendency equation for the thickness

$$(25.3) \qquad \frac{\partial}{\partial t}(d\Phi) = \frac{\partial}{\partial t}(d\Phi_1)\, U(t-\sigma) + \frac{\partial}{\partial t}(d\Phi_2)\, U(\sigma-t) +$$
$$+ (d\Phi_1 - d\Phi_2)\, \delta(t-\sigma) =$$
$$= \left[\frac{\partial T_1}{\partial t}\, U(t-\sigma) + \frac{\partial T_2}{\partial t}\, U(\sigma-t) + (T_2-T_1)\, \delta(t-\sigma)\right] d\,(\ln P)$$

Elimination of the terms $\frac{\partial T_1}{\partial t}$ and $\frac{\partial T_2}{\partial t}$ is now done by a known device consisting of the application of the first law of thermodynamics in the form

$$(25.4)\qquad \frac{dT}{dt} = \frac{dT_1}{dt}\,U(t-\sigma) + \frac{dT_2}{dt}\,U(\sigma-t) =$$

$$= \left[\frac{\partial T_1}{\partial t} + \boldsymbol{u}_1\cdot\nabla T_1 + \omega_1\frac{\partial T_1}{\partial P}\right]U(t-\sigma) +$$

$$+ \left[\frac{\partial T_2}{\partial t} + \boldsymbol{u}_2\cdot\nabla T_2 + \omega_2\frac{\partial T_2}{\partial P}\right]U(\sigma-t) =$$

$$= \frac{1}{c_p}\frac{dq}{dt} + \frac{1}{\varrho c_p}\frac{dP}{dt} =$$

$$= \left[\frac{1}{c_{p1}}\frac{dq_1}{dt} + \frac{\omega_1}{\varrho_1 c_{p1}}\right]U(t-\sigma) + \left[\frac{1}{c_{p2}}\frac{dq_2}{dt} + \frac{\omega_2}{\varrho_2 c_{p2}}\right]U(\sigma-t),$$

which leads to

$$(25.5)\qquad \frac{\partial}{\partial t}[d\Phi] = \left[R\left\{\frac{1}{c_{p1}}\frac{dq_1}{dt} - \boldsymbol{u}_1\cdot\nabla T_1 + \omega_1\left(\frac{1}{\varrho_1 c_{p1}} - \frac{\partial T_1}{\partial P}\right)\right\}U(t-\sigma)+\right.$$

$$+ R\left\{\frac{1}{c_{p2}}\frac{dq_2}{dt} - \boldsymbol{u}_2\cdot\nabla T_2 + \omega_2\left(\frac{1}{\varrho_2 c_{p2}} - \frac{\partial T_2}{\partial P}\right)\right\}U(\sigma-t)+$$

$$\left. + R\,(T_1-T_2)\,\delta(t-\sigma)\right]d\,(\ln P).$$

The thickness advection is given by

$$(25.6)\qquad \boldsymbol{u}\cdot\nabla\,(d\Phi) = R\,\boldsymbol{u}\cdot\nabla\,[T_1\,U(t-\sigma) + T_2\,U(\sigma-t)]\,d\,(\ln P) =$$

$$= R\,[\boldsymbol{u}_1\cdot\nabla T_1\,U(t-\sigma) + \boldsymbol{u}_2\cdot\nabla T_2\,U(\sigma-t) +$$

$$+ (T_2-T_1)\,\boldsymbol{u}\cdot\nabla\sigma\,\delta(\sigma-t)]\,d\,(\ln P)$$

which gives for (25.5)

$$(25.7)\qquad \frac{\partial}{\partial t}[d\Phi] = -\,\boldsymbol{u}\cdot\nabla(d\Phi) +$$

$$+ R\left[\left\{\frac{1}{c_{p1}}\frac{dq_1}{dt} + \omega_1\left(\frac{1}{\varrho_1 c_{p1}} - \frac{\partial T_1}{\partial P}\right)\right\}U(t-\sigma) +\right.$$

$$+ \left\{\frac{1}{c_{p2}}\frac{dq_2}{dt} + \omega_2\left(\frac{1}{\varrho_2 c_{p2}} - \frac{\partial T_2}{\partial P}\right)\right\}U(\sigma-t) +$$

$$\left. + (T_1-T_2)\,(1 - \boldsymbol{u}\cdot\sigma\nabla)\,\delta(t-\sigma)\right]d\,(\ln P)$$

By using (17.6) this equation may be simplified to

$$(25.8)\qquad \frac{\partial}{\partial t}[d\Phi] = -\boldsymbol{u}\cdot\nabla(d\Phi) +$$

$$+R\left[\left\{\frac{1}{c_{p1}}\frac{dq_1}{dt}+\omega_1\left(\frac{1}{\varrho_1 c_{p1}}-\frac{\partial T_1}{\partial P}\right)\right\}U(t-\sigma)+\right.$$

$$+\left\{\frac{1}{c_{p2}}\frac{dq_2}{dt}+\omega_2\left(\frac{1}{\varrho_2 c_{p2}}-\frac{\partial T_2}{\partial P}\right)\right\}U(\sigma-t)+$$

$$\left.+(T_1-T_2)\,\omega\frac{\partial\sigma}{\partial P}\,\delta(t-\sigma)\right]d(\ln P).$$

This is the basic equation for the theory of differential analysis. Consider first the advective term $A$ in (25.8)

$$(25.9)\qquad A=\left\{\frac{\partial}{\partial t}[d\Phi]\right\}_{adv}=-\boldsymbol{u}\cdot\nabla(d\Phi).$$

In geostrophic approximation this is equivalent to

$$(25.10)\qquad A=-\boldsymbol{u}_g\cdot\nabla(d\Phi)=\frac{1}{f}[\nabla(d\Phi\wedge\nabla\Phi)]\cdot\boldsymbol{k}.$$

Consider the identity

$$(25.11)\qquad \frac{1}{f}[\nabla(d\Phi)\wedge\nabla\Phi)\cdot\boldsymbol{k}=\frac{1}{f}[\nabla(\Phi+d\Phi)\wedge\nabla\Phi]\cdot\boldsymbol{k}=$$

$$=\frac{1}{f}[\nabla(d\Phi)\wedge\nabla(\Phi+d\Phi)]\cdot\boldsymbol{k}$$

which gives the justification of the following well-known theorem: In order to calculate the advective part of the thickness tendency one may advect the thickness with either the wind at the level $P_{(1)}$ or the wind at the level $P_{(2)}$, the difference being zero in geostrophic approximation on account of the fact that the vector difference between these winds is in geostrophic approximation parallel to the thickness lines and cannot therefore contribute to the thickness advection. Emphasis must be laid upon the fact according to (25.11) this remains true in the presence of discontinuities. However, if one writes $A$ in the developed form:

$$(25.12)\qquad A=-\boldsymbol{u}\cdot\nabla(d\Phi)=-\boldsymbol{u}_1\nabla(d\Phi_1)\,U(t-\sigma)-\boldsymbol{u}_2\cdot\nabla\Phi\,U(\sigma-t)+$$

$$+\nabla(\Phi_1-\Phi_2)\cdot\boldsymbol{u}\frac{\partial\sigma}{\partial P}dP\,\delta(t-\sigma)$$

one must be so cautious not to consider the last term independently if one wishes to calculate the singular contribution to the tendency, since also the last term of (25.8) gives a singular contribution to the thickness tendency.

The total singular thickness tendency is given by

$$\text{(25.13)} \qquad \left\{\frac{\partial}{\partial t}(d\Phi)\right\}_{sing} =$$

$$= \left\{\nabla(\Phi_1 - \Phi_2)\cdot \boldsymbol{u}\frac{\partial\sigma}{\partial P}\,dP + R\,(T_1 - T_2)\,\omega\frac{\partial\sigma}{\partial P}\frac{dP}{P}\right\}\delta(t-\sigma).$$

In geostrophic approximation $\omega = 0$ and $\boldsymbol{u}$ is perpendicular to $\nabla\Phi_1$ and $\nabla\Phi_2$ so that in geostrophic approximation there is no singular contribution to the thickness advection. Although this violates the criterion given in section **23**, the geostrophic approximation may be considered as relatively harmless since it is consistent in the following sense: In geostrophic approximation there is not only no singular thickness tendency but also the advective part of it is zero. So the fact that in geostrophic approximation the singular thickness tendency is zero is in this case not caused by mutual annihilation of infinite additive components.

Next consider the remaining terms of the thickness tendency equation (25.8)

$$\text{(25.14)} \qquad R\left\{\frac{1}{c_{p1}}\frac{dq_1}{dt}U(t-\sigma) + \frac{1}{c_{p2}}\frac{dq_2}{dt}U(\sigma-t)\right\}d\,(\ln P) =$$

$$= R\left\{\frac{1}{\omega_1}\left(\frac{1}{\varrho_1 c_{p1}} - \frac{\partial T_1}{\partial P}\right)U(t-\sigma) + \frac{1}{\omega_2}\left(\frac{1}{\varrho_2 c_{p2}} - \frac{\partial T_2}{\partial P}\right)U(\sigma-t)\right\}d\,(\ln P)$$

They describe the effects of non-adiabatic heating and of difference in vertical stability in the "air masses" on the thickness tendency. Since their discussion would not reveal things not known from the non-discontinuous case reference may be made to literature *e.g.* SUTCLIFFE and FORSDYKE [72].

Incidentally it may however be remarked that according to (25.4) no singular release of heat takes place in the SD, a conclusion which remains valid if phase transitions are allowed.

## 26. Potential vorticity

**26.1** In this section the criterion constructed in section **23** will be applied to some approximations related to the concept of *potential vorticity*. Some remarkable facts will be found which seem to be entirely unknown.

The vorticity equation in Cartesian coordinates has already been given

$$\text{(26.1)} = \text{(16.8)} \qquad \frac{d\boldsymbol{Q}}{dt} - \boldsymbol{Q}\cdot\nabla\boldsymbol{v} + \boldsymbol{Q}\,\mathrm{div}\,\boldsymbol{v} + \nabla\frac{1}{\varrho}\wedge\nabla P = 0.$$

Using the continuity equation (26.1) may be also written as

$$\varrho \frac{d}{dt}\left(\frac{\boldsymbol{Q}}{\varrho}\right) = \boldsymbol{Q} \cdot \nabla \boldsymbol{v} + \nabla P \wedge \nabla \frac{1}{\varrho}. \tag{26.2}$$

For every function $F$ one obtains, using (26.2) the relation

$$\begin{aligned}
\varrho \frac{d}{dt}\left(\frac{\boldsymbol{Q}\cdot\nabla F}{\varrho}\right) &= \boldsymbol{Q}\frac{d}{dt}(\nabla F) + \varrho \nabla F \cdot \frac{d}{dt}\left(\frac{\boldsymbol{Q}}{\varrho}\right) = \\
&= \boldsymbol{Q} \cdot \left[\frac{\partial}{\partial t}(\nabla F) + \boldsymbol{v} \cdot \nabla\nabla F\right] + \\
&+ \nabla F \cdot \left[\boldsymbol{Q} \cdot \nabla \boldsymbol{v} + \nabla P \wedge \nabla \frac{1}{\varrho}\right] = \\
&= \boldsymbol{Q} \cdot \left[\nabla\left\{\frac{dF}{dt} - \boldsymbol{v} \cdot \nabla F\right\} + \boldsymbol{v} \cdot \nabla\nabla F\right] + \\
&+ \nabla F \cdot \left[\boldsymbol{Q} \cdot \nabla \boldsymbol{v} + \nabla P \wedge \nabla \frac{1}{\varrho}\right] = \\
&= \boldsymbol{Q} \cdot \left[\nabla \frac{dF}{dt} - \nabla \boldsymbol{v} \cdot \nabla F\right] + \\
&+ \nabla F \cdot \left[\boldsymbol{Q} \cdot \nabla \boldsymbol{v} + \nabla P \wedge \nabla \frac{1}{\varrho}\right]
\end{aligned} \tag{26.3}$$

or

$$\varrho \frac{d}{dt}\left(\frac{\boldsymbol{Q}\cdot\nabla F}{\varrho}\right) = \boldsymbol{Q} \cdot \nabla \frac{dF}{dt} + \nabla F \cdot \left(\nabla P \wedge \nabla \frac{1}{\varrho}\right). \tag{26.4}$$

This formula is usually called ERTEL's *Wirbelsatz* (*circulation theorem*) (cf. ERTEL [26]).

If one takes $F = \boldsymbol{x}$ one obtains the vorticity equation (26.2) back. By taking however $F = \Theta$ one obtains an important new result

$$\varrho \frac{d}{dt}\left(\frac{\boldsymbol{Q}\cdot\nabla \Theta}{\varrho}\right) = \nabla \Theta \cdot \left(\nabla P \wedge \nabla \frac{1}{\varrho}\right). \tag{26.5}$$

This equation may be simplified as follows. Since $\Theta$ may be considered as a function of $P$ and $\varrho$

$$\Theta = \Theta(P, \varrho) \tag{26.6}$$

one has

$$\frac{\partial\left(\Theta; P; \frac{1}{\varrho}\right)}{\partial(x; y; z)} = \nabla \Theta \cdot \left(\nabla_{\imath} \quad \wedge \nabla \frac{1}{\varrho}\right) = 0. \tag{26.7}$$

So one obtains

$$(26.8) \qquad \frac{d}{dt}\left(\frac{\boldsymbol{Q}\cdot\nabla\Theta}{\varrho}\right) = 0.$$

Formula (26.8) is the exact description of the theorem of *conservation of potential vorticity*. The original formulation of this theorem by ROSSBY [64] is of an approximative nature and may be found from the exact equation by following a device described by VAN MIEGHEM [52, 53].

In good approximation one has

$$(26.9) \qquad \frac{\boldsymbol{Q}\cdot\nabla\Theta}{\varrho} \approx \frac{1}{\varrho}\frac{\partial\Theta}{\partial z}Q_z = -g\frac{\partial\Theta}{\partial P}Q_z,$$

so that in this approximation

$$(26.10) \qquad \frac{d}{dt}\left(\frac{\partial\Theta}{\partial P}Q_z\right) = 0.$$

This is equivalent to ROSSBY's formulation, which says that

$$(26.11) \qquad \frac{Q_z}{\triangle P}$$

is a conservative quantity if $\triangle P$ is the pressure difference between two surfaces of constant potential temperature (isentropic surfaces).

For further discussion one needs the expression for the conservation theorem in $(x, y, P, t)$-coordinates. Remembering the vorticity equation

$$(26.12) = (19.5) \qquad \frac{d\eta}{dt} + \eta D + \boldsymbol{k}\cdot\left(\nabla\omega\wedge\frac{\partial\boldsymbol{u}}{\partial P}\right) = 0$$

one obtains for an arbitrary function $F$ the counterpart of ERTEL's circulation theorem as follows:

$$(26.13) \qquad \frac{d}{dt}\left(\eta\frac{\partial F}{\partial P}\right) = \frac{d\eta}{dt}\frac{\partial F}{\partial P} + \eta\frac{d}{dt}\left(\frac{\partial F}{\partial P}\right) =$$

$$= \frac{d\eta}{dt}\frac{\partial F}{\partial P} + \eta\left[\frac{\partial^2 F}{\partial t\,\partial P} + \boldsymbol{u}\cdot\nabla\frac{\partial F}{\partial P} + \omega\frac{\partial^2 F}{\partial P^2}\right] =$$

$$= \frac{d\eta}{dt}\frac{\partial F}{\partial P} + \eta\left[\frac{\partial}{\partial P}\left(\frac{\partial F}{dt}\right) - \frac{\partial\boldsymbol{u}}{\partial P}\cdot\nabla F - \frac{\partial\omega}{\partial P}\frac{\partial F}{\partial P}\right] =$$

$$= \frac{\partial F}{\partial P}\left[\frac{d\eta}{dt} - \eta\frac{\partial\omega}{\partial P}\right] + \eta\left[\frac{\partial}{\partial P}\left(\frac{\partial F}{dt}\right) - \frac{\partial\boldsymbol{u}}{\partial P}\cdot\nabla F\right] =$$

$$= \frac{\partial F}{\partial P}\left[\frac{d\eta}{dt} + \eta D\right] + \eta\left[\frac{\partial}{\partial P}\left(\frac{dF}{dt}\right) - \frac{\partial\boldsymbol{u}}{\partial P}\cdot\nabla F\right],$$

or

$$(26.14)\qquad \frac{d}{dt}\left(\eta\frac{\partial F}{\partial P}\right)=\eta\frac{\partial}{\partial P}\left(\frac{\partial F}{dt}\right)+\frac{\partial \boldsymbol{u}}{\partial P}\cdot\left[\frac{\partial F}{\partial P}\nabla\omega\wedge\boldsymbol{k}-\eta\nabla F\right].$$

Since this is ERTEL's theorem, one should be able te come back to (26.12) by substitution of a suitably chosen quantity for $F$. Indeed $F=P$ gives formula (26.12).

Now, substitution of $F=\Theta$ gives, taking $\frac{d\Theta}{dt}=0$ into account

$$(26.15)\qquad \frac{d}{dt}\left(\eta\frac{\partial\Theta}{\partial P}\right)=\frac{\partial \boldsymbol{u}}{\partial P}\cdot\left[\frac{\partial \boldsymbol{u}}{\partial P}\nabla\omega\wedge\boldsymbol{k}-\nabla\Theta\right].$$

The left-hand side of this equation is the ROSSBY approximation of the potential vorticity if one identifies $\eta$ with $Q_z$

$$(26.16)\qquad \frac{d}{dt}\left(\eta\frac{\partial\Theta}{\partial P}\right)\approx\frac{d}{dt}\left(Q_z\frac{\partial\Theta}{\partial P}\right)\approx\frac{d}{dt}\left(Q_z\frac{\Delta\Theta}{\Delta P}\right).$$

So the right-hand side of (26.15) gives the deviation of conservatism of the ROSSBY approximation of the potential vorticity. It seems that this explicite formula for the deviation of conservatism of the ROSSBY approximation of potential vorticity is new.

**26.2** After these preparations the main theorem of this section may be proved, which may be formulated as follows: In case a discontinuity is present the ROSSBY approximation of potential vorticity makes no longer any sense; however, the exact definition of potential vorticity still makes sense and the conservation theorem remains valid.

The proof of the first part of this theorem starts with the observation that $\eta$ is known from (19.6) to have the following structure

$$(26.17)\qquad \eta=\eta_1\,U(t-\sigma)+\eta_2\,U(\sigma-t)+\boldsymbol{k}.\,[\nabla\sigma\wedge(\boldsymbol{u}_1-\boldsymbol{u}_2)]\,\delta\,(t-\sigma),$$

whereas $\frac{\partial\Theta}{\partial P}$ has the structure

$$(26.18)\qquad \frac{\partial\Theta}{\partial P}=\frac{\partial\Theta_1}{\partial P}U(t-\sigma)+\frac{\partial\Theta_2}{\partial P}U(\sigma-t)+(\Theta_2-\Theta_1)\frac{\partial\sigma}{\partial P}\,\delta\,(t-\sigma).$$

Now if one tries to calculate the expression $\eta\frac{\partial\Theta}{\partial P}$ occuring in (26.16) one obtains a term

$$\boldsymbol{k}\cdot[\nabla\sigma\wedge(\boldsymbol{u}_2-\boldsymbol{u}_1)]\,(\Theta_2-\Theta_1)\frac{\partial\sigma}{\partial P}\,\delta^2\,(t-\sigma).$$

This term contains $\delta^2$ as a factor and from section **8.2** one knows that $\delta^2$ in not a distribution so that even in distribution theory this term makes no sense. This means that in a SD the ROSSBY approximation of potential vorticity becomes so singular that even the theory of singular functions cannot deal with it. To proof the second part of the theorem one starts from the equations:

(26.19) $$\boldsymbol{Q} = \boldsymbol{Q}_1 U(t-S) + \boldsymbol{Q}_2 U(S-t) + (\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \nabla S\, \delta(t-S)$$

(Cf. (11.8))

and

(26.20) $$\nabla \Theta = \nabla \Theta_1 U(t-S) + \nabla \Theta_2 U(S-t) + (\Theta_2 - \Theta_1) \nabla S\, \delta(t-S).$$

In the exact expression for the potential vorticity the product $\boldsymbol{Q} \cdot \nabla \Theta$ gives no difficulties since, because of

(26.21) $$(\Theta_2 - \Theta_1)\, [(\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \nabla S] \cdot \nabla S = 0$$

the term in $\delta^2$ cancels.

So in the presence of a SD one has

(26.22) $$\frac{d}{dt}\left(\frac{\boldsymbol{Q} \cdot \nabla\Theta}{\varrho}\right) = \frac{d}{dt}\left(\frac{\boldsymbol{Q}_1 \cdot \nabla\Theta_1}{\varrho_1}\right) U(t-S) + \frac{d}{dt}\left(\frac{\boldsymbol{Q}_2 \cdot \nabla\Theta_2}{\varrho_2}\right) U(S-t) +$$
$$+ \frac{d}{dt}\left[\frac{\nabla S}{\varrho} \cdot \{\nabla\Theta \wedge (\boldsymbol{v}_1 - \boldsymbol{v}_2) + \boldsymbol{Q}\,(\Theta_2 - \Theta_1)\}\right] \delta(t-S) = 0.$$

in which every term makes sense. The interpretation of the regular terms in (26.22) gives no difficulties. For the singular term one may write

(26.23) $$\frac{d}{dt}\left[\frac{\nabla S}{\varrho} \{\nabla\Theta \wedge (\boldsymbol{v}_1 - \boldsymbol{v}_2) + \boldsymbol{Q}\,(\Theta_2 - \Theta_1)\}\right] \delta(t-S) =$$
$$= \frac{d}{dt}\left[\frac{\nabla S}{\varrho} \{\operatorname{rot}(\Theta\,(\boldsymbol{v}_1 - \boldsymbol{v}_2)) + \boldsymbol{Q}_2 \Theta_2 - \boldsymbol{Q}_1 \Theta_1\}\right] \delta(t-S) = 0.$$

The importance of this singular part of the potential vorticity in the theory of hydrodynamic instability will be studied in the next section.

**26.3** This section will be concluded with a discussion of the concept of potential vorticity in Langrangian coordinates. As already known in these coordinates a quantity $Z_h$ is used to describe the vorticity field

(26.24) = (20.37) $$Z_h = \varepsilon_{hij}\, \boldsymbol{b} \cdot \frac{\partial}{\partial a_j} (\boldsymbol{v} + \Omega \wedge \boldsymbol{x}),$$

and the vorticity equation takes the form

(26.25) = (20.43) $$\frac{\partial Z_h}{\partial t} + \varepsilon_{hij} \frac{\partial}{\partial a_i}\left(\frac{1}{\varrho}\right) \frac{\partial P}{\partial a_j}.$$

It follows from these formulae that for an arbitrary function $F$ it is true that

(26.26) $$\frac{\partial}{\partial t}\left[Z_h \frac{\partial F}{\partial a_h}\right] = Z_h \frac{\partial^2 F}{\partial t \partial a_h} - \frac{\partial F}{\partial a_h} \varepsilon_{hij} \frac{\partial}{\partial a_i}\left(\frac{1}{\varrho}\right) \frac{\partial P}{\partial a_j} =$$
$$= Z_h \frac{\partial^2 F}{\partial t \partial a_h} - \frac{\partial \left(F; \frac{1}{\varrho}; P\right)}{\partial (a_1; a_2; a_3)}.$$

This is again ERTEL's circulation theorem and again the vorticity equation may be found back by a substitution which in this case happens to be $F = a_k$
Now take for $F$ an arbitrary conservative quantity $C$. Then one gets

(26.27) $$\frac{\partial}{\partial t}\left[Z_h \frac{\partial C}{\partial a_h}\right] + \frac{\partial \left(C; \frac{1}{\varrho}; P\right)}{\partial (a_1; a_2; a_3)} = 0.$$

In the special case of $C$ being $\Theta$ this equation is simplified to

(26.28) $$\frac{\partial}{\partial t}\left[Z_h \frac{\partial \Theta}{\partial a_h}\right] = 0,$$

which is the theorem of conservatism of the potential vorticity in Lagrangian coordinates.
However, when using Langrangian coordinates, there is no reason at all why one should not choose one of the $a_i$ or particle coordinates to be the potential temperature. So let $a_\Theta$ be written for the particle coordinate which is identified with the potential temperature. Then

(26.29) $$\frac{\partial \Theta}{\partial a_h} = \delta_{h\Theta}$$

and (26.28) takes the surprisingly simple form

(26.30) $$\frac{\partial Z_\Theta}{\partial t} = 0.$$

It is believed that a more concise formulation of the conservation theorem for the potential vorticity is impossible.

However, it must be verified that the quantities introduced above make sense in the presence of discontinuities.

Write

(26.31) $$\boldsymbol{x} = \boldsymbol{x}_1\, U(\Psi) + \boldsymbol{x}_2\, U(-\Psi).$$

Since $\Psi$ is independent of the time one has

(26.32) $$\frac{\partial^k \boldsymbol{x}}{\partial t^k} = \frac{\partial^k \boldsymbol{x}_1}{\partial t^k} U(\Psi) + \frac{\partial^k \boldsymbol{x}_2}{\partial t^k} U(-\Psi)$$

and

(26.33) $$\boldsymbol{V} = \frac{\partial^2 \boldsymbol{x}}{\partial t^2} + \vec{\Omega} \wedge \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{V}_1\, U(\Psi) + \boldsymbol{V}_2\, U(-\Psi)$$

from which it follows that

(26.34) $$\frac{\partial \boldsymbol{V}}{\partial a_j} = \frac{\partial \boldsymbol{V}_1}{\partial a_j} U(\Psi) + \frac{\partial \boldsymbol{V}_2}{\partial a_j} U(-\Psi) + (\boldsymbol{V}_1 - \boldsymbol{V}_2) \frac{\partial \Psi}{\partial a_j}\, \delta(\Psi).$$

Furthermore one has

(26,35) $$\boldsymbol{b}_i = \frac{\partial \boldsymbol{x}}{\partial a_i} = \frac{\partial \boldsymbol{x}_1}{\partial a_i} U(\Psi) + \frac{\partial \boldsymbol{x}_2}{\partial a_i} U(-\Psi) + (\boldsymbol{x}_1 - \boldsymbol{x}_2) \frac{\delta \Psi}{\partial a_i}\, \delta(\Psi).$$

So in

(26.36) $$Z_h = \varepsilon_{hij}\, \boldsymbol{b}_i \cdot \frac{\partial \boldsymbol{V}}{\partial a_i}$$

no term in $\delta^2$ occurs since

(26.37) $$\varepsilon_{hij} \frac{\partial \Psi}{\partial a_i} \frac{\partial \Psi}{\partial a_j} = 0.$$

Next, the formulae

(26.38) $$Z_h = Z_{h1}\, U(\Psi) + Z_{h2}\, U(-\Psi) + \varepsilon_{hij}\, \boldsymbol{b}_i \cdot (\boldsymbol{V}_1 - \boldsymbol{V}_2) \frac{\partial \Psi}{\partial a_j}\, \delta(\Psi)$$

and

(26.39) $$\frac{\partial \Theta}{\partial a_h} = \frac{\partial \Theta_1}{\partial a_h} U(\Psi) + \frac{\partial \Theta_2}{\partial a_h} U(-\Psi) + (\Theta_1 - \Theta_2) \frac{\delta \Psi}{\partial a_h}\, \delta(\Psi)$$

reveal that on the same base (26.37) also in $Z_h \dfrac{\partial \Theta}{\partial a_h}$ no term in $\delta^2$ occurs.
One obtains

(26.40) $$Z_h \frac{\partial \Theta}{\partial a_h} = Z_{h1} \frac{\partial \Theta_1}{\partial a_h} U(\Psi) + Z_{h2} \frac{\partial \Theta_2}{\partial a_h} U(-\Psi) +$$
$$+ \left[ Z_h (\Theta_1 - \Theta_2) \frac{\partial \Psi}{\partial a_h} + \varepsilon_{hij}\, \boldsymbol{b}_i \cdot (\boldsymbol{V}_1 - \boldsymbol{V}_2) \frac{\partial \Theta}{\partial a_h} \right] \delta(\Psi).$$

By specialization of this formula according to (26.29) and (26.30) one finally obtains

(26.41) $$Z_\Theta = Z_{\Theta 1}\, U(\Psi) + Z_{\Theta 2}\, U(-\Psi) +$$
$$+ \left[ Z_\Theta (\Theta_1 - \Theta_2) \frac{\partial \Psi}{\partial \Theta} + \varepsilon_{\Theta i h}\, \boldsymbol{b}_i \cdot (\boldsymbol{V}_1 - \boldsymbol{V}_2) \frac{\partial \Psi}{\partial a_h} \right] \delta(\Psi),$$

which shows a complete analogy to (26.23).

Although the use of Lagrangian coordinates did not give rise to new discoveries from the point of view of consistency of approximations — all questions of approximations indeed have been discussed with Eulerian coordinates – it has given us the very interesting formulae (26.28) and (26.30) both being valid in discontinuous fields also, and both apparently new.

## 27. Hydrodynamic Instability

In this section it will be shown that the concept of potential vorticity is of great importance for the theory of (hydro)dynamic instability and for the theory of inertia oscillations. One knows that both theories gave rise to a frequency equation, which has been found using different approaches by SOLBERG [71], KLEINSCHMIDT [38] and VAN MIEGHEM [50, 51, 52]. The frequency equation may be written as

(27.1) $$\nu^4 - B\,\nu^2 + C = 0,$$

with

(27.2) $$B = 2\vec{\Omega} \cdot \mathbf{Q} - \nabla\Theta \cdot \nabla\Pi$$

(27.3) $$C = -2\,(\vec{\Omega} \cdot \nabla\Pi)\,(\mathbf{Q} \cdot \nabla\Theta),$$

$\Pi$ being the EXNER function

(27.4) $$\Pi = C_p\,(P\,P_0^{-1})^{\varkappa} \qquad \varkappa = \mathrm{R}/C_p$$

in which $P_0$ is some arbitrarily chosen reference pressure, usually $P_0 = 100$ cb.
Writing

(27,5) $$\text{sign } F = \frac{F}{|F|} = \begin{Bmatrix} +1 \text{ for } F > 0 \\ -1 \text{ for } F < 0 \end{Bmatrix} = U(F) - U(-F),$$

one may summarize the results of the theories mentioned in the form of the following scheme [50]:

| Sign B | Sign C | Equilibrium | | Inertia oscillations | |
|---|---|---|---|---|---|
| | | Hydrostatic | Hydrodynamic | Short periods | Long periods |
| + 1 | + 1 | stable | stable | stable | stable |
| + 1 | — 1 | stable | instable | stable | instable |
| — 1 | + 1 | instable | instable | instable | instable |
| — 1 | — 1 | instable | stable | instable | stable |

In the case of hydrostatic equilibrium the terms stable and instable refer to the stability of the vertical equilibrium against a perturbation consisting in the vertical displacement of an air quantum from its equilibrium position. In the case of hydrodynamic equilibrium these terms refer to the stability of the geostrophic equilibrium against a perturbation consisting in the displacement of an air quantum from its geostrophic equilibrium motion in which the displacement vector lies in an isentropic surface or makes a small angle with the isentropic surface. Inertia oscillations are said to have long periods if these periods have an order of magnitude of half a pendulum day, whereas the short periodic oscillations have periods of order of $10^{-2}$ pendulum day. (cf. VAN MIEGHEM [50]).

Now, one has

$$(27.6) \qquad \vec{\Omega} \cdot \boldsymbol{Q} = \vec{\Omega} \cdot \boldsymbol{Q}_1 \, U(t-S) + \vec{\Omega} \cdot \boldsymbol{Q}_2 \, U(S-t) + \\ + \vec{\Omega} \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) \wedge \nabla S \, \delta(t-S),$$

and

$$(27.7) \qquad \nabla \Theta \cdot \nabla \Pi = \nabla \Theta_1 \cdot \nabla \Pi_1 \, U(t-S) + \nabla \Theta_2 \cdot \nabla \Pi_2 \, U(S-t) + \\ + (\Theta_2 - \Theta_1) \nabla \Pi \cdot \nabla S \, \delta(t-S),$$

in which is taken into account the fact that $\nabla \Pi$ has no singular term according to the dynamic boundary condition, and

$$(27.8) \qquad \Omega \cdot \nabla \Pi = \vec{\Omega} \cdot \nabla \Pi_1 \, U(t-S) + \vec{\Omega} \cdot \nabla \Pi_2 \, U(S-t),$$

$$(27.9) \qquad \boldsymbol{Q} \cdot \nabla \Theta = \boldsymbol{Q}_1 \cdot \nabla \Theta_1 \, U(t-S) + \boldsymbol{Q}_2 \cdot \nabla \Theta_2 \, U(S-t) + \\ + \nabla S \cdot [\nabla \Theta \wedge (\boldsymbol{v}_1 - \boldsymbol{v}_2) + \boldsymbol{Q}(\Theta_2 - \Theta_1)] \, \delta(t-S). \qquad \text{(Cf. 26.19)}$$

Using these formulae the coefficients of the frequency equation are readily found to be equal to

$$(27.10) \qquad B = B_1 \, U(t-S) + B_2 \, U(S-t) + B_3 \, \delta(t-S),$$

with

$$(27.11) \qquad B_3 \, \delta(t-S) = [2 \vec{\Omega} \wedge (\boldsymbol{v}_1 - \boldsymbol{v}_2) + (\Theta_1 - \Theta_2)] \nabla S \, \delta(t-S)$$

and

$$(27.12) \qquad C = C_1 \, U(t-S) + C_2 \, U(S-t) + C_3 \, \delta(t-S),$$

with

$$(27.13) \qquad C_3 \, \delta(t-S) = -2(\vec{\Omega} \cdot \nabla \Pi) \nabla S \cdot [\Theta \wedge (\boldsymbol{v}_1 - \boldsymbol{v}_2) + \\ + \boldsymbol{Q}(\Theta_2 - \Theta_1)] \, \delta(t-S).$$

The equation for the geostrophic wind may be written using $\Theta$ and $\Pi$ to replace $\varrho$ and $P$ as

(27.14) $$2\vec{\Omega} \wedge \boldsymbol{v}_g + \Theta \nabla \Pi = 0.$$

So in geostrophic approximation $B_3$ is given by

(27.15) $$B_3\, \delta(t-S) = [\Theta_2 \nabla \Pi_2 - \Theta_1 \nabla \Pi_1 + (\Theta_1 - \Theta_2) \cdot \nabla \Pi]\, \nabla S\, \delta(t-S) = \\ = \Theta (\nabla \Pi_2 - \nabla \Pi_1) \cdot \nabla S\, \delta(t-S).$$

Consider any quantity $F$ of the following structure

(27.16) $$F = F_1\, U(t-S) + F_2\, U(S-t).$$

This means that

(27.17) $$\begin{cases} F = F_1 \text{ for } t > S & \text{i.e. after SD passage} \\ F = F_2 \text{ for } t < S & \text{i.e. before SD passage.} \end{cases}$$

so that the ascendent $\nabla S$ is directed from the air mass behind the SD to the air mass in front of the SD.

This follows also from the formula

(27.18) = (9.3) $$(1 - \boldsymbol{v} \cdot \nabla S)\, \delta\, (t-S) = 0.$$

Under the fairly safe hypothesis that the geostrophic approximation of $B_3$ does not have a sign different from that of the exact value of $B_3$ it now folows from (27.15) that

(27.19) $$\text{sign } B_3 = \text{sign } [(\nabla \Pi_2 - \nabla \Pi_1) \cdot \nabla S] = \text{sign } [(\nabla P_2 - \nabla P_1) \cdot \nabla S] = \\ = \text{sign } \left[\left(\frac{\partial P_2}{\partial z} - \frac{\partial P_1}{\partial z}\right) \frac{\partial S}{\partial z}\right] = \text{sign } \left[(\varrho_1 - \varrho_2)\, \frac{\partial S}{\partial z}\right].$$

Here use is made of the fact that in both the air masses $\nabla P$ is almost exactly vertical and directed downwards. In figure 6 the situation is shown in vertical cross section for a warm front (right hand side) and a cold front (left hand side). From (27.19) and figure 6 one infers that independent of the character of the frontal surface one has

(27.20) $$\text{sign } B_3 = +1.$$

The table given after (27.5) then shows that independent of the character of the frontal surface the atmosphere is at the SD in stable equilibrium, as one would expect it to be. This is exactly true only for an exact discontinuity. If one considers a frontal zone the situation, however, cannot change significantly. Since by passing to the limit of an exact discontinuity one must obtain for $B$ a value

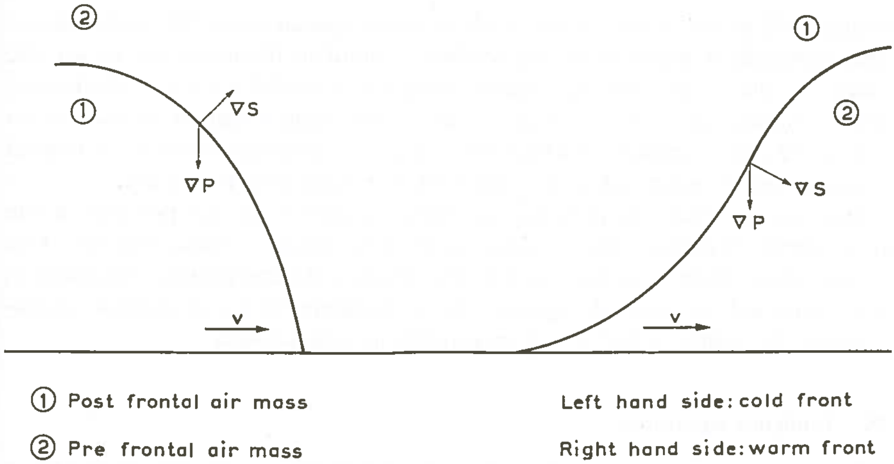


Figure 6

that is positively infinite, $B$ must become large and positive in a strong frontal zone or a layer of strong baroclinity. In general the atmosphere will be in stable static equilibrium in a frontal zone or a layer of strong baroclinity.

Now, supposing $B$ to be positive, consider $C_3$ (27.13). In geostrophic approximation the vertical component of $\vec{\Omega}$ is supposed to be the only component different from zero. Therefore one has

$$\text{(27.21)} \qquad \text{sign}\,(\vec{\Omega} \cdot \nabla \Pi) = \text{sign}\ \frac{\partial \Pi}{\partial z} = -1,$$

so that

$$\text{(27.22)} \qquad \text{sign}\ C_3 = + \text{sign}\ [\nabla S \cdot \{ \nabla \Theta \wedge (\boldsymbol{v}_1 - \boldsymbol{v}_2) + \boldsymbol{Q}\,(\Theta_2 - \Theta_1)].$$

Comparing this formula with (26.22) one observes that the sign of $C_3$ is that of the singular potential vorticity, whereas (27.3) shows that the sign of $C$ is to that of the total potential vorticity. The sign of $B$ being fixed (positive) the conclusion follows that in any case, continuous as well as discontinuous, the character of the hydrodynamic stability is fully determined by the sign of the potential vorticity. This remains true as long as $B > 0$ *i.e.* as long as the baroclinity is strong enough to be comparable to that in a frontal zone.

It is surprising to note that for the exact form of the relation between hydrodynamic instability and potential vorticity no reference could be found in literature. Formula (27.3) has already long been known, and it is a rather simple manipulation to introduce the potential vorticity in it. Moreover, that there should be *some* relation of general validity between hydrodynamic instability and a suitable chosen parameter characterizing the vorticity field is suggested by the results of the study of many special cases. The only reasonable explanation seems to be the tendency found in literature not to use the exact formula for the potential temperature but to prefer the use of the ROSSBY approximation $\boldsymbol{Q}_z\,(\triangle P)^{-1}$. Since however this approximation breaks down just at the circumstances in which the concept of potential vorticity is needed it cannot be of much value in studies on hydrodynamic instability.

One knows, that the potential vorticity is almost always positive in the atmosphere, negative values being confined to small isolated regions. This implies that, given a stable vertical stratification hydrodynamic instability is also restricted to isolated regions, the atmosphere being in general hydrodynamically stable, a fact which empirically is well known.

## 28. Tendency equations

In this section some remarks are made pertaining to the use of tendency equations. The discussion will show some resemblance to the discussion of the isallobaric winds (section **24**) which is in the nature of the subjects.

Quite a lot of tendency equations have been derived by different authors. It seems that the tendency equation constructed by BIJVOET [2] is the one in which using the minimum of simplifying assumtions, the maximum of results is obtained. So this equation will be used for discussion. However, the same method could be applied to any other tendency equation.

In contemporary meteorology tendency equations are not set up for the surface pressure but for the geopotential of some suitably chosen level of constant pressure usually 100 cb. However, since there are no essential differences between these two types of tendency equations the method to investigation remains the same for both types.

BIJVOET's tendency equation is one of a highly elaborated type. The geopotential ("height") tendency is split up in a sum of 7 terms. It will not be necessary to discuss all 7 terms. Following BIJVOET the discussion may be restricted to a term which is considered by BIJVOET as the most important one. In the notation of the present study this term may be written as

$$(28.1) \qquad -\frac{R}{\varrho_0 f^3}\int_0^{P_0} \boldsymbol{k}\cdot[\nabla(\triangle\,\Phi)\wedge\nabla T]\,dP.$$

The subscript 0 means "at the earth's surface". The discussion may be further restricted to the quantity $d\Psi$ given by

$$(28.2) \qquad d\Psi = -\boldsymbol{k}\cdot[\nabla(\triangle\,\Phi)\wedge\nabla T]\,dP,$$

which means that the same technique is followed which already has been applied in section **25** in which an infinitisimal layer is chosen instead of a layer of finite vertical extension.

Now it must be investigated if the quantity $d\Psi$ may be used in the case of a frontal surface being present in the atmosphere.

Using

$$\triangle \Phi = \triangle \Phi_1 \, U(t-\sigma) + \triangle \Phi_2 \, U(\sigma-t) + \\ + (\nabla \Phi_2 - \nabla \Phi_1) \cdot \nabla\sigma \, \delta(t-\sigma) \tag{28.3}$$

and

$$\nabla T = \nabla T_1 \, U(t-\sigma) + \nabla T_2 \, U(\sigma-t) + (T_2 - T_1) \nabla\sigma \, \delta(t-\sigma) \tag{28.4}$$

one readily observes that in $d\Psi$ a term in $\delta^2$ will occur.

So one finds, that at a SD the use of $d\Psi$ breaks down. At first sight this seems to imply the incorrectness of the tendency equation. This would however be a too hastely drawn conclusion. In reality the tendency equation is completely correct. For if one follows conscientious the method by which BIJVOET obtained his equation one observes that it has been derived by operations which cannot give rise to inconsistencies. He starts with equations like the continuity equation which are completely sound. The manipulations further used consist in writing the operator $\frac{d}{dt}$ in a rather complicated form, taking a divergence and making an integration, none of which can generate a $\delta^2$-term. So the tendency equation as a whole must be considered as being completely correct.

However, what in the present discussion has been done, *viz.* the isolation of a single term (28.1) is not correct. Since the complete equation is correct it must possess another term (or other terms) containing $\delta^2$ which compensates the $\delta^2$ occurring in (28.1). The conclusion must be made that one is not entitled to consider the term (28.1) in splendid isolation but that it is a term which is strongly coupled with at least one other term in the whole formula for the height tendency. But this also means that, at least on a SD or inside a frontal zone, one may not consider this term as the most important term.

The reason why investigators are fond of discussing terms of an equation separately is that they hope to be able to draw conclusions from their discussion of a qualitative nature which could be framed into rules to be employed by forecasters using the traditional subjective forecasting techniques. Each term is then interpreted as representing a certain process, which is supposed to be independent of the processes represented by the other terms. The forecaster using the subjective techniques so operates with processes which he considers to be independent of one another, the relative importances of which he estimates. It has become clear from the discussion given above that frontal surfaces or layers of strong baroclinity act as coupling elements between different processes, which therefore may not be considered as independent processes and from which it may be very difficult to single out one as being the most important. This is a strong argument in favour of numerical weather prediction but this very argument shows that real progress is to be expected only from baroclinic models.

## 29. The equation of Navier-Stokes

Discontinuities are no discovery of meteorologists, they have been introduced as long ago as the beginning of the 20th century by hydrodynamicists as DUHEM [21] and HADAMARD [31]. One of the conclusions arrived at by these scientists has been that no discontinuities can exist in a fluid the motion of which is governed by the NAVIER-STOKES equation. DUHEM [21] reached this result by an extremely long and tedious manipulation with boundary conditions. However, the conclusion may be drawn using the formalism developed in the present study without almost any calculation.

Consider the NAVIER-STOKES equation

$$\frac{d\boldsymbol{v}}{dt} + \nabla \Phi + \frac{1}{\varrho} \nabla P + 2\vec{\Omega} \wedge \boldsymbol{v} = \lambda \triangle \boldsymbol{v} + \frac{\lambda}{3} \nabla \operatorname{div} \boldsymbol{v}. \tag{29.1}$$

Now one knows that according to

$$\frac{d\boldsymbol{v}}{dt} = \frac{d\boldsymbol{v}_1}{dt} U(t-S) + \frac{d\boldsymbol{v}_2}{dt} U(S-t), \tag{29.5}$$

the accelerations cannot have a term with a $\delta$-function. However, in the right-hand side of (29.1) a $\delta$-function will occur if one introduces into it a discontinuous velocity field. So one must have $\lambda = 0$, which completes the proof.

There is a remark of DUHEM which runs as follows in modern terminology: If one considers the NAVIER-STOKES equation as the equation governing the mean motion of a turbulent fluid, then in order to be consistent one should assume an eddy heat conductivity. This gives a turbulent transfer of heat which destroys discontinuities in the temperature field. Since pressure cannot be discontinuous the destruction of temperature discontinuities implies the destruction of density discontinuities *i.e.* the total destruction of discontinuities. However, the proof given above shows that even if one does not take eddy transfer of heat into account discontinuities cannot exist in any fluid whose motion is governed by the NAVIER-STOKES equation, independent of the interpretation one may give to the type of motion described by the NAVIER-STOKES equation.

**30. Numerical weather prediction**

In this section some questions related to the problems of numerical weather prediction are discussed. In contrast to all preceding sections no exact results will be presented and the discussion will be of a more speculative nature.

It is well known and has been emphasized by different authors like *e.g.* CHARNEY and N. PHILLIPS [14] that the use of the geostrophic approximation in barotropic forecasts is compatible whit the assumption that the flow is governed by the laws of conservation of potential temperature and of potential vorticity. As has been shown in preceding sections these laws remain valid in discontinuous fields provided that the exact expression for the potential vorticity is used. Leaving aside the question whether in numerical prediction one has always been so cautious to use the exact formula for the potential vorticity, one should study what happens if the differential equations which govern the motion are replaced by schemes of finite difference equations, as one is obliged to do in numerical work.

It is obvious that in finite differences techniques the concept of a discontinuous field makes no sense at all. However, the finite differences techniques should be able to describe adequately layers of strong baroclinity, frontal zones or jet stream zones. Layers of strong baroclinity, frontal zones and jet stream zones are known to have narrow belts of rapid variation of significant parameters. So in order to be able to describe adequately the rapid variations which occur in these narrow belts the grid size to be used for the finite differences schemes should be at least one order of magnitude smaller than the horizontal dimensions of the zone of rapid variation of significant parameters. As a matter of fact the grid sizes in current use are almost never chosen smaller than say 300 km, which is of the same order of magnitude as the width of frontal zones, jet stream zones *etc.* So the conclusion follows that in order to describe adequately the structure of zones of rapid variation the grid sizes should be chosen 10 times as small, which implies that one should use $10^2$ times as much grid points in the numerical work. Then in order to avoid computational instability one should also decrease the time steps in the calculations by a factor $10^{-1}$ which amounts to 10 times as much time steps as are used nowadays. So the total number of calculations to be made by the machine becomes multiplied by at least $10^3$ and the number of data to be stored in the memory is also multiplied by at least $10^3$. This is a rather conservative estimate, since the number of levels necessary to describe adequately a baroclinic state should perhaps also be increased. It is clear that for the moment this is an impossible program for even without the factor $10^3$ the memories of even the best machines are hardly large enough to store the data needed for baroclinic forecasting. One wonders, however, that in current research so much labour is spent in constructing baroclinic models and calculation schemes, without paying attention to the question of the grid sizes needed, except for the demands of computational stability. However, this may be an explanation for the otherwise incomprehensible fact that the baroclinic models give results no better than the barotropic model, since if the grid sizes are too large to describe the baroclinic zones adequately this amounts to an underestimation of baroclinity with the effect of a decrease in the difference between barotropic and baroclinic forecasting.

There may be a way to come out of the difficulty which was described above. Among the more recent models for numerical prediction there are some in which a semi-Lagrangian approach is used *e.g.* the models constructed by ØKLAND [57] and by ELIASSEN [24]. They are more or less the machine counterpart of the graphical technique first described bij FJØRTOFT [27]. Their common feature is that if the initial data are given for points in a square grid, the grid after a few time steps becomes deformed. There is some empirical evidence that this gives rise to a non-uniform distribution of grid points over the map with a tendency for clustering of grid points in regions of mass convergence. So one gets more points exactly in those regions in which a more detailed description is necessary. Perhaps it may be possible by a judicious choice of a non-uniform grid point distribution from the very beginning to decrease the factor $10^3$ considerably. This might *e.g.* be done by superposing two grids, one the traditional square grid and a supplementary grid with smaller grid size in regions in which it seems to be necessary. The use of two grids is not prohibitive since from the work of N. PHILLIPS [60] one knows that the simultaneous use of two grids gives no unsurpassable difficulties. However, according to a remark made by SHUMAN [69], there are limitations to the quasi-Lagrangian approach for the same tendency for clustering of grid points in certain regions sets a limit to the forecasting period which may not be chosen too large in order to avoid too much uncertainty in those parts of the map in which the density of grid points becomes low. Moreover the question arises whether the upper air observational network is dense enough for this purpose.

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* References [24], [57] and [69] not yet published. Opinions quoted in the text are given according to the author's notes made at the Tōkyō Symposium. The original authors cannot be held responsible for their opinions quoted, prior to confirmation with their papers to be published in the Symposium Proceedings.

# SUMMARY


The present study consists of three parts. There is a General Introduction, which together with the Introductions to the parts I and III constitutes a supplementary summary.

The first part gives an introduction to the theory of distributions, and is purely mathematical. The original theory of SCHWARTZ has been given first, followed by other and more elementary approaches to this theory. It has been tried to come through with as less sophistication as possible to keep the exposition within the limits of the normal mathematical knowledge of meteorologists. Being mainly a carefully prepared presentation of existing knowledge on some modern branch of mathematics this part does not contain essentially new material.

The second part is devoted to the study of the basic equations of dynamic meteorology for discontinuous fields. The equations are given in Eulerian and Lagrangian coordinates, the former both with geometric height as with pressure as an independent variable. Attention may be drawn to section **20** in which a new technique for dealing with Lagrangian coordinates is developed and in which also a presentation of the vorticity equation in Lagrangian coordinates is given, which is lacking in existing meteorological literature using Lagrangian coordinates.

In part three the theory is used for various applications. After a study of the energy balances (section **22**) a criterion has been constructed to discuss the consistency of approximations in discontinuous fields. With the aid of this criterion various approximative theories are studied in the following sections, the results of which may summarized as follows:

The approximations involved in the theory of the ageostrophic winds are correct in discontinuous fields provided that some caution is taken into account (section **24**). The same may be said on the theory of differential analysis (section **25**). However, the ROSSBY approximation of the potential vorticity is proved to be incorrect in discontinuous fields (section **26**). The same section contains a new description of the theorem of conservatism of the potential vorticity in Lagrangian coordinates. In the next section the relation between potential vorticity and hydrodynamic instability is studied and some remarkable results are given which are valid both for continuous as discontinuous fields and which seem hitherto have escaped notice even in the continuous case. Section **28** gives a discussion of some points of interest pertaining to tendency equations. Attention is drawn to some dangers, which easily present themselves by uncritical use of tendency equations.

The next section contains a remarkable short proof of a theorem of DUHEM regarding the impossibility of discontinuities in fluids the motion of which is governed by the NAVIER-STOKES equation. In the last section some remarks are made on the subject of numerical weather prediction.

There is a list of references in which it has been tried to give full bibliographical documentation on the items referred to since many of them are usually not found in meteorological libraries.

# CONTENTS

PART III. APPLICATIONS

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