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CONTRIBUTION TO THE THEORY OF INTERNAL WAVES

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Summary

The problem of internal waves is dealt with theoretically for certain continuous density distributions of the general type shown in fig. 1. The relative variation of density is supposed to be small. The fluid is supposed to be incompressible and to be at rest in the non-perturbed state; the internal waves are treated as small perturbations.

If we describe the simple harmonic, basic waves by means of a streamfunction

$$\varphi(x, z, t) = \varphi(z) \exp i(\mu x - \nu t),$$

it appears that $\varphi(z)$ may be found with, in general, sufficient accuracy as a solution of the equation

$$\frac{d^2}{dz^2}\left(\frac{\varphi}{\sqrt{S}}\right) + \left(\frac{g\, dS/dz}{c^2 S_0} - \mu^2\right)\frac{\varphi}{\sqrt{S}} = 0,$$

where $S(z)$ = specific volume in the equilibrium state, S_0 = mean specific volume, $c = \nu/\mu$ = velocity of propagation.

Together with the boundary conditions, this equation gives an eigen-value problem, solution of which gives relations between wave-length and period.

When using, as an analytical representation of the density distribution, the function

$$S(z) = S_0 + \tfrac{1}{2} \triangle S \operatorname{tgh}(2z/b),$$

where b is a measure of the thickness of the transition layer (see fig. 1) and $\triangle S$ is the total variation of the specific volume, we may solve the above differential equation analytically by means of hypergeometric series. When the fluid is sufficiently deep on both sides of the transition layer, the relation between the wave length $L = 2\pi\lambda$ and the period $T = 2\pi\tau$ is given by

$$\frac{g\triangle S}{bS_0}\tau^2 = n(n+1)\left(\frac{2\lambda}{b}\right)^2 + (2n+1)\left(\frac{2\lambda}{b}\right) + 1,$$

where λ is *positive*; n has one of the values 0, 1, 2, 3, etc. (any integer) and represents the order of the mode of oscillation, which is equal to the number of zeros of the corresponding solution $\varphi_n(z)$.

When $L \to 0$, the period approaches a minimum value, which is independent of n, viz:

$$T_{min} = 2\pi\sqrt{\frac{bS_0}{g\triangle S}} = \frac{2\pi}{\sqrt{g(S^{-1}dS/dz)_{max}}}.$$

The existence of this lower limit of the period of internal waves appears to be a general feature, not restricted to the special type of density distribution assumed here.

The theory is extended so as to include the earth's rotation. In this case the same relation as exists between τ and λ in the previous (non-rotating) case, now exists between τ and $\lambda\sqrt{1-(2\omega_z\tau)^2}$, ω_z being the vertical component of the angular velocity of rotation.

Symbols

$a = 1/b$.
b = thickness of transition layer, as defined by figure 1.
c = velocity of propagation.
F = hypergeometric series.
$f = d \log S_1/dz$.
g = acceleration of gravity.
$h = f/(\sigma b^{-1} + f)$.
L = wave lenght.
$m = b\mu/2 = b/2\lambda = \pi b/L$.
n = order of mode of oscillation.
P = unperturbed pressure.
p = local pressure perturbation.
$q = 2\lambda/b$, or λ expressed in $b/2$ as unit of length.
$r = \tau^2 g \triangle S/bS_0 = (\tau/\tau_1)^2$.
S = specific volume in the equilibrium state.
S_0 = value of S at the level $z = 0$.
$S_1 = S(z) - \frac{1}{2} \triangle S \,\mathrm{tgh}\, 2az$.
s = perturbation of specific volume.
t = time.
T = period of oscillation.
u = velocity component in the x-direction.
v = velocity component in the y-direction.
w = velocity component in the z-direction.
x = coordinate in the direction of propagation of the waves.
y = horizontal coordinate perpendicular to x.
z = vertical coordinate.
$Z = 2z/b$, or z expressed in $b/2$ as unit of length.
$\varepsilon = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu_0}{2a}}$
$\eta = \varphi/\sqrt{S}$
$\lambda = L/2\pi$
$\mu = 1/\lambda$
$\mu_0 = \frac{g \triangle S}{2 S_0 c^2}$
$\nu = 2\pi/T$
$\xi = (\cosh Z)^2$
ϱ = density
$\sigma = \triangle S/S_0$ = total relative variation of specific volume
$\tau = T/2\pi$
φ = stream function
$\psi = \eta (\cosh Z)^\varepsilon$
ω_z = vertical component of angular velocity of rotation.

CONTRIBUTION TO THE THEORY OF INTERNAL WAVES

Contents

1. *Introduction.*

The behaviour of internal waves in vertically inhomogeneous fluids has been studied theoretically by various authors, from the point of view of general hydrodynamics as well as of meteorology and of physical oceanography (see the list of references, which, however, is not meant to be complete).

For mathematical reasons most of them assumed discontinuities at certain levels, either in the density ϱ or in its first derivative with respect to the vertical coordinate, z. Any transition layer was often assumed to be thin in comparison with the wave length. FJELDSTAD [7], on the other hand, by using numerical integration, succeeded in giving an approximate method of solving the problem for certain general density-distributions, such as may actually occur in the sea, a method, however, which is only applicable for very *long* waves.

Besides this restriction to long waves only, FJELDSTAD's method has one other disadvantage, *viz.* of not directly yielding general rules or relations between the properties of the internal waves and certain parameters of the density-distribution.

It is therefore, that we have gone back to somewhat more special density distributions, which are perfectly continuous with respect to ϱ and $d\varrho/dz$ (as are FJELDSTAD's density-distributions), but which appear to be capable of an *analytical* treatment; furthermore, the results are also valid for *small* wavelengths. The density distribution is of the general type shown in fig. 1, where we have a graph of the specific volume against depth.

In the present paper we shall only deal with fluids extending to infinity both upwards and downwards. At first sight this seems rather unrealistic. We know, however, that the wave-motions are always confined to a certain layer, above and below which they are negligibly small, so that, if only the boundaries of the fluid fall without this layer, they will not interfere essentially with the solutions we shall find here. The thickness of this layer depends on the wave length (see section 5).

For the rest, it is quite possible to introduce a free surface and a rigid bottom, if necessary. This will make the computations much more complicated and laborious, however.

2. *Derivation of the basic equations.*

Let the fluid be at rest, in the equilibrium state, and let in the perturbed state the internal wave motion be propagated in the x-direction, all static, kinematic and dynamic properties being assumed to be independent of the y-coordinate. The symbols u and w denote respectively the x- and the y-component of velocity (perturbation velocity).

As the earth's rotation is neglected, at least for the time being (see section 6), the y-component of the perturbation velocity vanishes, on account of the suppositions made.

The pressure and density fields will then be described by the following scalars:

$$\text{pressure} = P(z) + p(x, z, t), \tag{1}$$

$$\text{specific volume} = S(z) + s(x, z, t), \tag{2}$$

where p and s denote the local changes brought about respectively in the pressure and the specific volume, by the wave perturbation.

As the fluid is supposed to be *incompressible* [1]), the wave motion is now governed by the following equations:

$$\frac{\partial u}{\partial t} + S\frac{\partial p}{\partial x} = 0, \tag{3}$$

$$\frac{\partial w}{\partial t} + S\frac{\partial p}{\partial z} + s\frac{\partial P}{\partial z} = 0, \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \text{ (continuity equation)}, \tag{5}$$

$$\frac{\partial s}{\partial t} + w\frac{\partial S}{\partial z} = 0 \text{ (incompressibility)}. \tag{6}$$

These equations are obtained by the usual linearization with respect to the *small* quantities u, w, p and s (see *f.i.* V. BJERKNES c.s. [8] page 300).

Now we introduce the stream function φ by writing

$$u = \partial\varphi/\partial z, \quad w = -\partial\varphi/\partial x. \tag{7}$$ ([illegible] !)

As equation (5) is now automatically satisfied, we are left with

$$\frac{\partial^2\varphi}{\partial t\partial z} + S\frac{\partial p}{\partial x} = 0, \tag{8}$$

$$-\frac{\partial^2\varphi}{\partial t\partial x} + S\frac{\partial p}{\partial z} + s\frac{\partial P}{\partial z} = 0, \tag{9}$$

$$\frac{\partial s}{\partial t} - \frac{\partial\varphi}{\partial x}\frac{\partial S}{\partial z} = 0. \tag{10}$$

We may write the simple harmonic, basic solutions of this system of equations, which we are looking for, in the following form:

$$\varphi(x, z, t) = \varphi(z) \exp i(\mu x - \nu t); \; p(x, z, t) = p(z) \exp i(\mu x - \nu t); \; s(x, z, t) = s(z) \exp i(\mu x - \nu t);$$

so that we may write, symbolically: $\partial/\partial t \equiv -i\nu$, $\partial/\partial x \equiv i\mu$; the velocity of propagation is then $c = \nu/\mu$. Finally we can substitute: $\partial P/\partial z = -g/S$.

We obtain then (equations (8) and (10) are divided by $i\mu$):

$$-c\varphi' + Sp = 0, \tag{11}$$

$$-\mu^2 c\varphi + Sp' - gs/S = 0, \tag{12}$$

$$cs + \varphi S' = 0. \tag{13}$$

Here a prime denotes a differentation with respect to z. Besides, the quantities φ, p and s will be considered as functions of z only (as S is), the common factor exp $(i\mu x - i\nu t)$ being left out, for the present.

From (11) and (13) we derive:

$$p = c\varphi'/S, \quad s = -\varphi S'/c. \tag{14}$$

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[1]) If we should want to take the compressibility into account we might use the potential density ϱ_{pot} instead of the actual density. For sea water we can write, with sufficient approximation, $\varrho_{pot} = 1 + 10^{-3}\sigma_t$, where σ_t has the usual meaning.

Substituting this into (12) and dividing by c, we get:

$$\varphi'' - \varphi' \frac{S'}{S} + \varphi \left(\frac{gS'}{c^2 S} - \mu^2 \right) = 0. \tag{15}$$

In order to get rid of the first derivative of φ, we put

$$\varphi = \eta \sqrt{S} \tag{16}$$

Then η must satisfy the equation:

$$\eta'' + \eta \left(\tfrac{1}{2} \frac{S''}{S} - \tfrac{3}{4} \frac{S'^2}{S^2} + \frac{gS'}{c^2 S} - \mu^2 \right) = 0. \tag{17}$$

Now we shall suppose the fluid layer to be infinitely high. Then the „boundary" conditions which should be satisfied by η are: η and η' must remain finite for $z \to +\infty$ as well as for $z \to -\infty$.

We wish to find solutions of the present problem for vertical density distributions of the general type represented in fig. 1, which we shall describe analytically by the function

$$S = S_0 + \tfrac{1}{2} \triangle S \operatorname{tgh} 2az, \tag{18}$$

where $\triangle S$ denotes the total variation of the specific volume and $a^{-1} = b$ may roughly be taken as the thickness of the transition layer.

Now it may easily be seen that, when we use the function (18), the first two terms of the form in brackets in eq. (17) may be neglected compared with the third one, if

$$\frac{2ac^2}{g} << 1 \text{ and } \frac{3ac^2 \triangle S}{4gS} << 1. \tag{19}$$

The question as to the extent to which the above conditions (19) will be actually satisfied will be discussed later (page 12). It will appear that the relation between wave length, period and velocity of propagation, which we shall derive, is accurate in most cases. It may still be noted here, that, as in the atmosphere and in the ocean $\triangle S/S << 1$, the second one of the inequalities (19) will automatically be satisfied if the first one is, when we are dealing with atmospheric or oceanic internal waves.

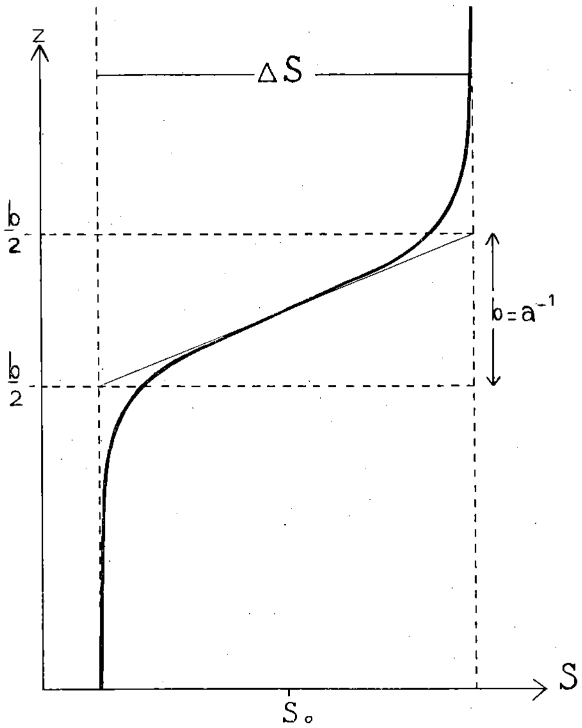

Fig. 1. Distribution of specific volume S as a function of z, according to the formula $S = S_0 + \frac{1}{2} \triangle S$ tgh $2az$.

Considering the fact that the *relative* variation of S is small in the density distributions concerning us here, we may, finally, replace S'/S by S'/S_0 without introducing any appreciable error[1]) and we obtain:

$$\eta'' + \eta \left(\frac{ga \triangle S}{c^2 S_0 (\cosh 2az)^2} - \mu^2 \right) = 0, \tag{20}$$

or:

$$\eta'' + \eta \left(\frac{2a \mu_0}{(\cosh 2az)^2} - \mu^2 \right) = 0, \tag{20}$$

where

$$\mu_0 = \frac{g \triangle S}{2 S_0 c^2} = \frac{2\pi}{L_0}, \tag{21}$$

[1]) We may avoid this error entirely, if we like, by just assuming: $S = S_0 \exp \left(\frac{\triangle S}{2 S_0} \operatorname{tgh} 2az \right)$ instead of (18); as $\triangle S/S_0$ is supposed to be rather small (of the order of 0.01), the general picture of the density distribution remains the same.

L_0 being the wave length to which belongs a velocity of progagation c of internal waves along a surface of discontinuity between homogeneous fluids with specific volumes $S_0 + \frac{1}{2} \triangle S$ and $S_0 - \frac{1}{2} \triangle S$; or, in other words: μ_0 is the value of μ to which would belong a velocity of propagation c *if* the thickness of the transition layer were *zero*.

Now by a final transformation

$$2\,az = \frac{z}{b/2} = Z \tag{22}$$

equation (20) may be written as follows:

$$\frac{d^2\eta}{dZ^2} + \eta \left[\frac{\mu_0/2a}{(\cosh Z)^2} - \left(\frac{\mu}{2a}\right)^2\right] = 0. \tag{23}$$

If we assume the fluid to extend to infinity both upwards and downards, we have as „boundary"-conditions that φ, and consequently η, must remain finite or become zero when $z \to +\infty$ as well as when $z \to -\infty$.

If $S(z)$ is given, we have as variable parameters in equation (23) the quantities μ and μ_0, or, in other words: the wave number and the velocity of propagation (c).

Now, equation (23) does possess solutions satisfying the above „boundary"-conditions *only* for special combinations of values of c and μ; in other words: our problem is an *„eigen-value"-problem.*

This will yield the relations between wave-length and velocity of propagation, or between wave length and period.

3. *Solving the eigen-value-problem.*

Equation (23) is a differential equation of the general type

$$\frac{d^2Y}{dZ^2} + Y\left[\frac{k\,(k-1)}{(\cosh Z)^2} - \frac{l\,(l-1)}{(\sinh Z)^2} - m^2\right] = 0,$$

which may be solved by the substitution

$$Y = \psi(\xi)\,(\cosh Z)^k\,(\sinh Z)^l,\quad \xi = (\cosh Z)^2,$$

yielding:

$$\psi''\,\xi(\xi-1) + \psi'\,[(k+l+1)\,\xi - (k+\tfrac{1}{2})] + \psi\left[\left(\frac{k+l}{2}\right)^2 - \left(\frac{m}{2}\right)^2\right] = 0,$$

the latter equation being a hypergeometric differential equation:

$$\psi''\,\xi(\xi-1) + \psi'\,[(\alpha+\beta+1)\,\xi - \gamma] + \alpha\beta\psi = 0, \tag{24}$$

with

$$\alpha = \frac{k+l+m}{2},\quad \beta = \frac{k+l-m}{2},\quad \gamma = k + \tfrac{1}{2}\ \ (m > 0); \tag{25}$$

a prime now stands for one differentiation with respect to Z.

In order to solve (23) we put

$$k\,(k-1) = \frac{\mu_0}{2a},\quad l = 0,\quad m = \frac{\mu}{2a}\ (m > 0), \tag{26}$$

the first of these relations yielding:

$$k = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + \frac{\mu_0}{2a}};$$

we take the negative root:

$$k = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + \frac{\mu_0}{2a}} = -\varepsilon\ (\varepsilon > 0). \tag{27}$$

Substituting this in (25) we get

$$\alpha = \frac{-\varepsilon + m}{2},\quad \beta = \frac{-\varepsilon - m}{2},\quad \gamma = -\varepsilon + \tfrac{1}{2}, \tag{28}$$

where ε and m are defined by (27) and (26), respectively; for $m = \mu/2a$ we may also write $\pi b/L$, which means that m is π times the ratio transition-layer thickness : wave-length (L).

The general solution of equation (23) is now

$$\eta = (\cosh Z)^{-\varepsilon}\ \psi(\xi),\ \ \xi = (\cosh Z)^2,$$

where $\psi(\xi)$ should satisfy (24), α, β and γ being given by (28). The solution of (24) for $\xi \geq 1$ may be expressed in terms of hypergeometric series of ascending powers of ξ^{-1}, which we shall generally denote by

$$F(A, B, C; \xi^{-1}) = 1 + \frac{A \cdot B}{1 \cdot C}\xi^{-1} + \frac{A(A+1)\,B(B+1)}{1\cdot 2\cdot C\,(C+1)}\xi^{-2} + \ldots\ldots$$

The general solution for $\xi \geq 1$ reads:

$$\psi(\xi) = C_1 \xi^{-\alpha} F(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \xi^{-1}) + C_2 \xi^{-\beta} F(\beta, \beta-\gamma+1, \beta-\alpha+1; \xi^{-1}) =$$

$$= C_1 \xi^{\frac{\varepsilon-m}{2}} F\left(\frac{-\varepsilon+m}{2}, \frac{\varepsilon+m+1}{2}, m+1; \xi^{-1}\right) + C_2 \xi^{\frac{\varepsilon+m}{2}} F\left(\frac{-\varepsilon-m}{2}, \frac{\varepsilon-m+1}{2}, 1-m; \xi^{-1}\right) =$$

$$= C_1 \xi^{\frac{\varepsilon-m}{2}} F_1(\xi^{-1}) + C_2 \xi^{\frac{\varepsilon+m}{2}} F_2(\xi^{-1}).$$

Hence

$$\eta = (\cosh Z)^{-\varepsilon}\,\psi = \xi^{-\varepsilon/2}\,\psi = C_1 \xi^{-m/2} F_1(\xi^{-1}) + C_2 \xi^{m/2} F_2(\xi^{-1}). \tag{29}$$

As $\xi \to \infty$ when $Z \to +\infty$ or $Z \to -\infty$, while $F_2(0) = 1$, this solution is infinite for $z = +\infty$ or $z = -\infty$ *unless* $C_2 = 0$. So we are left with

$$\eta = C_1 \xi^{-m/2} F_1(\xi^{-1}) = C_1 (\cosh Z)^{-m} F\left[\frac{-\varepsilon+m}{2}, \frac{\varepsilon+m+1}{2}, m+1; (\cosh Z)^{-2}\right]. \tag{30}$$

For $Z = 0$, $\xi \to 1+0$ or $\xi^{-1} \to 1-0$; the value of the series F_1 then converges[1]) towards

$$F_1(1) = \frac{\Gamma(1+m)\,\Gamma(\frac{1}{2})}{\Gamma\left(1+\frac{\varepsilon+m}{2}\right)\Gamma\left(\frac{-\varepsilon+m+1}{2}\right)}. \tag{31}$$

Now we must bear in mind that (30) represents only half of the solution we need, either for the positive half of the Z-axis, or for the negative half. Indeed, as, in general, the derivative of (30) with respect to Z, for $Z = 0$, will not be equal to *zero*, taking (30) as a solution for both halves of the Z-axis would mean that the solution would have a discontinuous derivative at $Z = 0$. According to equation (7), however, φ as well as $\partial\varphi/\partial z$ must be everywhere continuous; according to (16) the same is true for η.

Hence, it follows, that, if (30) is the solution for, say, the positive half of the Z-axis, we have, for the other half, to find the analytical continuation of this half of the solution. This continuation will, in general, be of the type (29) with $C_2 \neq 0$, so that it will have no finite value for $Z = -\infty$, *unless* either

$$\eta'(0) = 0, \qquad \text{(I)}$$

or

$$\eta(0) = 0. \qquad \text{(II)}$$

In case I (the prime denotes differentiation with respect to Z, here) our solution is simply even and is described by (30) with the same value of C_1 for both halves of the Z-axis.

In case II the continuation for $Z < 0$ is obtained by taking for the cofactor C_1 the opposite of the value used for $Z > 0$; the solution is then odd.

[1]) The criterium for convergence of $F(A, B, C; X)$ for $X = 1$ is: $A + B - C < 0$, $F(A, B, C; 1)$ being then $\frac{\Gamma(C)\,\Gamma(C-A-B)}{\Gamma(C-A)\,\Gamma(C-B)}$.

Case I.

$$\eta' = -mC_1 (\cosh Z)^{-m-1} \sinh Z \cdot F_1 [(\cosh Z)^{-2}] - 2C_1 (\cosh Z)^{-m-3} \sinh Z \cdot F_1' [(\cosh Z)^{-2}].$$

The first term in the above form vanishes for $Z = 0$. As to the second term, we have, generally:

$$F_1' (X) = \frac{d}{dX} F (A, B, C; X) = \frac{AB}{C} F (A + 1, B + 1, C + 1; X),$$

where

$$A = \frac{-\varepsilon + m}{2}, \quad B = \frac{\varepsilon + m + 1}{2}, \quad C = m + 1.$$

For the latter hypergeometric series, the condition for convergence (see footnote on page 9) is not satisfied, here. It can, however, be proved[1]) that

$$\lim_{X=1-0} (1-X)^{A+B-C+1} F (A+1, B+1, C+1; X) = \frac{\Gamma (C+1)\, \Gamma (A+B-C+1)}{\Gamma (A+1)\, \Gamma (B+1)}.$$

Now, in our case $(1-X)^{A+B-C+1}$ means $(\mathrm{tgh}\, Z)^{2(A+B-C+1)} = \mathrm{tgh}\, Z = \sinh Z/\cosh Z$. According to this we have

$$\lim_{Z=0} \sinh Z \,.\, F_1' [(\cosh Z)^{-2}] = \frac{(-\varepsilon + m)\, (\varepsilon + m + 1)}{4\, (m+1)} \cdot \frac{\Gamma (m+2)\, \Gamma (\frac{1}{2})}{\Gamma \left(\frac{-\varepsilon + m}{2} + 1\right) \Gamma \left(\frac{\varepsilon + m + 3}{2}\right)}.$$

Hence, $\eta' (Z = 0)$ is finite; as both ε and m are positive, it will only *vanish* if either $-\varepsilon + m = 0$, making

$$m = \varepsilon, \tag{32}$$

or $\Gamma\left(\frac{-\varepsilon + m}{2} + 1\right) = \infty$, the latter being the case if $\frac{-\varepsilon + m}{2} + 1 = 0, -1, -2, -3, \ldots\ldots$, or

$$(0 <) m = \varepsilon - 2, \varepsilon - 4, \varepsilon - 6, \ldots\ldots \text{etc.} \tag{33}$$

By (32) and (33) a series of eigen-values of $\mu = 2am$ is given, determining a series of possible wavelengths for any given value of the velocity of propagation c. Since, however, m should be positive, while

$$\varepsilon = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \frac{\mu_0}{2a}},$$

it follows, that (33) yields one or more eigen-values (but always only a finite set) *only if* $\mu_0/2a > 6$. The larger $\mu_0/2a$ is, the larger is the number of possible solutions. As, according to (21), μ_0 is inversely proportional to c^2, the foregoing statement means only that, the smaller c is, the larger is the number of solutions, if the other constants are fixed. The simplest solution of type I is given by (32), making

$$\frac{\mu}{2a} = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \frac{\mu_0}{2a}},$$

or:

$$b\mu = -1 + \sqrt{1 + 2b\mu_0}. \tag{34}$$

On account of the conditions expressed by (19) this result may not be applied for too small values of b. Nevertheless equation (34) yields the correct limiting value of μ when b tends towards *zero*: $\lim_{b=0} \mu = \mu_0$. Power series development of the right hand menber of (34) would, however, yield: $\mu = \mu_0 (1 - \frac{1}{2} b\mu_0 + \ldots)$, whereas a correct approximation for small transition layer thicknesses yields: $\mu = \mu_0 (1 - b\mu_0 + \ldots)$. [2])

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[1]) See: Whittaker and Watson, A course of modern analysis, Cambridge 1935, Ch XIV, ex. 8,18.

[2]) This may f.i. be derived from an equation, obtained by Haurwitz [5] for the case of two infinite homogeneous layers seperated by a thin transition layer, viz. equation (12a) of paragraph 9 of the paper referred to.

Case II.

According to equations (30) and (31), $\eta(0)$ will be *zero*, if $\frac{-\varepsilon+m+1}{2} = 0, -1, -2, -3, \ldots\ldots,$ or:

$$(0 <)\, m = \varepsilon - 1, \varepsilon - 3, \varepsilon - 5, \varepsilon - 7, \ldots\ldots \text{etc.} \qquad (36)$$

In order to make possible one or more solutions of this type, $\mu_0/2a$ should be larger than 2 (m being positive).

4. *Relation between wave-length and period.*

The relations found above between wavelength and velocity of propagation may be transformed into relations between wavelength and period (the latter being a more directly measurable quantity). On account of equations (32), (33) and (36) we may write:

$$b\mu = 2m = 2(\varepsilon - n) = -(2n+1) + \sqrt{1+2b\mu_0} = -(2n+1) + \sqrt{1+g\sigma b c^{-2}}, \quad n = 0, 1, 2, 3, \ldots\ldots,$$

where $\sigma = \Delta S/S_0$ = the *relative* variation of the specific volume; σ is a *small* number. As $c = \nu/\mu$ it follows, that

$$b\mu + 2n + 1 = \sqrt{1 + g\sigma b\mu^2\nu^{-2}},$$

or:

$$b^2\mu^2 + 2(2n+1)\,b\mu + (2n+1)^2 - 1 = g\sigma b\mu^2\nu^{-2},$$

or:

$$g\sigma\tau^2/b = 4n(n+1)(\lambda/b)^2 + 2(2n+1)(\lambda/b) + 1, \qquad (37)$$

where $\lambda = L/2\pi = \mu^{-1}$ and $\tau = \nu^{-1} = T/2\pi$, L and T being the wavelength and the period, respectively.

By writing

$$q = 2\lambda/b, \; r = g\sigma\tau^2/b,$$

we get:

$$r = n(n+1)q^2 + (2n+1)q + 1. \qquad (37)$$

Figure 2 shows a set of graphs, giving r as a function of q, and thus, implicitly, τ^2 as a function of λ, for a set of values of n. The corresponding formulas are written down below:

$(n = 0)\;\; r = q + 1$
$(n = 1)\;\; r = (2q+1)(q+1)$
$(n = 2)\;\; r = (3q+1)(2q+1)$
$(n = 3)\;\; r = (4q+1)(3q+1)$
$(n = 4)\;\; r = (5q+1)(4q+1)$

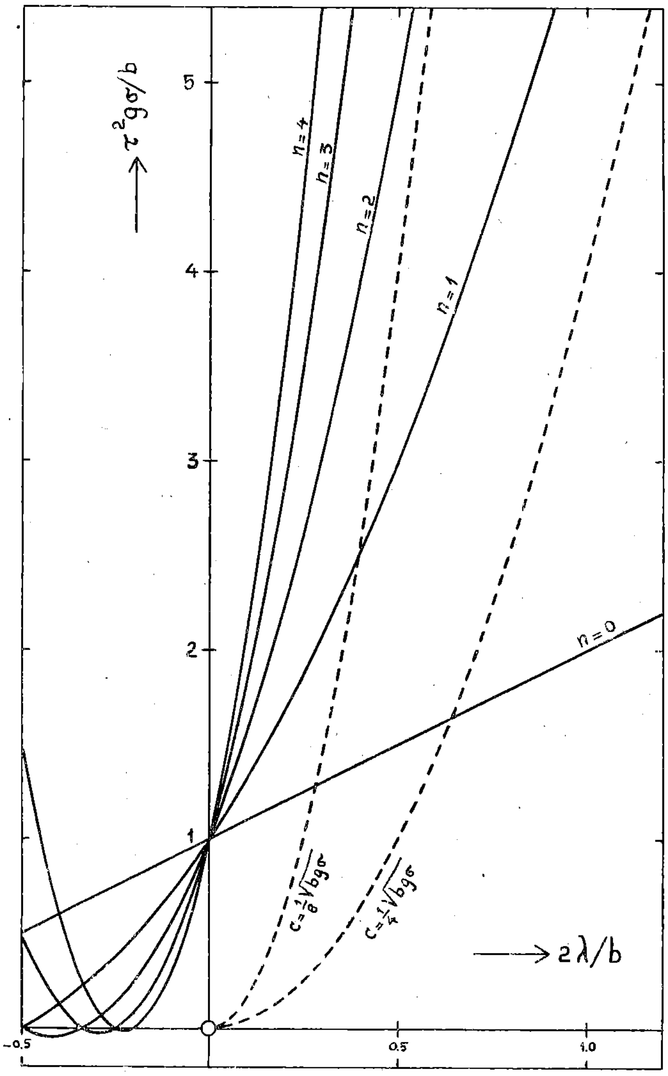

Fig. 2. Relations between wave-length ($2\pi\lambda$) and period ($2\pi\tau$) of internal waves.

It may be noticed that both variables q and r are pure numbers, which have a simple meaning, q being obtained when we simply express λ in terms of $b/2$, or half the transition layer thickness, as a unit of length, and r being equal to $(\tau/\tau_1)^2$, where $\tau_1 = \sqrt{b/\sigma g}$ has been used as a unit of time. The quantities $b/2$ and τ_1 are the two characteristics of the fluid system we must know in order to be able to use fig. 2.

Except for $n = 0$, which gives a straight line, all curves are parabolas, only part of which, however, must be used, *viz.* the points *to the right of* and *above* the point $q = 0, r = 1$. This is a consequence of the fact that by agreement λ should be positive, as m is positive, according to (26)[1]).

As may be seen from the formulas as well as from fig. 2, we have always (for $\lambda \neq 0$) : $g\sigma\tau^2/b > 1$, or:

$$\tau > \tau_1 = \sqrt{\frac{bS_0}{g\triangle S}} = \sqrt{\frac{S_0}{g\,(dS/dz)_{\max}}}, \tag{38}$$

the latter value appearing to be *the lower limit of the period of any internal wave in the given density distribution.* This lower limit, which the period approaches when the wave-length tends to zero, is exactly equal to the „period of free oscillation" of a fluid particle belonging to the level of the greatest vertical density gradient. We shall see in the next section, that the thickness of the layer within which, practically speaking, the wave motions are confined, becomes small proportionally to the square root of the wave-length, when the latter becomes very small, so that a free surface and a rigid bottom donot make themselves felt for the shortest internal waves, which have the shortest periods.

Numerical example: For $S_0^{-1}\,(dS/dz)_{\max} = 0.0001\ \mathrm{m}^{-1}$ the limiting value of the period is $T_1 = 2\pi\tau_1 = 3.3$ min. We shall return to the matter in the last section.

Another general result which follows from the above formulas is, that, *when* b and λ, or: *the thickness of the transition layer and the wave-length, are multiplied by the same factor* (the total density variation remaining the same), *the period is multiplied by the square root of that factor.*

The *velocity of propagation* is given by $c = \lambda/\tau$; curves $c =$ constant in figure 1 are represented by $\tau^2 = \lambda^2/c^2$, or $r = (bg\sigma/4c^2)q^2$. These are parabolas touching the q-axis in the origin. Two of them are drawn, viz. $c = \frac{1}{4}\sqrt{bg\sigma}$ and $c = \frac{1}{8}\sqrt{bg\sigma}$.

It should not be forgotten that the conditions (19) might impose a certain restriction on the use of our solutions. From the first one of the inequalities (19) we get: $2c^2/gb << 1$, or:

$$\sigma \,.\, \frac{2(\lambda/b)^2}{g\sigma\tau^2/b} = \sigma \,.\, \frac{q^2}{2r} << 1.$$

Now, for $n = 1$ or higher, the factor $q^2/2r$ of this inequality is always < 0.25, so that here this condition is automatically satisfied, σ being a small number.

For $n = 0$, however, it might imply a real restriction by excluding too large values of λ/b.

Now it should not be forgotten that the left hand members of the inequalities (19) are the *maxima* of the ratios of the terms which we neglected in equation (17) to the term gS'/c^2S_0, which we used, so that we cannot just say, that the value of $\sigma q^2/2r$ means the order of magnitude of the error brought about in, say, the computed wave lenght (as a function of the period) by the neglect of the term $\frac{1}{2}S''/S$; *nor* can we say that this error must needs be much larger than the one brought about by the neglect of the term $-\frac{3}{4}(S'/S)^2$, although the *maximum* of the latter term is only about $\triangle S/S_0$ times the maximum of the former one.

As a matter of fact, it appears that for the *larger* wave-lenghts (the only ones for which the errors need any consideration) both terms taken separately imply relative errors which are of the *same* order of magnitude, namely about $0.1 \cdot \sigma^2 q$.

Since σ is supposed to be small, those errors might become important only for the very largest wave-lenghts; for $\sigma = 0.01$, for instance, $0.1 \cdot \sigma^2 q$ is still only 0.01 for $q = 1000$, or L as large as $3000b$.

But, as the errors have opposite signs, they compensate each other for large values of L/b or small values of b/L, for which the transition layer acts like a discontinuity; as we have seen when discussing (34), our solution gives again the *exact* relation between wave length

[1]) Of course, this is an arbitrary agreement, but we must stick to it, because, otherwise our solution (30) would become infinite for $z = +\infty$ and $z = -\infty$.

and period for the limiting case, that $b/L \to 0$. This result may also easily be obtained from equation (37) by putting $n = 0$ and neglecting the term 1 with respect to $2\lambda/b$.

That the term $\frac{1}{2} S''/S$ involves only such a small error, even when $2\lambda/b$ is as large as 1000, is explained by the fact that $\frac{1}{2} S''/S_0$, to which this term is nearly equal, is, as a function of z, *odd* with respect to z, $S''(-z)$ being equal to $-S''(z)$.

The exact method of computing the effect of small terms in a differential equation upon the eigen-value-problem, determined by this differential equation and certain boundary-conditions, is called *perturbation-calculus* [1]).

Let the differential equation *without* „perturbation" be written in the following symbolical form:

$$(H_{op} - E)\,\eta = 0,$$

where H_{op} is a so-called Hermiteian operator, which in the case of equation (20) is described by

$$H_{op} = \frac{d^2}{dz^2} + \frac{ga \triangle S}{c^2 S_0 (\cosh 2az)^2},$$

where as E is a parameter, the eigen-values of which are to be obtained; E corresponds to μ^2 in equation (20).

Let the boundary conditions be

$$\eta\,(z = z_1) = 0, \qquad \eta\,(z = z_2) = 0,$$

and let the eigen-values of E and the corresponding eigen-functions, which are supposed to be *real*, be denoted by

$$E_0,\ E_1,\ E_2, \ldots\ldots$$

and

$$\eta_0(z),\ \eta_1(z),\ \eta_2(z), \ldots\ldots$$

respectively. Then the eigen-value problem defined by the differential equation

$$(H_{op} + f(z) - E)\,\eta = 0,$$

where $f(z)$ is a small „perturbation", and by the *same* boundary conditions as above, has eigen-values

$$E_0 + \delta E_0,\ E_1 + \delta E_1,\ E_2 + \delta E_2, \ldots\ldots,$$

and eigen-functions

$$\eta_0(z) + \delta\eta_0(z),\ \eta_1(z) + \delta\eta_1(z),\ \eta_2(z) + \delta\eta_2(z), \ldots\ldots,$$

where, in first approximation,

$$\delta E_n = \frac{\int_{z_1}^{z_2} \eta^2{}_n f(z) dz}{\int_{z_1}^{z_2} \eta^2{}_n dz}. \tag{39}$$

In the case we are dealing with, namely equation (17), the function $f(z)$ is

$$f(z) = \tfrac{1}{2}\frac{S''}{S} - \tfrac{3}{4}\left(\frac{S'}{S}\right)^2, \tag{40}$$

and the eigen-function we are concerned with (see section 5) is

$$\eta_0 = (\cosh 2az)^{-m}.$$

---

1) The term „perturbation" is used here in the classical meaning in which it is used in celestial mechanics and quantum mechanics.

Since H_{op}, as defined above, contains c as a factor, formula (39) gives the (small) variation brought about in the eigen-value $\mu^2 = E_0$ by the function $f(z)$ when *c is kept constant.*

Since η_0^2 is an even function of z, it follows from formula (39) that any term of $f(z)$, which is odd with respect to z, gives no contribution to the integral occurring in the numerator and, consequently, to the result of (39). Now we can write:

$$\tfrac{1}{2}\frac{S''}{S} = \tfrac{1}{2}\frac{S''}{S_0\,(1+\tfrac{1}{2}\sigma \operatorname{tgh} 2az)} = \tfrac{1}{2}\frac{S''}{S_0} - \tfrac{1}{4}\sigma\frac{S''}{S_0}\operatorname{tgh} 2az,$$

the first term of which is by far the larger one, but *odd* with respect to z. This term gives a contribution to $\delta(\mu^2)$ only in the *second* approximation and thus becomes, as far as the effect upon μ or λ is concerned, of the same importance as the second one and as the term $-\tfrac{3}{4}(S'/S)^2$, which we still have in $f(z)$ (40).

We shall not enter into further details of the perturbation-calculus which we used[1]), but confine ourselves to giving the result, as stated above[2]).

5. *Stream functions and velocity fields.*

The stream functions corresponding to the solutions found, are given by (16) and (30), where $\varepsilon = m + n$, and $m = b/2\lambda = 1/q$. For η, which we may call the „reduced stream function", we get, as the most general solution,

$$\eta = \sum_{n=0,1,2,\ldots} C^{(n)}\,\eta_n,$$

with

$$\eta_n = \frac{F\left[-\frac{n}{2},\, m+\frac{n}{2}+\tfrac{1}{2},\, m+1;\, (\cosh 2z/b)^{-2}\right]}{(\cosh 2z/b)^m}. \tag{41}$$

Here m is always positive.

For $n = 0$ the numerator in (41) is equal to unity, so that

$$\eta_0 = (\cosh 2z/b)^{-m}$$

For $n = 1$ we have

$$F[-\tfrac{1}{2}, m+1, m+1; (\cosh 2z/b)^{-2}] = \sqrt{1-(\cosh 2z/b)^{-2}} = \operatorname{tgh} 2z/b,$$

so that

$$\eta_1 = (\operatorname{tgh} 2z/b)\,(\cosh 2z/b)^{-m}.$$

For $n = 2$ we have

$$F\,[-1,\, m+1\tfrac{1}{2},\, m+1;\, (\cosh 2\,z/b)^{-m}] = 1 - \frac{m+1\tfrac{1}{2}}{m+1}(\cosh 2z/b)^{-2},$$

so that

$$\eta_2 = \left[1 - \frac{m+1\tfrac{1}{2}}{m+1}(\cosh 2z/b)^{-2}\right](\cosh 2z/b)^{-m}.$$

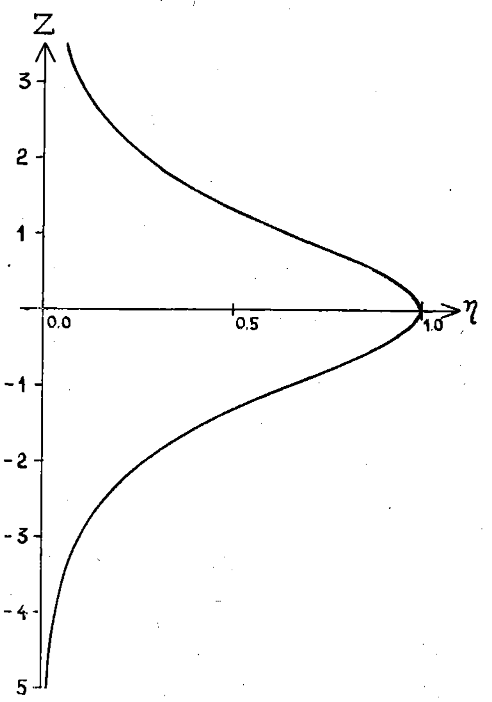

Fig. 3. Reduced stream function (η) as a function of Z, for $n = 0$, $\lambda = b/2$.

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1) For further details the reader is referred to any textbook on the matter. A very clear and concise explanation of „perturbation-theory" may be found in H. A. Kramers, Die Grundlagen der Quantentheorie, Leipzig 1933, Chapter 5.

2) As to the relative variation of λ when τ is fixed, instead of c, we have the relation

$$\lambda^{-1}(\delta\lambda)_{\tau\ \mathrm{const.}} = (1 - 2r/q)\,\lambda^{-1}(\delta\lambda)_{c\ \mathrm{const.}},$$

which may easily be proved.

For $n = 0$ this relation yields:

$$\lambda^{-1}(\delta\lambda)_{\lambda\ \mathrm{const.}} = -(1 + b/\lambda)\,\lambda^{-1}(\delta\lambda)_{c\ \mathrm{const.}}$$

The figures 3, 4, 5 show graphs of the reduced stream functions for these three cases. Here we have used $Z = 2z/b$ as the vertical coordinate; this means simply that we have again used $b/2$ as the unit of length. The value of $m = 1/q$ was chosen to be 1.

As regards the solutions belonging to higher values of n we confine ourselves to stating, that the number of *zeros* of the stream function is apparently n[1]; that for any *even* value of n ($n = 2N$) the hypergeometric series in (41) degenerates into a polynomial in $(\cosh 2z/b)^{-2}$; and that in all solutions the argument of the hypergeometric function, *viz.* $(\cosh 2z/b)^{-2}$ tends to *zero* when $z \to \pm\infty$, so that then the numerator in (41) tends to unity, whereby all solutions η_n tend to zero in the same way, when $z \to \pm\infty$, *viz.* as $(\cosh 2z/b)^{-m}$.

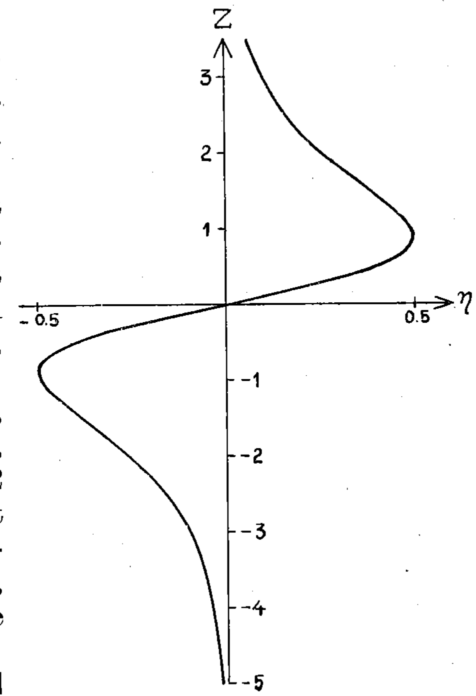

Fig. 4. Reduced stream function (η) as a function of Z, for $n = 1$, $\lambda = b/2$.

From the last statement it may be easily derived that, if b is small enough compared with L, the wave motion is, practically speaking, confined between the limits $z = L/2$ and $z = -L/2$, the amplitudes being then reduced to about $e^{-\pi} \approx 4\%$ at $z = \pm L/2$. We shall call the interval of z, within which the wave motion is, practically speaking, confined, the „wave thickness". Thus, if b is sufficiently small, the wave thickness is $2 \cdot L/2 = L$.

If b is larger than λ, the wave thickness will also depend upon b, more or less. For $b = L$, for instance, the factor $(\cosh 2z/b)^{-m}$ referred to above will amount to about 0.26 for $z = L/2$ and we must go to $z = 0.85 \cdot L$ in order to find the value 0.04 again. The wave thickness is here, consequently, $1.7 \cdot L$. Finally, it may be proved, that for large values of b/λ the wave thickness amounts to about $\sqrt{2bL}$, the factor $(\cosh 2z/b)^{-m}$ again being reduced to about 4% ($\approx e^{-\pi}$) at $z = \pm\sqrt{bL/2}$.

From the above it follows that, in general, the solutions just found may also be applied to fluid systems which are not infinitely deep on both sides of the transition layer, if only the bottom and the free surface are not closer to the middle of the transition layer than about half of the wave thickness, which may vary between $L/2$ and $\sqrt{bL/2}$ according as b is small or large relatively to λ.

Inasmuch as the smallest periods belong to the smallest wave lengths, the result of the preceding section concerning the lower limit of the period of internal waves appears to be independent of the assumption of infinitely deep fluid layers on both sides of the transition layer.

Velocity field. The velocities u and w are now determined by the stream function according to equation (7), yielding:

$$u = \frac{\partial\,(\eta\sqrt{S})}{\partial z}\, e^{i\,(\mu x - \nu t)},\quad w = -i\mu\eta\sqrt{S}\,.\,e^{i\,(\mu x - \nu t)}. \tag{42}$$

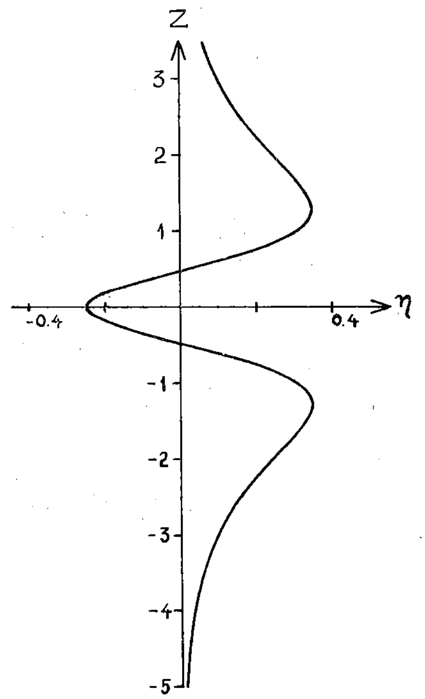

Fig. 5. Reduced stream function (η) as a function of Z, for $n = 2$, $\lambda = b/2$.

It appears from equations (42) that u and w differ in phase by 90°, the velocity vectors describing ellipses; the *w-axis* and the *u-axis* of these ellipses are in the ratio $\mu\eta\sqrt{S} : \partial(\eta\sqrt{S})/\partial z$, or $\eta\sqrt{S} : \lambda\partial(\eta\sqrt{S})/\partial z$,

[1]) This may be proved directly from the form of the differential equation, without calculating the solutions explicitly.

or nearly as $\eta : \lambda \partial\eta/\partial z$, or $m\eta : \partial\eta/\partial Z$. The latter ratio can for any given value of m immediately be derived from such curves as given in fig. 3, 4 or 5. It appears that near the points where η possesses a maximum or a minimum (as, for instance, in the middle of the transition layer when $n = 0, 2, 4, \ldots$) the fluid particles move simply up and down, while at the zeros of η they oscillate horizontally. Besides, it may easily be derived, that, when $z \to \pm\infty$, the ellipses, becoming smaller and smaller, change into circlets.

Fig. 6 and fig. 7 show the stream line patterns corresponding to the solutions for $n = 0$ and $n = 1$, for a certain moment; $m = 1/q$ was the same as in fig. 3 and fig. 4, *viz.* 1.

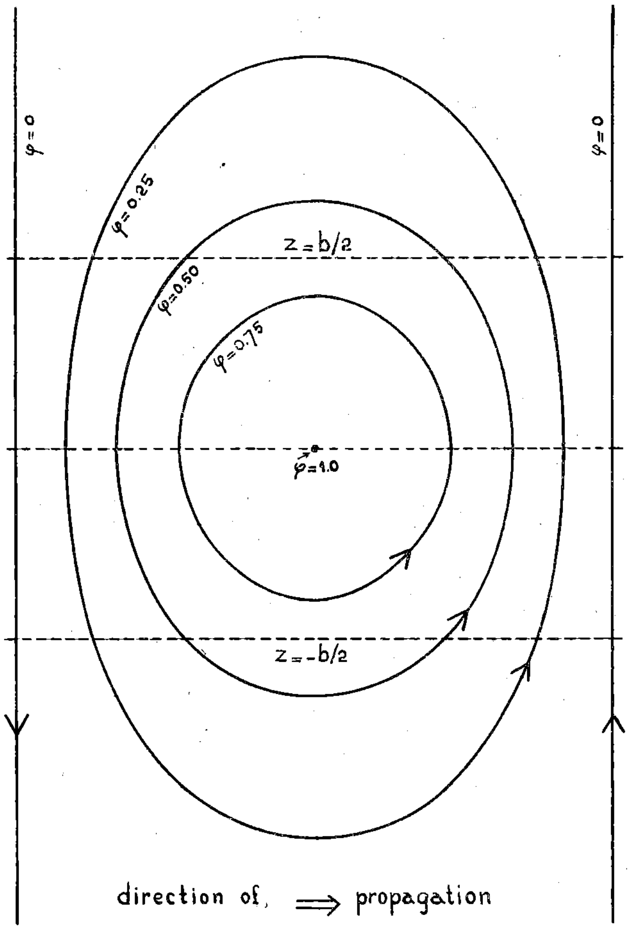

Fig. 6. Stream line pattern of internal wave; $n = 0$, $\lambda = b/2$.

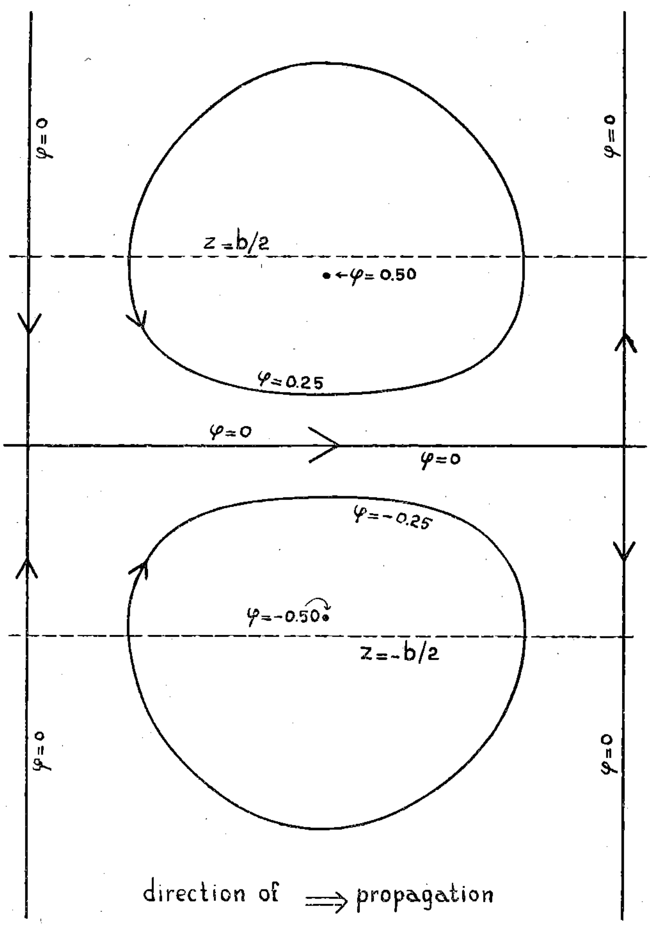

Fig. 7. Stream line pattern of internal wave; $n = 1$, $\lambda = b/2$.

Finally, it can easily be shown that the amplitudes of the vertical and of the horizontal displacements are $c^{-1} |\varphi|$ and $\tau |\varphi'|$, respectively.

6. *Influence of rotation.*

If the whole fluid system rotates in such a way that we need only take into account the vertical component of the rotation vector ω, equation (3) must be replaced by

$$\frac{\partial u}{\partial t} + S\frac{\partial p}{\partial x} - 2\omega_z v = 0, \tag{3*}$$

while for the y-component of the motion the following equation must be added:

$$\frac{\partial v}{\partial t} + 2\omega_z u = 0. \tag{3**}$$

The equations (4), (5) and (6) remain unaltered, $\partial/\partial y$ being, here also, equal to zero.

By differentiating equation (3*) with respect to t and substituting for $\partial v/\partial t$ what follows from (3**), viz. $\partial v/\partial t = -2\omega_z u$, we obtain:

$$\left(\frac{\partial^2}{\partial t^2} + 4\omega_z^2\right) u + S\frac{\partial^2 p}{\partial x \partial t} = 0.$$

Introducing again the stream function φ and substituting $\partial/\partial t = -i\nu$, $\partial/\partial x = i\mu$, we find the following equation:

$$-c\left(1 - \frac{4\,\omega_z^2}{\nu^2}\right)\varphi' + Sp = 0. \qquad (11^*)$$

The only difference with equation (11) is the appearance of the factor $1 - 4(\omega_z/\nu)^2 = 1 - (T/D)^2$, where $T = 2\pi/\nu$ is the period of the wave motion, while $D = \pi/\omega_z = \frac{1}{2}$ pendulum day.

Solving (11*) for p and substituting the result in (12) we obtain now (after dividing by c, again):

$$\varphi''\left[1 - (2\omega_z\tau)^2\right] - \varphi'\frac{S'}{S}\left[1 - (2\omega_z\tau)^2\right] + \varphi\left(\frac{gS'}{c^2 S} - \mu^2\right) = 0, \qquad (15^*)$$

instead of (15). Dividing now by $1 - (2\omega_z\tau)^2$ and substituting $\frac{\lambda}{\tau}$ for c, $\frac{1}{\lambda}$ for μ, we obtain:

$$\varphi'' - \varphi'\frac{S'}{S} + \varphi\left\{\left[\frac{\tau}{\lambda\sqrt{1-(2\omega_z\tau)^2}}\right]^2\frac{gS'}{S} - \left[\frac{1}{\lambda\sqrt{1-(2\omega_z\tau)^2}}\right]^2\right\} = 0.$$

It appears that the only difference, brought about by the earth's rotation, is such, that in the basic differential equations, written down in terms of λ and τ, λ has been replaced by $\lambda\sqrt{1-(2\omega_z\tau)^2}$.

This means that in the final result we have the same relations, here, between $\lambda\sqrt{1-(2\omega_z\tau)^2}$ and τ as we previously had between λ and τ. In other words: if we write the relation between τ and λ, which is implicitly described by equations (37) and fig. 2, in the form

$$\lambda = F_n(\tau), \text{ (without rotation)},$$

we have now:

$$\lambda = \frac{F_n(\tau)}{\sqrt{1-(2\omega_z\tau)^2}} \text{ (with rotation)}.$$

From the last formula, it appears that when T approaches to D, the wave length tends to infinity, half a pendulum day apparently being the upper limit of the period of free internal waves. For very small values of T/D, on the other hand, the influence of the rotation will be negligible.

If we again draw r, q-curves, all curves show a bending to the right for sufficiently large values of q, now, approaching asymptotically to the straight line $r = (D/2\pi\tau_1)^2$.

As to the stream functions, it appears that, for the same value of τ, any $\varphi_n(z)$ is the same as was derived in the previous sections, m being now equal to $b/2\lambda\sqrt{1-(2\omega_z\tau)^2}$ and having (for any n) the same numerical value as before, if τ is the same.

From (3**) follows, that the y-component of the motion, occurring here, is given by the equation

$$-i\nu v + 2\omega_z u = 0,$$

or

$$v = -i\frac{2\omega_z}{\nu}u = -i\frac{T}{D}u.$$

This means that v is 90° behind u, in phase. Any velocity vector describes an ellipse in a sloping plane, intersecting a horizontal plane along a line in the x-direction and making an angle of arc tg $\frac{D\,\varphi(z)}{T\,\lambda\,d\varphi/dz}$ with it.

The horizontal projections of the velocity vectors describe ellipses, the shorter axis (the v-axis) and the longer axis (the u-axis) of which are to each other as $2\omega_z$ to ν, or as T to D; the rotation is *cum sole* (clock-wise on the northern hemisphere).

5. *Somewhat more general type of density distribution.*

A final generalization may still be given to our treatment by taking for S a function of the type

$$S = S_1(z) + \tfrac{1}{2} \triangle S \operatorname{tgh} 2az, \tag{44}$$

where

$$S_1(z) = S_0\, e^{fz} \approx S_0 (1 + fz) \text{ for } z_1 < z < z_2,$$

fz being *small* everywhere within the interval of z to which the use of this function is limited; this interval is either the total depth of the fluid system or else may be taken to coincide with the wave thickness; the *total* relative variation of S is again supposed to be small within the interval of z we are concerned with; f is positive.

Then, writing down the term gS'/c^2S of equation (17), getting

$$\frac{gS_1'}{c^2S} + \frac{ga \triangle S}{c^2S\,(\cosh 2az)^2},$$

we may again replace S in the denominator of the second term by S_0, whereas in the first term we may write $S_1'/S_1 = f$ for S_1'/S.

Introducing this into our derivations we obtain, instead of equation (20), the following equation:

$$\eta'' + \eta \left[\frac{ga \triangle S}{c^2S_0\,(\cosh 2az)^2} + \frac{gf}{c^2} - \mu^2\right] = 0. \tag{45}$$

On further proceeding as before, the only difference in the result will be that the eigen-values of μ, found above, will now be the eigen-values of $\sqrt{\mu^2 - gf/c^2}$, *when the value of c is the same* in both cases, so that we obtain

$$\tfrac{1}{2}\, b\mu^\dagger = \tfrac{1}{2}\, b \sqrt{\mu^2 - gf/c^2} = \varepsilon - n,$$

ε being defined as before. Hence,

$$b\mu^\dagger + 2n + 1 = \sqrt{1 + g\sigma b\mu^2/\nu^2},$$

from which we may easily derive the following equation

$$g\sigma\tau^2/b = 4n\,(n+1)\left(\frac{\lambda}{b}\right)^2 + 2\,(2n+1)\,\frac{\lambda}{b}\sqrt{1 - gf\tau^2} + 1 - gf\tau^2, \tag{46}$$

or

$$\left(\frac{\tau}{\sqrt{1 - gf\tau^2}}\right)^2 g\sigma/b = n\,(n+1)\left(\frac{2\lambda}{b\sqrt{1 - gf\tau^2}}\right)^2 + (2n+1)\,\frac{2\lambda}{b\sqrt{1 - gf\tau^2}} + 1.$$

Writing, for the moment, $\tau/\sqrt{1 - gf\tau^2} = \tau^\dagger$ and $\lambda/\sqrt{1 - gf\tau^2} = \lambda^\dagger$, we have the same relation between $\tau^\dagger$ and $\lambda^\dagger$ as previously found between τ and λ for the case where f was *zero*, so that, in order to calculate the value of λ corresponding to a given value of τ, we may proceed as follows: first, find $\tau^\dagger$ by dividing τ by $\sqrt{1 - gf\tau^2}$; then enter into the diagram of fig. 2 with the value $g\sigma\tau^{\dagger 2}/b$ for r and find the value of q of the corresponding point of the appropriate curve; then, putting $q = 2\lambda^\dagger/b$ find $\lambda^\dagger$; finally find λ by multiplying $\lambda^\dagger$ by $\sqrt{1 - gf\tau^2}$.

As $\lambda^\dagger/\tau^\dagger = \lambda/\tau = c$, the velocity of propagation of waves with period $2\pi\tau$ or wave length $2\pi\lambda$ is the same as we find from fig. 2 for $\tau^\dagger$ or $\lambda^\dagger$. In other words: if we represent the period-wave-length relation for any value of n in a q-r-diagram as before, each point $[q(\lambda), r(\tau)]$ of the curve lies with the point $[q(\lambda^\dagger), r(\tau^\dagger)]$ of the corresponding curve for the case $f = 0$ (fig. 2) on the same equal-velocity parabola.

From the form (46) of our equation it may easily be seen that for any fixed value of λ the corresponding value of τ must be smaller here than in the previous case, where f was *zero*. Furthermore the values of τ larger than $1/\sqrt{gf}$ are here ruled out; the corresponding limiting value of λ obtained from (46) turns out $\frac{1}{2}b\sqrt{\sigma/n(n+1)bf}$.

Meanwhile, it should be borne in mind, that just for large values of λ or, more generally speaking, for large values of $\lambda^\dagger$ (on which the wave thickness depends, as we shall see), where the deviation of the results of this section from those of section 2 (fig. 2) becomes important, the increasing wave thickness may prevent us from applying them to the cases we are concerned with, *viz.* cases of limited depth of the fluid on both sides of the transition layer and of limited interval of validity of formula (44).

The minimum value of τ is here:

$$\tau_{\min} = \frac{1}{\sqrt{g\,(\sigma/b + f)}}.$$

The quantity $\sigma/b + f$ occurring in the above formula is again the maximum of the rate of relative variation of S, found at the middle of the transition layer.

For $n = 0$ we can separate the variables (τ and λ) and find an explicit formula for λ as a function of τ, *viz.*

$$\frac{g\,(\sigma/b + f)\,\tau^2 - 1}{\sqrt{1 - gf\tau^2}} = \frac{2\lambda}{b}.$$

A graph of this relation is shown in fig. 8, where now $r_f = g(\sigma/b + f)\,\tau^2 = (\tau/\tau_{\min})^2$ has been used as the ordinate; the abscis is $q = 2\lambda/b$, as before. In terms of r_f and q the equation of this curve reads as follows:

$$\frac{r_f - 1}{\sqrt{1 - hr_f}} = q,$$

where

$$h = \frac{f}{\sigma/b + f}$$

is the ratio of the lapse rate of the term S_1 in formula (44) at the level $z = 0$ to the total lapse rate of S at the same level, the latter being the maximum lapse rate of S. The curve of fig. 9 has $h = 0.1$. The maximum value[1]) of r_f is $1/h$; the minimum value of r_f is 1, as was the minimum of r for $f = zero$. The straight line $r = q + 1$ (38) found previously for that case has been added in fig. 8 (dashed line).

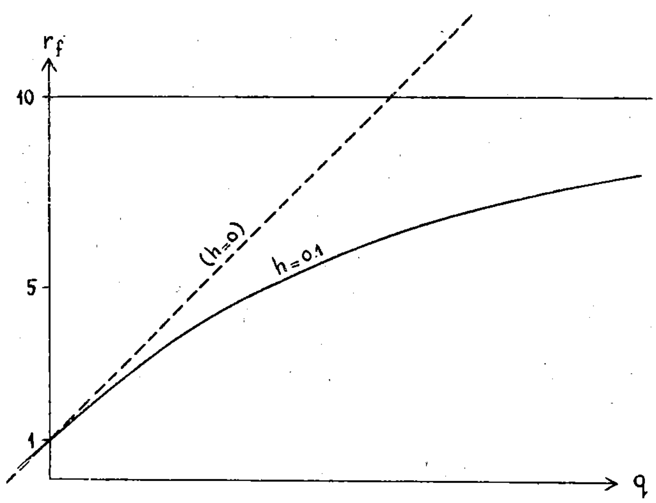

Fig. 8. Relation between $(\tau/\tau_{\min})^2$ and $2\lambda/b$.

For all values of n the reduced stream function corresponding to a certain solution is, here also, given by formula (41), where now, however,

$$m = \tfrac{1}{2}b\mu\sqrt{1 - gf\tau^2} = \tfrac{1}{2}b\mu^\dagger = \frac{b}{2\lambda^\dagger},$$

this being exactly the inverse of the value of q found on entering into the diagram of fig. 2 with the value $g\sigma\tau^{\dagger 2}/b$ for r, as described above.

The wave thickness belonging to a solution (41), when b is small enough, was, in terms of b and m, given by $\pi b/m$. This yields: wave thickness $= L/\sqrt{1 - gf\tau^2} = L^\dagger = 2\pi\lambda^\dagger$, $\lambda^\dagger$ being the quantity most easily calculated when starting from τ, as described above.

If, however, b is large compared with $\lambda^\dagger$, the wave thickness may be put equal to $\sqrt{2bL^\dagger}$.

[1]) *Cf.*, however, the remark made above as to the limited applicability of the formulas of this section for large values of $\lambda^\dagger$.

If, in addition to the complication dealt with in this section, we wish to take into account a rotation of the whole fluid system, we have to substitute $\lambda\sqrt{1-(2\omega_z\tau)^2}$ for λ in all equations. The upper limit of τ will now be the lower one of the two values $1/\sqrt{gf}$ and $1/2\omega_z$.

8. *General proof of the existence of the lower bound of periods of interval waves.*

In this section we shall drop our special assumptions concerning the density distribution and the absence of boundaries.

The density distribution is only supposed to be stable and to have a finite density gradient everywhere inside the fluid.

A lower rigid boundary may be present at a level $z = z_0$, so that, since the vertical velocity must, at any time, be zero there, the stream function vanishes at this level, or:

$$\varphi(z_0) = 0, \tag{47}$$

$\varphi(z)$ being defined as before (section 2).

If the fluid is infinitely deep, we may put $z_0 = -\infty$.

When dealing with *internal* gravitational waves we always have still at least one other level, $z = z_1$, where $\varphi(z)$ vanishes:

$$\varphi(z_1) = 0 \tag{48}$$

(where, eventually, z_1 may have the „value" $+\infty$), for, *either* there is a rigid upper boundary at $z = z_1$, *or* there is a free surface at $z = z_2$, but in the latter case we know that *internal* waves are supposed to have at least one „node"-level between the surface and the bottom.

The general, exact equation for $\varphi(z)$ is:

$$\varphi'' - \varphi'\frac{S'}{S} + \varphi\mu^2\left(\tau^2\frac{gS'}{S} - 1\right) = 0, \tag{49}$$

which is identical with equation (15).

Now, if $\varphi(z)$ is to vanish at $z = z_0$ and at $z = z_1$, it must needs, somewhere in between, have at least one maximum, where it is positive, or a minimum, where it is negative, or both, since $\varphi(z)$ is a continuous function of z and is not identically equal to *zero*. This means, since φ' and φ'' exist everywhere between z_0 and z_1, that, at a certain level $z = z^*$ $(z_0 < z^* < z_1)$,

$$\left.\begin{aligned} \varphi'(z^*) &= 0, \\ \frac{\varphi''(z^*)}{\varphi(z^*)} &< 0. \end{aligned}\right\} \tag{50}$$

From (50) and (49) follows, since S' is finite, that $\mu^2(\tau^2 gS'/S - 1)$ is positive at $z = z^*$. This means, that $\tau^2 gS'/S$ must become larger than unity somewhere inside the fluid; in other words: $\tau^2 g(S'/S)_{\max} > 1$, or:

$$\tau^2 > \frac{1}{g\,(S'/S)_{\max}},$$

or:

$$T > \frac{2\pi}{\sqrt{g\,(S'/S)_{\max}}} = T_1. \tag{51}$$

This exact result, which is independent of any special assumptions with regard to the density distribution (even if $S' = \infty$ at a certain level, (51) is true, because it gives $T_1 = 0$, then) differs from what was arrived at in section 4 (38) only in so far as (38) has $S'_{\max}/S_0$ instead of $(S'/S)_{\max}$, but in section 4 the difference between S'/S and S'/S_0 was discarded, as we remember.

The period T_1 is exactly equal to what is sometimes called the „period of free oscillation" of a fluid particle belonging to the level of greatest stability.

The „period of free oscillation" of any fluid particle in a stable stratification is understood to be the period of the oscillating motion the said fluid particle is supposed to assume, under the influence of the Archimedian force, when it is removed from the level it belongs to, without, however, removing the surrounding fluid particles from their equilibrium levels. Now this is a rather hypothetical experiment. Perhaps we might best put it this way, that (51) gives some sense to this „period of free oscillation", but only for the level of greatest stability, in so far as this period, computed for the level of greatest stability, is the lower limit of the periods of internal waves in the wole fluid layer considered.

In the table alongside we have given values of T_1 for various values of $(S'/S)_{max}$, or $(-\varrho'/\varrho)_{max}$.

$(S'/S)_{max}$	T_1
0	∞
$1 . 10^{-6}\,m^{-1}$	33 . 4 min.
$4 . 10^{-6}\,m^{-1}$	16 . 7 min.
$1 . 10^{-5}\,m^{-1}$	10 . 6 min.
$4 . 10^{-5}\,m^{-1}$	5 . 3 min.
$1 . 10^{-4}\,m^{-1}$	3 . 3 min.
$4 . 10^{-4}\,m^{-1}$	$1\frac{2}{3}$ min.
$1 . 10^{-3}\,m^{-1}$	63 sec.
$4 . 10^{-3}\,m^{-1}$	32 sec.
$1 . 10^{-2}\,m^{-1}$	20 sec.
∞	0

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