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NOTES ON MOTIONS NEAR DISCONTINUITY SURFACES, WHICH ARE NOT IN EQUILIBRIUM

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Introduction and Survey

The present paper is the result of a study of certain standing wave motions on surfaces of discontinuity between homogeneous imcompressible fluids, with a view of drawing herefrom, if possible, some conclusions about air motions in the vicinity of atmospheric frontal surfaces. These motions interest us in the first place as far as their vertical components are concerned, in connection with the formation and dissolution of clouds and the formation of precipitation.

It is a well-known fact, that frontal systems, that are in dynamic equilibrium, are rather "inactive" as to cloud formation and precipitation. It has generally been understood, therefore, that for studying these frontal phenomena we have more especially to consider systems, that are not in equilibrium, thereby causing the frontal surfaces to alter their slopes, or, more adequately speaking, causing the air-masses in question to "flow out" or "retreat" relatively. We shall deal with the problem here in connection with certain oscillations or standing wave motions of the whole system about its equilibrium state. The same has been tried by N. Kotschin, 1932 [1]), but his results cannot be applied to the earth's atmosphere, for in one of the two air masses he neglects the true boundary conditions on the ground.

The simplest waves of the type, here studied, have already been treated elsewhere in hydrodynamic literature. When we consider them all the same in some detail (section 2) it will be, *primo*, because they form the starting point for studying less simple situations, *secundo*, because we want a somewhat more precise analysis of the fields of motion connected with these "waves".

As to the less simple wave motions, *viz.* waves in plan parallel fluid layers in a rotating system of reference (section 3) and waves in wedge-shaped fluid layers (section 4), our problems resemble in some respect those studied, among others, by Solberg, 1928, Bjerknes c.s., 1933, and Godske, 1935. The difference, however, is that these authors considered in particular waves propagated horizontally along the discontinuity surface, while our solutions are independent of the coordinate in this direction; moreover, here again it is not so much our intention to study the wave as a whole [2]) as to consider the resulting motions in the two fluids.

It must be admitted, that as regards one part (section 3) these problems are of a somewhat academic character, because the true boundary conditions for the atmosphere are not taken into account, and as regards the other part (section 4) were not solved quantitatively; this is somewhat unsatisfactory. However, considering that anyhow the real conditions in the atmosphere are still much more complicated by other factors, the results obtained may nevertheless give a fair idea of the motions interesting us.

We confine ourselves to the treatment of homogeneous incompressible fluids. This has the important advantage, that the Eulerian method, applied to our hydrodynamical problems, gives us rather simple differential equations, the solutions of which we may study conveniently by means of stream functions and stream lines.

Real atmospheric "fluids" on both sides of frontal surfaces are not homogeneous and incompressible, that is to say: they do not in general have an indifferent stratification, but a stable one. These and other complications are considered very briefly at the end of the paper (section 5).

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[1]) References are indicated by mentioning the year of publication, see list of literature at the end of the paper.

[2]) In the following pages we shall constantly use the term „wave" in the sence of „*standing* wave". For general use we have preferred this term here to the term „oscillation" because the latter suggests more or less the motion of a *single* particle or that of a material system considered *as a whole*.

We start (section 1) with a somewhat general treatment of some special aspects of our subject.

1. General remarks. If accelerations of the air are absent, we have for the inclination of atmospheric surfaces of discontinuity the well-known formula of Margules:

$$tg\,\alpha = \frac{2\omega_z}{g}\,\frac{\varrho V - \varrho' V'}{\varrho - \varrho'},$$

where ϱ and V, resp. ϱ' and V', are the density and the horizontal velocity parallel to the front of the cold resp. of the warm air, and ω_z is the vertical component of the earth's rotation (the horizontal component has been neglected).

When this relation is not satisfied, we must conclude that accelerations are present and that the frontal surface may alter its inclination. This is not *necessary*, however, as witnessed by the case of stationary curvilinear currents, in which *stationary* frontal surfaces can exist (the equilibrium inclination then depends upon the radius of curvature, cf. Margules 1906). In the following we shall confine ourselves, however, to rectilinear currents; in that case, when Margules' formula is not satisfied, a frontal surface is not in equilibrium, at least when it is not unlimited in every direction (cf. Koschmieder 1941, page 266).

Here two questions present themselves, viz:

1°. that of the motions in the two air masses on both sides of the surface, in so far as these motions are the result of this non-equilibrium position;

2°. that of the effect of the state of things mentioned upon the pressure field and the energy balance (cf. Margules 1905).

Of these two problems only the first one will be discussed here.

Neglecting the vertical accelerations in comparison with the acceleration of gravitation [1]), we find the following relation as to the horizontal accelerations in a direction perpendicular to the front (the deduction of this relation [2]) is based on the so-called dynamical boundary-surface condition, i. e.: the continuity of pressure at the surface of discontinuity):

$$\varrho \frac{dU}{dt} - \varrho' \frac{dU'}{dt} = g\,(\varrho - \varrho')\,(tg\,\alpha - tg\,\vartheta). \qquad (1)$$

Here α is the equilibrium angle, determined by Margules' formula, and ϑ the actually occurring angle of inclination of the frontal surface. In this as well as in all other formulas the accented symbols refer to the lighter air (fluid), the corresponding unaccented symbols to the heavier air (fluid).

When, for example, $\vartheta > \alpha$, it follows from (1), that $\varrho' \frac{dU'}{dt} > \varrho \frac{dU}{dt}$. When both accelerations are positive, this means, that also $\frac{dU'}{dt} > \frac{dU}{dt}$; when, on the other hand, both are negative, it *may* be that $\frac{dU'}{dt} > \frac{dU}{dt}$ (the warmer air has a smaller acceleration towards the left), but it is not necessary.

Vertical accelerations.

The so-called kinematical boundary-surface condition implies the continuity of the velocity component normal to the surface of discontinuity. In order to deduce from this a relation concerning the accelerations, we suppose that we have to deal with a state of

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[1]) We should bear in mind here, that in the deduction of formula (1) the gravitational acceleration figures with a reduction-factor of the order of $(\varrho - \varrho')/\varrho$ as compared to vertical accelerations. By making a reasonable estimate as to these vertical accelerations (see e.g. Hesselberg and Friedmann: „Die Grössenordnung der meteorologischen Elemente", etc., Veröff. Geophys. Inst. Leipzig, Spezialarb., 1, p. 147), it appears that this neglect is nevertheless permitted, considering the degree of inaccuracy we must accept in calculations of this sort.

[2]) See for instance D. Brunt, Physical and Dynamical Meteorology, Cambridge 1939, par. 128 or H. Koschmieder, Dynamische Meteorologie, Leipzig 1941, page 266, where also further literature is given.

things differring not too much from an equilibrium state, so that we may treat the difference as a small perturbation:

$$U = U_o + u,\ V = V_o + v,\ W = W_o + w,\ U' = U_o' + u',\ V' = V_o' + v',\ W' = W_o' + w',\ tg\,\vartheta = tg\,\vartheta_o + \delta.$$

Here tg ϑ_o is the inclination determined by the formula of Margules when we substitute V_o and V_o' for V and V'. The perturbation velocities are denoted by small types, the unperturbed (equilibrium) quantities are indicated by the suffix $_o$. In the unperturbed state no vertical velocity components occur and the whole system performs a translation U_o in the x-direction [1])

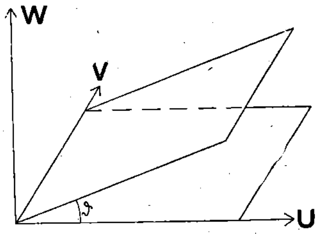

Figure 1

$$(W_o = W_o' = 0;\ U_o = U_o').$$

The kinematical boundary-surface condition is as follows:

$$W - W' = (U - U')\, tg\,\vartheta,$$

or

$$w - w' = (u - u')\, tg\vartheta_o + (u - u')\,\delta,$$

or, when, according to the method of small perturbations, we neglect the second term in the second member, as being small of the second order:

$$w - w' = (u - u')\, tg\vartheta_o \tag{2}$$

By giving to our system of reference a velocity in the x-direction [2]), equal to that of the unperturbed system, we make once for all the velocity U_o *zero*. Now w, w', u and u' are functions of x, y, z and t; equation (2), however, applies to points of the surface of discontinuity. Consequently, for such points we have also:

$$\frac{\partial w}{\partial t} - \frac{\partial w'}{dt} = \left(\frac{\partial u}{dt} - \frac{\partial u'}{dt}\right) tg\vartheta_o.$$

The vertical accelerations are given by:

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + (V_o + v)\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z},$$

$$\frac{dw'}{dt} = \frac{\partial w'}{\partial t} + u'\frac{\partial w'}{\partial x} + (V_o' + v')\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z}.$$

In this study we only concern ourselves with solutions, which are independent of y, so that we may put $\partial/\partial y = 0$. By neglecting once more, according to the method of small perturbations, the quantities that are small of the second order, we find the vertical accelerations, i. e. the individual derivatives of w with respect to time, to be equal to the local derivatives. The same is true for the horizontal velocity components, so that finally we obtain the following relation between the vertical and the horizontal components of the accelerations:

$$\frac{dw}{dt} - \frac{dw'}{dt} = \left(\frac{du}{dt} - \frac{du'}{dt}\right) tg\vartheta_o \tag{3}$$

These relations can therefore be made actually to hold to any arbitrary degree of accuracy by simply taking the perturbance sufficiently small.

Finally, if we like, we may replace the factor tg ϑ_o in (3) by tg ϑ, the change in the second member of (3), implied by this substitution, again being small of the second order.

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[1]) In the atmosphere such a translation of the system will be kept up by the pressure field. As to the complication implied by this, see section 5 (a).

[2]) The apparent force (Coriolis force), which is thereby induced in our system of reference, is conceived to be balanced by the pressure field, see preceding note.

Equations (1) and (3) give us two relations between the horizontal and the vertical accelerations of air-particles, bordering on the frontal surface and situated opposite each other on either side of the surface; in (1) we may now replace U and U' by u and u', while we should bear in mind that in general α is not the same angle as ϑ_o.

For calculating the various acceleration components with the aid of (1) and (3) we need still two more independent relations between these quantities. Exner[1]), 1924, for his calculations of these accelerations, introduced the suppositions, that $du/dt = - du'/dt$ and $dw/dt = - dw'/dt$. The argumentation for this suppositions was purely qualitative, the reasoning being about as follows: when the frontal surface changes its slope, warm air in the lower parts of the system is replaced by cold air, or vice versa, whereas in the upper parts cold air is replaced by warm air, or vice versa. It is clear, however, that the matter is not so simple. This supposition cannot possibly be proved in a general way, the motions in such a system still depending on the boundary conditions of the whole. What we can state is only that, if two fluids together fill a limited and invariable region, their *centres of gravity* will perform motions in nearly opposite directions.

If the fluids are incompressible we can even prove exactly, that the integral of the velocity vector $\vec{v}$ for one fluid over the region R, occupied by it, is the opposite of the corresponding integral for the other fluid (denoted by accented symbols) over the region R':

$$\iiint_R \vec{v}\,(x, y, z)\, dx\, dy\, dz = - \iiint_{R'} \vec{v}'\,(x, y, z)\, dx\, dy\, dz. \tag{4}$$

To prove this, we write (4) in the equivalent form

$$\iiint_{R + R'} \underline{\vec{v}}\,(x, y, z)\, dx\, dy\, dz = 0, \tag{4*}$$

where now $\underline{\vec{v}}\,(x, y, z)$ denotes the velocity field, defined in R by $\vec{v}\,(x, y, z)$, in R' by $\vec{v}'\,(x, y, z)$. This field has the following two porperties:

first, that in all points of the boundary of $R + R'$ the normal component of $\underline{\vec{v}}$ vanishes (for this boundary was agreed to be invariable);

secondly, that the integral of the normal component of $\underline{\vec{v}}$ over the boundary of any arbitrary part R_1 of the region $R + R'$ vanishes (this is a consequence of the incompressibility of the fluids). If now we understand by $O\,(x)$ the intersection of a surface $x =$ constant and the region $R + R'$, it follows easily from the combination of the above statements, that

$$\iint_{O(x)} \underline{u}\,(x, y, z)\, dy\, dz = 0, \tag{5a}$$

where $\underline{u}$ stands for the x-component of $\underline{\vec{v}}$. Likewise the other components satisfy the equations

$$\iint_{O(y)} \underline{v}\,(x, y, z)\, dx\, dz = 0, \qquad \iint_{O(z)} \underline{w}\,(x, y, z)\, dx\, dy = 0. \tag{5b, c}$$

By multiplying the first members of equations (5) resp. by dx, dy and dz and integrating, we find (4*).

If the region $R + R'$ is unlimited in the y-direction and bounded by a cylindrical surface parallel to the y-axis, and if the velocities are independent of y, our problem becomes an essentially two-dimensional one. Instead of equations (5) we obtain:

$$\int_{L(x)} \underline{u}\,(x, z)\, dz = 0, \qquad \int_{L(z)} \underline{w}\,(x, z)\, dx = 0, \tag{6}$$

[1]) For the horizontal accelerations Exner derived a formula, which is equivalent to (1); as to the vertical accelerations, however, he set up an erroneous relation, for the deduction of which he used the static equation for pressure so far as it depends upon z!

where $L(x)$ resp. $L(z)$ are the intersections of lines $x = \text{const.}$ resp. $z = \text{const.}$ and the (two-dimensional) region $G + G'$ (see fig. 2). On integrating (6) we find, analogous to (4):

$$\iint_G u\,dx\,dz = -\iint_{G'} u'\,dx\,dz, \quad \iint_G w\,dx\,dz = -\iint_{G'} w'\,dx\,dz. \qquad (7)$$

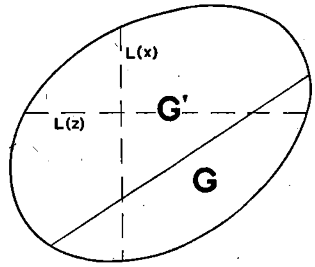

Figure 2

If, finally, the regions G and G' are equal, it follows from (7) that the averages taken over the velocity fields in the two fluids are opposite.

By differentiating the equations (7) with respect to time and equating the partial (local) derivatives of the velocities to the accelerations (this is permitted within the limits of small perturbations), we find equations, analogous to (7), for the accelerations instead of the velocities. If again the regions G and G' are equal, it follows, that the averages taken over the acceleration fields in the two fluids are opposite (for small perturbations).

But this is still far from being equivalent to Exner's supposition stating the accelerations of the two fluids in a point of the frontal surface to be opposite. Moreover, (7)is only valid for fluids filling an invariable region (and being incompressible — but this is, for our problem, the less serious difference from real atmospheric conditions).

In order to learn more about the field of motion on both sides of a surface of discontinuity we shall have to combine the equations of motion with certain external boundary conditions (the kinematical and dynamical conditions at surfaces of discontinuity might be called internal boundary conditions).

Before passing on to these we shall first give a brief analysis of the general nature of the solutions of the equations of motion we wish to obtain. *These considerations apply not only to the more simple conditions of the following section but quite generally to all fields of motion studied in this paper.*

It has already been said that we shall confine ourselves to small perturbations of the equilibrium of a flow system. Let the perturbation field of motion be designed by $\vec{v}\,(x, y, z, t)$ and the perturbation pressure by $p\,(x, y, z, t)$. Then we can develop in a Fourier-integral:

$$\vec{v}\,(x, y, z, t) = \vec{v}_o\,(x, y, z) + \int_{-\infty}^{+\infty} \vec{v}_\nu\,(x, y, z)\, e^{i\nu t}\, d\nu,$$

$$p\,(x, y, z, t) = p_o\,(x, y, z) + \int_{-\infty}^{+\infty} p_\nu\,(x, y, z)\, e^{i\nu t}\, d\nu.$$

Now, by small perturbations we shall understand perturbations during which not only the perturbation velocities but also the displacements of the individual particles from their equilibrium positions remain small. This implies that the term $\vec{v}_o\,(x, y, z)$ in the above development is zero, that is to say: a time-independent part is not permitted in the perturbation velocity field [1]).

If we substitute the above developments into the *linear* differential equations being valid for small perturbations (see, for a general exposure of the hydrodynamical theory of small pertrubations, e.g. Bjerknes c.s., Physikalische Hydrodynamik, 1933, Chapter VII, sections 75 and 83; as to our problems, see our equations (8) and (24)), it then appears, that any one of the vector fields $\vec{v}_\nu\,(x, y, z)\, e^{i\nu t}$ and the corresponding $p_\nu\,(x, y, z)\, e^{i\nu t}$ has separately to satisfy these differential equations, whereas p_0 can at most be a constant,

[1]) The reason for this restriction lies in the fact, that real atmospheric „fluids" have a stable stratification; now, for a stable stratification the above restriction lies in the nature of small perturbations (for a homogeneous and incompressible fluid it does not).

which, besides, does not matter. In other words: the general perturbation may be treated as a superposition of motions, all being harmonic with respect to time and each separately satisfying the perturbation equations; these motions may therefore be considered as the fundamental solutions of the problem.

In any actual case the boundary conditions limit the number of fundamental solutions, so that, as a rule, a set of discrete possible values ν_n of the frequency is selected out of the continuum of values from $-\infty$ to $+\infty$; this causes the general solution to be a series instead of an integral:

$$\vec{v}\ (x, y, z, t) = \sum_n \vec{v}_n\ (x, y, z)\ e^{i\nu_n t}$$

Finally we shall, on the basis of the foregoing, prove a vector field $\vec{v}\ (x, y, z, t)$, representing a small perturbation of an equilibrium position for a system of one or more homogeneous incompressible fluids, to be irrotational within each of them:

$$rot\ \vec{v} \equiv 0, \qquad \text{or:} \qquad \oint_C \vec{v}\ \vec{ds} = 0,$$

C standing for any arbitrary circuit within one fluid.

The proof follows from the well-known theorem of classical hydrodynamics stating the constancy of circulation. Since, however, each of the components (fundamental solutions) $\vec{v}_n$ of a field $\vec{v}$ satisfies the equations of motion, this theorem implies, that [1])

$$rot\ \frac{\partial \vec{v}_n}{\partial t} \equiv 0, \qquad \text{or:} \qquad rot\ i\,\nu_n\,\vec{v}_n \equiv 0, \qquad \text{or:} \qquad rot\ \vec{v}_n \equiv 0,$$

whence the above statement follows immediately.

This implies, that, if the component of the field of motion in a plane surface can be represented by means of stream lines, no closed stream lines can exist within *one* fluid. On the other hand: if a closed stream line does exist, it must pass through at least two fluids. Evidently all closed stream lines inside this stream line must likewise pass through both fluids. On the boundary surface, therefore, at least one point exists, where the velocities on both sides are directed opposite to each other; see e.g. fig. 12.

2. Rectangular basin. Since the atmospheric conditions are rather complicated, it is useful first to consider a simpler case, e.g.: that of two fluids in a rectangular basin or channel (see fig. 3), without rotation. The close analogy of this case to our atmospheric case appears immediately from the fact, that here with respect to the horizontal accelerations a relation may be deduced, arising from (1) by putting tg $\alpha = 0$ in this formula (the equilibrium position of the surface of discontinuity is horizontal); tg ϑ is the slope of the surface of discontinuity in the point considered (the intersection of the boundary surface and a surface $y =$ const. is supposed to be independent of y, see fig. 3). Here again, however, we cannot give a *general* statement about the individual accelerations. To find these we must in each separate case take into account the external boundary conditions.

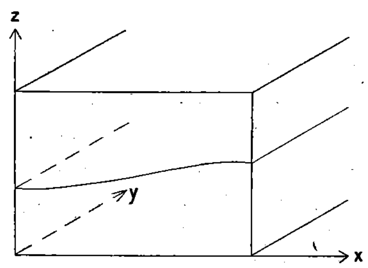

Figure 3

Of the two forms in which the hydrodynamical equations of motion can be written, the Lagrangian form and the Eulerian form, we prefer the latter, since it gives the less complicated forms here.

[1]) Here d/dt has again been put equal to $\partial/\partial t$ as before.

If the unperturbed state of motion be that of rest or be a flow in the y-direction with a velocity constant with respect to place and time, the equations for small perturbations are [1]):

$$\dot{u} + S\frac{\partial p}{\partial x} = 0, \quad \dot{v} + S\frac{\partial p}{\partial y} = 0, \quad \dot{w} + S\frac{\partial p}{\partial z} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where u, v and w are the components of the perturbation velocity field in the medium considered, which has a specific volume $S = \varrho^{-1}$; p is the perturbation pressure.

In our case, these equations can still be simplified, since we are only concerned with solutions independent of y (cf. fig. 3), so that the derivatives with respect to y vanish and we are left with a two-dimensional problem. Now we can introduce the perturbation stream function ψ by putting

$$u = \frac{\partial \psi}{\partial z}, \qquad w = -\frac{\partial \psi}{\partial x},$$

thereby making the equation of continuity identically satisfied if only ψ be a continuous function of x and z. We obtain:

$$\frac{\partial^2 \psi}{\partial z\, \partial t} + S\frac{\partial p}{\partial x} = 0, \qquad -\frac{\partial^2 \psi}{\partial x \partial t} + S\frac{\partial p}{\partial z} = 0. \tag{8}$$

If the fluid in question is anywhere bounded by a rigid wall, parallel to the y-direction we have the boundary condition: $\psi(x, z) =$ const. along the intersection of this wall and the surface $y =$ const., or, in other words: this intersection must be a stream line.

In the case of two fluids, of densities ϱ and ϱ', one underneath the other, we have to solve two sets of equations (8) and (8′), where (8′) is obtained from (8) by replacing S, ψ, p by S', ψ', p', respectively representing the specific volume, the stream function and the perturbation pressure in the upper fluid. The dynamical and kinematical boundary-surface conditions now require the permanent continuity of pressure and of the stream function, respectively, at the boundary-surface.

The former condition gives generally [2]):

$$\frac{\partial\,(p - p')}{\partial\, t} + \vec{V}\, \nabla\, (p - p') + \vec{v}\, \nabla\, (P - P') = 0, \tag{9}$$

which has to be satisfied at the *unperturbed* boundary-surface; here $\vec{V}$, P are the unperturbed velocity field and the unperturbed pressure field, respectively. If, as we have assumed, $\vec{V}$ has a component in the y-direction only, whilst our solutions are independent of y, the second term of the first member of (9) vanishes, so that

$$\frac{\partial\,(p - p')}{\partial\, t} + \vec{v}\,(\nabla P - \nabla P') = 0. \tag{10}$$

In the present case $\triangle P$ and $\triangle P'$ are both directed vertically and we obtain:

$$\frac{\partial p}{\partial t} - \frac{\partial p'}{\partial t} - g\, w\, (\varrho - \varrho') = 0, \tag{11}$$

which has to satisfied for $z = z_1$, z_1 being the level of the unperturbed discontinuity surface.

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[1]) See e.g. Bjerknes c.s., Physikalische Hydrodynamik, 1933, section **83**.

[2]) See previous note. The equation, analogous to (9), with $\vec{V}'$ and $\vec{v}'$ instead of $\vec{V}$ and $\vec{v}$ is equivalent to (9) as soon as the kinematical boundary-surface condition is satisfied at the surface $P = P'$.

The kinematical boundary-surface condition gives an analogous equation for the stream function, P, P', p, p' being replaced by Ψ, Ψ', ψ, ψ', with the only difference, that the unperturbed stream functions Ψ, Ψ' are constants, so that $\triangle \Psi$ and $\triangle \Psi'$ vanish. Hence we find:

$$\partial\psi/\partial t - \partial\psi'/\partial t = 0, \qquad \text{or} \qquad \psi - \psi' = 0, \tag{12}$$

which again applies to the unperturbed boundary-surface, here: $z = z_1$.

Solutions of (8), having one horizontal straight stream line $z = z_o$, are given by:

$$\psi = A \sinh \mu (z - z_0) \sin \mu x \sin (\nu t + \sigma), \tag{13a}$$

$$Sp = B \cosh \mu (z - z_0) \cos \mu x \cos (\nu t + \sigma), \tag{13b}$$

where A, B, μ and ν are constants; A, B are, in general, functions of μ and ν, μ and ν depending on internal and external boundary conditions. By substituting (13) in (8) we find:

$$B = \nu A.$$

Vertical stream lines are found for $x = k\pi/\mu$, k being an integer.

A more general solution, having one horizontal and at least one vertical straight stream line, at $z = z_o$ and $x = 0$ respectively, is obtained by superposition of solutions (13):

$$\psi = \sum_n A_n \sinh \mu_n (z - z_0) \sin \mu_n x \sin (\nu_n t + \sigma_n), \tag{14a}$$

$$Sp = \sum_n \nu_n A_n \cosh \mu_n (z - z_0) \cos \mu_n x \cos (\nu_n t + \sigma_n). \tag{14b}$$

The line $x = 0$ may coincide with a vertical rigid wall. In order to have a second one at $x = a$, it is necessary that, for all n's,

$$\mu_n = k_n \frac{\pi}{a}, \qquad k_n \text{ being an integer.}$$

If we have two fluids with densities $\varrho = S^{-1}$, $\varrho' = S'^{-1}$, we combine (14) with a corresponding solution (14') of (8') for the upper fluid. Let the unperturbed boundary-surface be found at $z = z_1$; $z = z_0$ coincides with the (rigid) bottom.

The kinematical boundary-surface condition (12) now requires:

$$\mu_n = \mu'_n, \qquad \nu_n = \nu'_n, \qquad \sigma_n = \sigma_n',$$

and

$$A_n \sinh \mu_n (z_1 - z_0) = A'_n \sinh \mu_n (z_1 - z'_{0n}), \tag{15}$$

where the constants z'_{0n} are determined by the upper boundary of the upper fluid.

The dynamical boundary-surface condition (11) gives:

$$\nu_n^2 [\varrho A_n \cosh \mu_n (z_1 - z_0) - \varrho' A'_n \cosh \mu_n (z_1 - z'_{0n})] =$$
$$= g (\varrho - \varrho') \mu_n A_n \sinh \mu_n (z_1 - z_0) =$$
$$= g (\varrho - \varrho') \mu_n A'_n \sinh \mu_n (z_1 - z'_{0n}),$$

or

$$\mu_n^{-1} \nu_n^2 [\varrho \coth \mu_n (z_1 - z_0) - \varrho' \coth \mu_n (z_1 - z'_{0n})] = g (\varrho - \varrho'). \tag{16}$$

For small perturbations, independent of y, we have the following relation referring to the vertical displacement (elevation) ζ:

$$w = \frac{\partial \zeta}{\partial t}. \tag{17}$$

From this, it easily follows (by integration) that:

$$\zeta_1 = \sum_n A_n \frac{\mu_n}{\nu_n} \sinh \mu_n (z_1 - z_0) \cos \mu_n x \cos (\nu_n t + \sigma_n), \tag{18}$$

where ζ_1 is the elevation of a point of the surface of discontinuity.

The constants A_n (A'_n) and σ_n can now be determined by giving, at a certain initial moment, the elevation $\zeta_1(x)$ as well as the vertical velocity $w_1(x)$ at the boundary-surface (the latter being the same for both fluids, according to the kinematical condition).

By differentiating (14a) and (14'a) with respect to z we find u and u' (see (20)). By differentiating these expressions with respect to time and putting $z = z_1$ we easily deduce:

$$\varrho \frac{du_1}{dt} - \varrho' \frac{du_1'}{dt} = -g(\varrho - \varrho') \frac{\partial \zeta_1}{\partial x} = -g(\varrho - \varrho')\, tg\vartheta \tag{19}$$

where u_1, u'_1 stand for u ($z = z_1$), u' ($z = z_1$); for our small perturbations d/dt has been put equal to $\partial/\partial t$, as before. The relation (19) is identical with the one we formulated in the beginning of this section by putting tg $\alpha = 0$ in (1).

As for the constants z'_{0n}: *if the upper boundary is a horizontal rigid wall* $z = z'_0$, then $z'_{0n} = z'_0$ for all n's. Since $z'_0 > z_1 > z_0$, it follows from (15), that in this case A_n and A'_n have opposite signs, and therefore, the same is true for the terms

$$\begin{aligned} u_n &= A_n \mu_n \cosh \mu_n (z - z_0) \sin \mu_n x \sin (\nu_n t + \sigma_n) \\ u'_n &= A'_n \mu_n \cosh \mu_n (z - z'_0) \sin \mu_n x \sin (\nu_n t + \sigma_n) \end{aligned} \tag{20}$$

by which the horizontal velocity components $u = \Sigma u_n$, $u' = \Sigma u'_n$ are built up.

Whether u and u' themselves, resp. their values at the boundary-surface u_1 and u'_1, are directed oppositely, cannot generally be said. If, however, $z'_0 - z_1 = z_1 - z_0$ (the fluids have equal depths) it follows from (15), that $A_n = -A'_n$, and therefore, that $u_1 = -u_1'$. Then (19) yields:

$$\frac{du_1}{dt} = -\frac{du_1'}{dt} = -g \frac{\varrho - \varrho'}{\varrho + \varrho'} \frac{\partial \zeta_1}{\partial x}. \tag{21}$$

If the fluids do not have equal depths, such a simple relation cannot be given. Then we can only state that a *simple wave* (13) (all but *one* of the coefficients A_n in (14) are zero) has horizontal velocity components u and u', directed oppositely to each other, for the same x; the same is true for the acceleration components du/dt and du'/dt, du/dt having the sign opposite to that of $\partial \zeta_1/\partial x$ (the inclination of the boundary-surface at the point considered), du'/dt having the same one: the heavier fluid has a tendency to "slide down", the lighter one to "slide up".

In the case of a *non-simple wave* the matter is much more complicated because of the continually changing phase differences between the components. If *either* one of these component simple waves predominates, *or* the difference between the depths of the two fluids is small (making the difference between A_n and $-A'_n$ small also), we may state that the above conclusions, referring to simple waves, are to a large extent valid here also. As to the restriction "to a large extent": we have to leave out the intervals of time, during which the horizontal velocity components resp. the horizontal acceleration components, to which our conclusions refer, differ from zero only slightly; during the remaining part of time, either the different component simple waves cooperate sufficiently, with respect to the quantity considered, or one of them is predominating (when accelerations are considered, the local inclination of the boundary-surface may not be too small).

If, on the other hand, the upper boundary is not a rigid wall, but a free surface, represented, in the unperturbed state, by $z = z_2$, we have the following dynamical boundary condition in this level [1]):

$$\frac{\partial p'}{\partial t} - g\, w' \varrho' = 0, \tag{22}$$

or:

$$\nu_n^2 \cosh \mu_n (z_2 - z'_{0n}) = g \mu_n \sinh \mu_n (z_2 - z'_{0n}).$$

[1]) See e.g. Bjerknes c.s., 1933, **83** (9).

Bij substituting the expression for $\mu_n^{-1}\nu_n^2$, following from this relation, in (16), we find:

$$\frac{\coth \mu_n (z_1 - z'_{0n}) - \coth \mu_n (z_2 - z'_{0n})}{\coth \mu_n (z_1 - z_0) - \coth \mu_n (z_2 - z'_{0n})} = \frac{\varrho}{\varrho'} > 1. \tag{23}$$

On the basis of this relation it is easily seen, that either:

$$(1^0)\ z_1 < z'_{0n} < z_2,$$

or:

$$(2^0)\ z_0 < z'_{0n} < z_1.$$

In the first case we have to deal with an "*internal wave*" [1]); as to the fluid motions between z_0 and z'_{0n}, this case resembles completely the above case of a rigid upper boundary at $z = z'_{0n}$: the horizontal velocity components on either side of the boundary-surface are at any moment directed opposite to each other; this is valid for a *simple* wave. Also the relation to the local inclination of the boundary-surface is in that case the same as above. Further, when $\varrho/\varrho' \to 1$, $z'_{0n} \to z_2$. See fig. 4*a*.

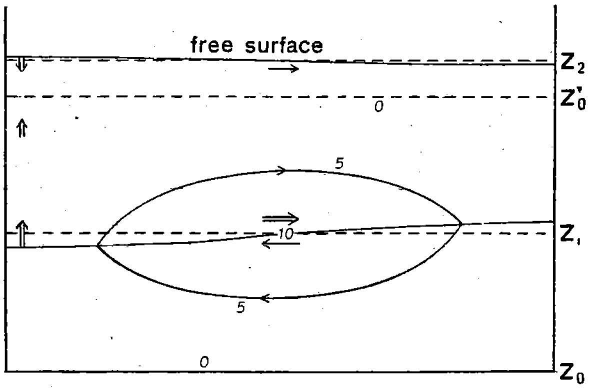

Figure 4*a*

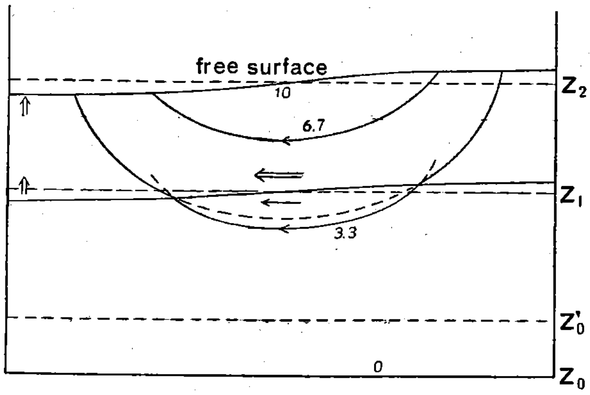

Figure 4*b*

A simple wave of the second type is a so-called "*external wave*" [1]). These solutions show quite a different structure, see fig. 4*b*. Now, the horizontal velocity components on either side of the boundary-surface are, at any moment, *directed in the same sense; that of the upper* (*lighter*) *fluid has the larger absolute value in a point of the boundary-surface:* seen from above, the stream lines are broken towards the normal (vertical), at this surface, or, more generally speaking: they are broken *in a direction opposite to their curvature* in the two media. These statements follow directly from (15) and (20). Quantitatively speaking, this "refraction" is small, if ϱ/ϱ' differs but slightly from unity, for, according to (23) the level z'_{0n} will in that case differ but slightly from z_0.

If $\varrho/\varrho' \to 1$, $u_1/u_1' \to 1$; in the limit $\frac{du_1}{dt} = \frac{du_1'}{dt} = -g\frac{\partial \zeta_1}{\partial x} = -g \operatorname{tg} \vartheta$. As we see, the accelerations are here by a factor of the order of magnitude $\frac{\varrho + \varrho'}{\varrho - \varrho'}$ larger than in the case of an internal wave, if the heights of the layers do not differ too much, cf. (21).

All this applies again to simple waves; in non-simple waves we have, as before, continually changing phase differences between the component simple waves, so that the above conclusions are no longer *generally* applicable. We may only assume them to be so to a large extent, when *either* one of the components of the superposition predominates, *or* ϱ/ϱ' differs but slightly from unity, so that according to (15) the differences between A_n and A'_n are also small. As to the restriction made ("to a large extent") we may refer to what has already been said above.

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[1]) Cf. Bjerknes c.s., l.c. page 320.

With repect to the accelerations we may formulate quite analogous conclusions (it is important to note, that the streamline-pattern of the velocity field may also be used to describe the acceleration field).

3. Rotating system. We shall now pass on to the case of a system of reference, rotating around the z-axis, ω_z being the angular velocity. If the two fluids in the unperturbed state have velocities V and V' in the y-direction, the perturbation equations, analogous to (8)[1], become, after eliminating y[2]:

$$\begin{aligned} &\frac{\partial^3 \psi}{\partial z \partial t^2} + 4\omega_z^2 \frac{\partial \psi}{\partial z} + S \frac{\partial^2 p}{\partial x \partial t} = 0, \\ &-\frac{\partial^2 \psi}{\partial x \partial t} + S \frac{\partial p}{\partial z} = 0, \end{aligned} \tag{24}$$

and two corresponding equations, with accented symbols, referring to the lighter fluid: (24′).

Now it is possible to write down solutions of (24) and (24′), analogous to (13) and the corresponding solution (13′) of (8′), which can satisfy the kinematical and the dynamical boundary-surface condition[3] at the surface of discontinuity $\mu_o x + \lambda_o z = c_1$, namely:

$$\begin{aligned} \psi &= A \sinh(\mu x_o + \lambda_o z - c) \sin(\mu_1 x + \lambda_1 z) \sin(\nu t + \sigma), \quad (25a) \\ Sp &= B \cosh(\mu_o x + \lambda_o z - c) \cos(\mu_1 x + \lambda_1 z) \cos(\nu t + \sigma), \quad (25b) \end{aligned} \tag{25}$$

$$\begin{aligned} \psi' &= A' \sinh(\mu_o x + \lambda_o z - c') \sin(\mu_1 x + \lambda_1 z) \sin(\nu t + \sigma), \quad (25'a) \\ S'p' &= B' \cosh(\mu_o x + \lambda_o z - c') \cos(\mu_1 x + \lambda_1 z) \cos(\nu t + \sigma). \quad (25'b) \end{aligned} \tag{25'}$$

On substitution in (24) it appears, that solutions of the form (25) resp. (25′) can only exist when

$$-\frac{\mu_o \mu_1}{\lambda_o \lambda_1} = -\frac{4\omega_z^2}{\nu^2} + 1 = m^2, \qquad \frac{\mu_1}{\lambda_o} = -\frac{\mu_o}{\lambda_1} = m, \tag{26}$$

unless $A = B = 0$ resp. $A' = B' = 0$. For B and B' we find[4]:

$$B = A\nu \sqrt{1 - \frac{4\omega_z^2}{\nu^2}}, \qquad B' = A'\nu \sqrt{1 - \frac{4\omega_z^2}{\nu^2}}. \tag{27}$$

Consequently, *real solutions of the type* (25) *resp.* (25′) *can only be found for* $|\nu| > 2\omega_z$. The value of ν is determined by the "frequency-equation", following from the dynamical boundary-surface condition, to be discussed below.

Since the slope of the boundary-surface $\mu_o x + \lambda_o z = c_1$ (*equilibrium* position) is given by the formula of Margules, we have:

$$-\frac{\mu_o}{\lambda_o} = \frac{2\omega_z}{g} \frac{\varrho V - \varrho' V'}{\varrho - \varrho'} = \frac{f}{g} = tg\alpha. \tag{28}$$

Denoting by tg β the slope of the equidistant straight streamlines $\mu_1 x + \lambda_1 z = k\pi$ (k being an integer), we find

$$tg\alpha\, tg\beta = \frac{4\omega_z^2}{\nu^2} - 1 = -m^2, \tag{29}$$

according to (26) and (28). It follows, that, if $\mu_o x + \lambda_o z = 0$ lies in the 1st and the 3rd quadrant, then $\mu_1 x + \lambda_1 z = 0$ in the 2nd and the 4th quadrant, and *vice versa;* further,

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[1] In the place of $\partial u/\partial t$ and $\partial v/\partial t$ we have $\partial u/\partial t - 2\omega_z v$ and $\partial v/\partial t + 2\omega_z u$ respectively.

[2] It appears, that the perturbation motions have a component in the y-direction also, which, however, does not concern us.

[3] More generally we might write down a solution
$\psi = \{ F \exp(\mu_o x + \lambda_o z) \sin(\mu_1 x + \lambda_1 z + \delta) + G \exp(-\mu_o x - \lambda_o z) \sin(\mu_1 x + \lambda_1 z + \varepsilon) \} \sin(\nu t + \sigma)$ and for ψ' a similar form with μ'_o, λ'_o, μ_1', λ_1', F', G', δ', ε', ν', σ'; corresponding solutions for p and p' can easily be formed on the basis of (24) and (24′). The kinematical and the dynamical boundary-surface condition, however, can, for non-trivial solutions, only be satisfied, when $\mu'_o/\mu_o = \lambda'_o/\lambda_o = \mu_1'/\mu_1 = \lambda_1'/\lambda_1 = \pm 1$, $\delta = \varepsilon + k\pi = \delta' + k'\pi = \varepsilon' + k''\pi$ (k, k' and k'' being integer numbers), $\nu' = \nu$, $\sigma' = \sigma + l\pi$ (l being an integer). Thus we can easily arrive at (25) and (25′) (if necessary, we may still perform a translation of the origin of coordinates).

[4] For m we take the positive root; this does not imply a real limitation as to the possible solutions.

that the equidistant straight streamlines $\mu_1 x + \lambda_1 z = k\pi$ are only then perpendicular to the boundary-surface and to the straight streamlines $\mu_o x + \lambda_o z = c$, $\mu_o x + \lambda_o z = c'$, when $\omega_z = 0$ (in that case we should obtain once more the formulas of the "rotationless" case).

If $\omega_z \neq 0$, we may advantageously use a system of non-rectangular coordinates, defined by

$$\mu_o x + \lambda_o z = \varkappa_o z^*, \qquad \varkappa_o = \sqrt{\mu_o^2 + \lambda_o^2},$$

$$\mu_1 x + \lambda_1 z = \varkappa_1 x^*, \qquad \varkappa_1 = \sqrt{\mu_1^2 + \lambda_1^2}.$$

Now we can write (25), (25′) as follows:

$$\begin{aligned} \psi &= A \sinh \varkappa_o (z^* - z_o^*) \sin \varkappa_1 x^* \sin (\nu t + \sigma), \\ Sp &= B \cosh \varkappa_o (z^* - z_o^*) \cos \varkappa_1 x^* \cos (\nu t + \sigma); \\ \psi' &= A' \sinh \varkappa_o (z^* - z'_o{}^*) \sin \varkappa_1 x^* \sin (\nu t + \sigma), \\ S'p' &= B' \cosh \varkappa_o (z^* - z'_o{}^*) \cos \varkappa_1 x^* \cos (\nu t + \sigma). \end{aligned} \tag{25*}$$

The velocity components u and w are derived from ψ by differentiating with respect to z and x respectively. The result may be written in the form:

$$\begin{aligned} \varkappa_1 u^* &= \mu_1 u + \lambda_1 w = C \cosh \varkappa_o (z^* - z_o^*) \sin \varkappa_1 x^* \sin (\nu t + \sigma), \\ \varkappa_o w^* &= \mu_o u + \lambda_o w = - C \sinh \varkappa_o (z^* - z_o^*) \cos \varkappa_1 x^* \sin (\nu t + \sigma), \end{aligned} \tag{30}$$

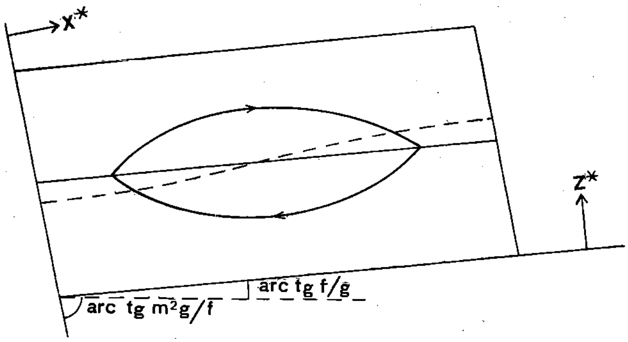


Figure 5

where $C = (\mu_1 \lambda_0 - \mu_0 \lambda_1) A$. Corresponding relations (30′), with $C' = (\mu_1 \lambda_0 - \mu_0 \lambda_1) A'$, may be written down for u'^* and w'^*. Formulas (30), (30′) give us the velocity components in the x^*- and in the z^*-direction.

If $z^* = z_1^*$ be the position of the (unperturbed) boundary-surface, the kinematical condition yields:

$$A \sinh \varkappa_o (z_1^* - z_o^*) = A' \sinh \varkappa_o (z_1^* - z'_o{}^*) \tag{31}$$

This is equivalent to: $w^* (z_1^*) = w'^* (z_1^*)$.

The dynamical condition may in this case be written:

$$\frac{\partial p}{\partial t} - \frac{\partial p'}{\partial t} + (\varrho - \varrho') (fu - gw) = 0, \qquad z^* = z_1^*,$$

or:

$$\frac{\partial p}{\partial t} - \frac{\partial p'}{\partial t} - (\varrho - \varrho') \sqrt{f^2 + g^2} . w^* = 0, \qquad z^* = z_1^*.$$

Working out this relation with the help of (26), (27), (28), (29) and (31), we obtain:

$$\nu^2 m \left[- \varrho \coth \varkappa_o (z_1^* - z_o^*) + \varrho' \coth \varkappa_o (z_1^* - z_o'^*) \right] + (\varrho - \varrho') \frac{\varkappa_o}{\sqrt{f^2 + g^2}} (m^{-1} f^2 + m g^2) = 0,$$

or:

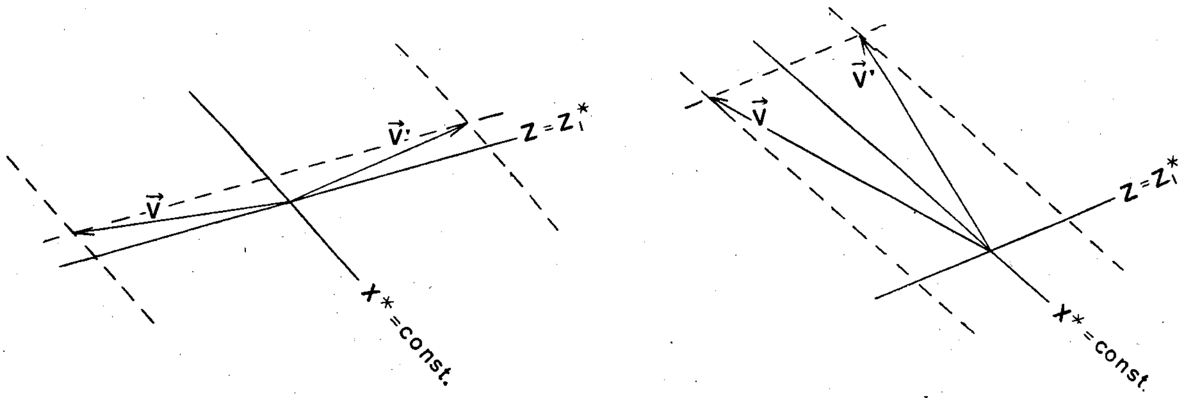
$$\frac{\varkappa_o (\varrho - \varrho')}{\varrho \coth \varkappa_o (z_1^* - z_o^*) - \varrho' \coth \varkappa_o (z_1^* - z_o'^*)} = \frac{4 \omega_z^2 \sqrt{f^2 + g^2}}{(m^{-1} - m)(m^{-1} f^2 + m g^2)}, \tag{32}$$

whilst, according to (26) and (28),

$$\varkappa_o = \varkappa_1 \sqrt{\frac{f^2 + g^2}{m^{-2} f^2 + m^2 g^2}}. \tag{33}$$

If $z_1^* - z_o^*$ and $z_1^* - z'_o{}^*$ be fixed (*for instance* by the presence of two rigid walls parallel to the equilibrium position of the surface of discontinuity, cf. fig. 5), the "frequency equation" (32) gives a relation between $\varkappa_o$ and ν or, via (33), between $\varkappa_1$ and ν. When, for instance, $\varkappa_1$ is fixed by the half "wave-length" (in the x^*-direction) $\pi/\varkappa_1$, the frequency ν can be solved; it should, as to its absolute value, be larger than $2\omega_z$; we shall return to this subject later on.

If we have to deal with an *internal wave*, that is, if $z_o^* < z_1^* < z'_o{}^*$, it follows from (31), that A and A', and consequently from (30), that $u^*(z_1^*)$ and $u'^*(z_1^*)$ have opposite signs. Provided $w^*(z_1^*) = w'^*(z_1^*)$ be small enough as compared with $u^*(z_1^*)$ and $u'^*(z_1^*)$, the same will hold for the vertical velocity components; see fig. 6*a*.



a *b*

Figure 6

This will certainly be true for points of the boundary-surface not too far from the middle between the straight streamlines $\varkappa_1 x^* = 0$, $\varkappa_1 x^* = \pi$ (see fig. 5). On the contrary, in points near these lines, the vertical components are directed in the same sense, see fig. 6*b*. *Everywhere*, however, the streamlines undergo a refraction at the surface of discontinuity, in the *same* sense as their curvature in the two media (away from the nearest straight streamline), compare fig. 5. We shall call such a refraction an "*inward*" one. Hence, if the motions be such to cause the boundary-surface to tilt downwards, as indicated by the arrows in fig. 5, then in a point of the boundary-surface *either* (I) the vertical component of motion of the heavier fluid will be negative, this fluid will "slide down", whereas that of the lighter one is positive, this fluid will "slide up", *or* (II) both will "slide down", but the lighter fluid less than the heavier one (on the left-hand side of fig. 5), *or* (III) both will "slide up", but the heavier fluid less than the lighter one (on the right-hand side of fig. 5). See fig. 7. If, however, the motions are such as to cause the boundary-surface to "erect" itself, in all three of these cases the opposite is true.

Corresponding conclusions may be formulated with regard to the accelerations; we may refer to the remark at the end of the foregoing section.

The elevation ζ^* of a point of the boundary-surface is defined here by its displacement from the equilibrium position of this surface, measured in the normal direction, that is, in the z^*-direction. From considerations analogous to those held for the "rotationless" case it follows, that in a point of the boundary-surface the acceleration component du'^*/dt of the *lighter* fluid has the same sign as the derivative of ζ^* with respect to x^*, with constant z^*, and consequently, also as the tangential derivative of ζ^*; by tangential we understand: in the direction of the line $z^* = z_1^*$, towards the right being taken as positive: this derivative is positive if the angle of inclination $\vartheta > \alpha$, α being the equilibrium angle. The acceleration component of the heavier fluid du^*/dt has the opposite sign. Now, for points not too close to a streamline $\varkappa_1 x^* = k\pi$, a positive acceleration in the x^*-direction also means a positive vertical acceleration component, if we have to deal with an equilibrium position as shown in fig. 5 (tg α positive); we call this an "upsliding" acceleration or a tendency to slide up. Likewise, a negative acceleration in the x^*-direction (again if tg α is positive) means a tendency to slide down for such points.

In points near one of the lines $\varkappa_1 x^* = k\pi$ a positive du^*/dt or du'^*/dt may be attended by a tendency to slide down, but implies at any rate a smaller one than the negative du'^*/dt resp. du^*/dt in the other medium does, whilst a negative du^*/dt or du'^*/dt implies at any rate a smaller tendency to slide up than the positive du'^*/dt resp. du^*/dt in the other medium does. *Thus:* if $\vartheta > \alpha$, the heavier fluid exhibits along the boundary-surface a tendency to slide down, c.q. a smaller tendency to slide up than the lighter fluid, whereas the latter exhibits a tendency to slide up c.q. a smaller tendency to slide down than the heavier fluid. If $\vartheta < \alpha$, the opposite is true.

All this applies to *internal* simple waves. For *external* waves, i. e. waves having $z'_o{}^* < z_1{}^*$, the above conclusions do not hold generally and certainly not in cases, where $z_o{}^* < z'_o{}^* < z_1{}^*$. The latter situation is analogous to the one we found in the "rotationless" case for external waves, when a free surface was present. In our rotating system of reference, solutions of the type (25*) with a *free surface* are only possible, when the velocities in the y-direction V and V' are the same for both fluids, and, consequently, the isobaric surfaces in both fluids run parallel to the boundary-surface [1]) (in the unperturbed state; the boundary-surface is in that case itself an isobaric surface). In a manner analogous to that used in section 2, it may then be proved that here again for an external wave $z'_o{}^* > z_o{}^*$.

We can easily see, that for $z'_o{}^* > z_o{}^*$ the refraction of the streamlines is again directed opposite to that in the case of an internal wave, viz. against their curvature in the two media, or: it is not an inward refraction but an "*outward*" one. The effect on the vertical velocity components and the vertical acceleration components on either side of the boundary-surface is therefore also opposite, see fig. 8 (case of a boundary-surface that is in the course of tilting downwards).

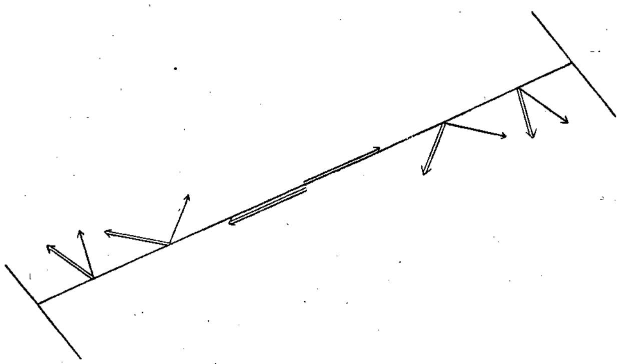


Figure 7

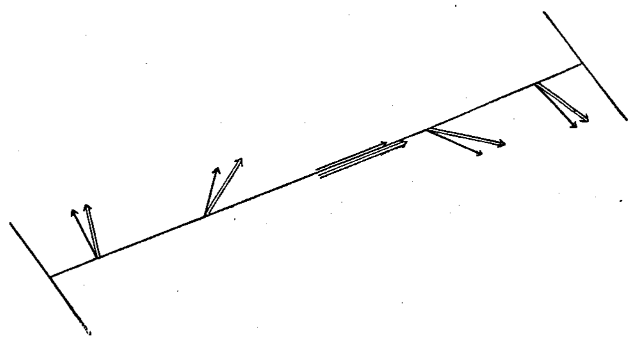


Figure 8

If, however, in the case of an external wave, $z'_o{}^* < z_o{}^*$ the matter lies in so far differently, that, the motions everywhere being directed in the same sense on either side, the x^*-components in the lighter fluid have now the larger absolute values.

From the above formulas, relation (1) might easily be derived as an approximation, if the slope of the boundary-surface is small, in the same manner as (19) was derived.

As to the absolute values of the accelerations we confine ourselves to stating that in the case of an external wave these accelerations in the middle of figures 7 and 8 are again by a factor of the order $\frac{\varrho + \varrho'}{\varrho - \varrho'}$ larger than in the case of an internal wave, when the deviation of the discontinuity surface from its equilibrium position is the same in both cases (and its slope is small). For the rest we may refer to the general formulas of section 1.

For complex waves the situation is again much more complicated. Statements with

[1]) If $V \neq V'$, the free surface $P = 0$ resp. $P' = 0$ must cut the boundary-surface, whilst it has a different slope in the two media. This appears to be incompatible with the dynamical boundary conditions (22) and (35) for the free surface, applied to (25*).

respect to the motions on either side of a discontinuity surface cannot generally be made for such a case unless all component simple waves are of the same type and then only with restrictions analogous to those formulated in section 2 for the "rotationless" case.

In the atmosphere, the air near frontal surfaces at some distance from the earth's surface may sometimes be subjected to motions of the type (25*), for instance in the form of an internal wave (fig. 5) in a planparallel layer extending on either side along the frontal surface.

As an example we may consider the special case, that $z'_o{}^* - z_1{}^* = z_1{}^* - z_o{}^* = d$. A simple discussion of the frequency equation (32) shows ν to satisfy the condition $\nu > 2\omega_z$ (making m real) if $d \geqq d_o$, where d_o, provided the slope $f/g = h$ of the frontal surface be small enough ($h^2 << 1$), may with sufficient accuracy be written

$$d_o = \frac{4\omega_z{}^2 L^2 h^2}{\pi^2 g} \frac{\varrho + \varrho'}{\varrho - \varrho'};$$

here L is half the wavelength, measured along the frontal surface, see fig. 9. Since the slope of the frontal surface is small, Lh may be put equal to the difference in height of the frontal surface over the half wavelength, denoted by b in fig. 9. Putting $2\omega_z = 10^{-4}\ \text{sec}^{-1}$ (which is the case for some middle latitude) we find with sufficient accuracy:

$$d_o = 10^{-10} b^2 \frac{\varrho + \varrho'}{\varrho - \varrho'},$$

if d_o and b are expressed in meters. We see that even for an unlikely large value of b, e.g. $b = 10^4$ m, and a large temperature jump, e.g. $(\varrho + \varrho')/(\varrho - \varrho') = (T' + T)/(T' - T) = 500°/5° = 100$, such a small value d_o is found (our example gives $d_o = 1$ m), that we may safely confine ourselves to the case $d \geqq d_o$, and consequently $\nu > 2\omega_z$.

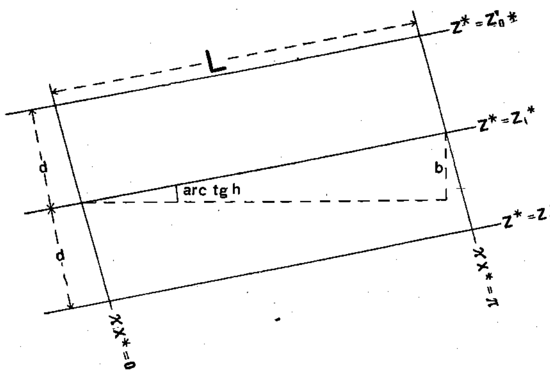


Figure 9

As an example we take a state of things in which $(\varrho + \varrho')/(\varrho - \varrho') = 200$ and a slope $h = 1/200$, further $d = 2.10^3$ m and $L = \pi.10^5$ m, whilst $2\omega_z = 10^{-4}\ \text{sec}^{-1}$ again. Then from the frequency equation we obtain: $m^2 = \frac{1}{2}$, or $\nu = 2\omega_z\sqrt{2} = 1{,}41.\ 10^{-4}\ \text{sec}^{-1}$.

The angle β between the straight streamlines $\varkappa_1 x^* = k\pi$ and the x-axis is now given by (29): tg $\beta = -100$, or $\beta = 90{,}6°$; so that these lines are, practically speaking, vertical.

One might be tempted to try that solution, where $\beta = 180°$, since then the boundary condition at the earth's surface might be satisfied by making it coincide with the plane $x^* = 0$. We see easily, however, that in that case $m = 0$ and $\nu = 2\omega_z$, and that all particles would describe horizontal orbits, viz. circles of inertia — an uninteresting case.

The solutions developed in this section — although at some distance from the earth's surface such a solution may sometimes be realized to a certain extent as a more or less isolated "wave" — are not appropriate as a basic form for perturbational motions along atmospheric frontal surfaces, as the boundary condition determined by the earth's surface cannot be fulfilled by the general type (25*) of these solutions.

Nevertheless, the rather ample discussion of these solutions is instructive, because it gives still a qualitative picture of the different types of motion near atmospheric frontal surfaces, when not in equilibrium. In passing from standing waves at boundary-surfaces in a non-rotating system of reference to analogous motions in a rotating system we found the essential qualitative features to be saved; likewise, the conclusions drawn in this section concerning sliding up and sliding down on either side of discontinuity surfaces might be transferred as regards their essential features to flow-patterns adjusted to atmospheric boundary conditions. In the following section we shall enter upon this somewhat more precisely.

When, in the preceding pages, we have described motions of a frontal surface as oscillations, this must be understood as a working hypothesis only. In reality, surface friction and eddy viscosity will cause such an oscillation to be damped to such a measure that it often does not survive one period. Added to this, however, a quarter of a period (the time taken to reach the equilibrium position) is, in general, small enough compared with the intervals of time in which substantial changes in the atmospheric conditions arise (in the example computed above, the period is about 12 hours, a quarter of a period, therefore, only 3 hours) to cause the situations to differ never so strongly from equilibrium situations as to give rise to considerable oscillations.

4. Wedge-shaped layers. In order to satisfy the true boundary condition we shall have to study the motions of fluids in and near wedge-shaped layers. Here also we may start from a much simplified model; as such we take the case of *one* fluid in a non-rotating system of reference, having a free surface (the latter being horizontal when in rest) and being bounded on the lower side by a sloping bottom.

On account of the mathematical difficulty of this problem, one has resorted to an approximation, where in the first instance vertical accelerations were neglected in the equations of motion, considering, that in hydrographical and meteorological situations vertical motions are generally an order of magnitude smaller than the horizontal motions. Following Bjerknes c.s., 1933, we shall call this the quasistatic approximation.

The perturbation equations (8) then reduce to

$$\frac{\partial^2 \psi}{\partial z \partial t} + S\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial z} = 0. \tag{34}$$

From (34) follows the well-known fact that *in this approximation, for small perturbations, p as well as $u = \partial\psi/\partial z$ turn out to be independent of z.*

If the sloping bottom be represented by $z + x \operatorname{tg} \alpha = 0$, the fundamental solution, being harmonic with respect to t and satisfying the boundary condition at this bottom (viz. that $z + x \operatorname{tg} \alpha = 0$ be a straight streamline), as well as the dynamic condition

$$\frac{\partial p}{\partial t} - g\, w\varrho = 0 \tag{35}$$

at the free surface $z = 0$, is as follows [1]):

$$\psi = A\,(z + x\, tg\alpha)\frac{J_1(2\sqrt{\gamma x})}{\sqrt{\gamma x}}\, sin\,(\nu t + \sigma), \tag{36}$$

$$Sp = B\, J_o\,(2\sqrt{\gamma x})\, cos\,(\nu t + \sigma), \tag{37}$$

where

$$\gamma = \frac{\nu^2}{g\, tg\, \alpha};$$

J_o and J_1 are Bessel's functions [2]) in the usual notation; A and B are constants. Substition in (34) yields:

$$B = \frac{\nu A}{\gamma} = \frac{A\, g\, tg\, \alpha}{\nu}.$$

---

[1]) If we denote bij ψ_0 the value of ψ at the bottom, being independent of x and z, we easily find by integrating (34): $\frac{\partial \psi}{\partial t} + z\,\frac{\partial\,(Sp)}{\partial x} = \frac{\partial \psi_0}{\partial t} - tg\alpha\,\frac{\partial (Sp)}{\partial x}$. Applying this to the free surface $z = 0$, then differentiating with respect to z and applying (35) we obtain a differential equation for Sp, which is satisfied by (37), etc.

[2]) The following developments may be written down here:

$$\frac{J_1\,(2\sqrt{\gamma x})}{\sqrt{\gamma x}} = 1 - \frac{\gamma x}{1.2} + \frac{(\gamma x)^2}{1.2.2.4} - \ldots\ldots$$

$$J_0\,(2\sqrt{\gamma x}) = 1 - \frac{\gamma x}{1^2} + \frac{(\gamma x)^2}{1^2.2^2} - \ldots\ldots$$

It is seen that the straight line $z + x$ tg $\alpha = 0$ is really a streamline ($\psi = 0$), whilst the zeroes of J_1 also determine a set of straight streamlines: $x = x_1, x_2, \ldots\ldots,$ provided ν and thereby γ be given; the latter streamlines are vertical. If a rigid wall be given at $x = a$, the equation

$$J_1 (2 \sqrt{\gamma a}) = 0$$

determines reciprocally a set of values of γ and consequently of the frequency ν.

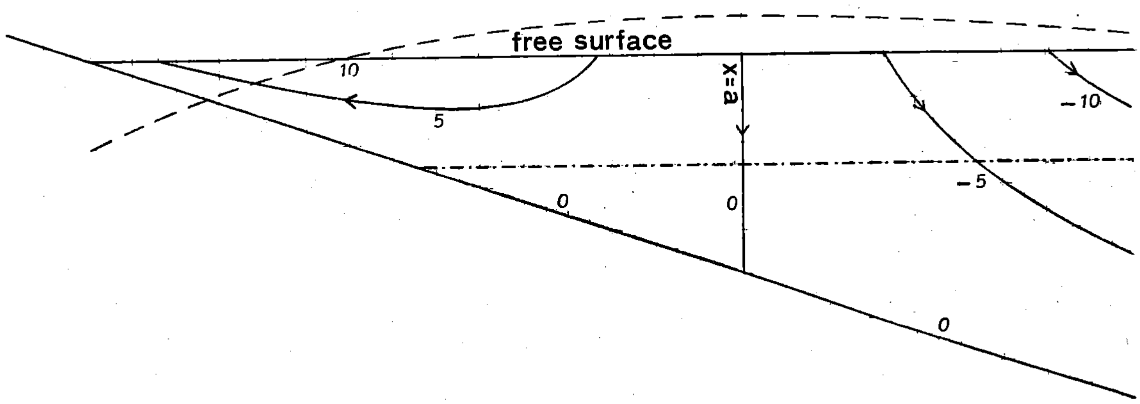


Figure 10

Fig. 10 gives an impression of the streamline pattern. If a rigid wall is present at $x = a$, we are only concerned with the rectangular triangle to the left of this line; the fundamental mode of oscillation of the system in that case is represented in the figure.

The elevation ζ has been drawn in fig. 10 on a rather large scale (too large for a small perturbation) as a dashed line. It is connected by a simple relation to the perturbation pressure p given by (37), the latter according to the quasistatic supposition being equal to $\varrho g \zeta$, so that

$$\zeta = Sp/g = \nu^{-1} A \, tg \, \alpha \, J_o (2 \sqrt{\gamma x}) \cos (\nu t + \sigma).$$

The vertical streamlines intersect the free surface at the extreme values of ζ.

When we now try to deal with the case of two fluids with a discontinuity surface, the deficiencies of the quasistatic approximation make themselves felt. For we can generally prove, that for the case of two homogeneous incompressible fluids separated by a discontinuity surface and bounded by a sloping plane bottom, the quasistatic approximation can only give solutions, in which the two stream functions ψ and ψ' of the two media are at any moment identical, but for an additive constant:

$$\psi (x, z, t) \equiv \psi' (x, z, t) + C (t). \qquad (38)$$

This follows directly from the boundary conditions. Indeed, let the discontinuity surface be denoted by $z = z_1 = 0$. Then along the two straight lines $z + x$ tg $\alpha = 0$ and $z = 0$ the function $\varphi (x, z, t) \equiv \psi (x, z, t) - \psi' (x, z, t)$ must have a value $\varphi_o(t)$ independent of x and z (because of the continuity of φ in the point $x = 0$, $z = 0$). Now let q stand for $Sp - S'p'$; then, on account of (34) and the corresponding equation (34') for the other medium, the following differential equations for φ and q are valid:

$$\frac{\partial^2 \varphi}{\partial z \partial t} + \frac{\partial q}{\partial x} = 0, \qquad \frac{\partial q}{\partial z} = 0.$$

Integrating the first equation we find, on account of the second one:

$$\frac{\partial \varphi}{\partial t} + z \frac{\partial q}{\partial x} = f (x, t),$$

where $f(x, t)$ is an unknown function independent of z. By applying this to the line $z = 0$, we obtain

$$\frac{\partial \varphi}{\partial t} + z \frac{\partial q}{\partial x} = \frac{\partial \varphi_o}{\partial t}. \tag{39}$$

By substitution of the coordinates of a point of the line $z + x \operatorname{tg} \alpha = 0$, (39) yields:

$$\frac{\partial \varphi_o}{\partial t} - x \, tg \, \alpha \frac{\partial q}{\partial x} = \frac{\partial \varphi_o}{\partial t},$$

or:

$$\frac{\partial q}{\partial x} = 0.$$

Substituted in (39), this yields:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi_o}{\partial t}$$

or:

$$\varphi = \varphi_o + \chi(x, z).$$

The presence of a time-independent term $\chi(x, z)$ in φ would, however, imply at least one of the stream functions ψ, ψ', and consequently at least one of the perturbation velocity fields $v(x, z, t)$, $v'(x, z, t)$ to have a time-independent part also, but according to the statements at the end of section 1 such a part is not permitted in small perturbations. We have, therefore:

$$\varphi \equiv \varphi_o \equiv C(t),$$

which is identical with (38).

This implies, that at the boundary-surface $z = z_1$ the streamlines are not broken: not only the normal component, but also the tangential component of the velocity is continuous there: $u(x, z_1) = u'(x, z_1)$.

For a system of two fluids, therefore, over a sloping, plane bottom, when the upper one has a free surface, the field of motion would for *both* fluids be given by (36); the motions at the boundary-surface $z = -d$ (—·—·— in fig. 10) would be obtained by substituting $-d$ for z in (36).

Apart from that, (38) is generally valid, independent of further special conditions (free surface or not, etc.) of the system considered, provided only the plane bottom be really "sloping", i. e. neither horizontal nor vertical: in those two cases our proof does not hold; indeed, the internal waves of section 2 can be approximated quasi-statically.

As to *external* waves the general result (38) may still be a certain approximation of reality, as in the foregoing sections we have seen that in the case of such waves (fig. 4*b*) the refraction of the streamlines is inconsiderable, if the relative difference of density is small. With respect to *internal* waves, however, we meet with insuperable difficulties in this way.

We may characterize an internal wave by the occurrence of closed streamlines in its flow pattern. These closed streamlines imply, however, that there are values of x, where for the same x but different z the horizontal velocity components are directed oppositely. Since, however, in the quasistatic approximation u is idependent of z — owing to (38) there is now no need to distinguish u and u' — this method is obviously incapable of giving such solutions.

In the exact solutions the tangential velocity component will *not* be continuous at the surface of discontinuity. In order to analyze quantitatively the deviation of (38) from reality, we start from the complete perturbation equations (8) and the corresponding equations (8′) relating to the lighter fluid. Subtracting (8′) from (8), integrating with respect to z, from z_1 to Z, and applying the result for $Z = z_1$, we obtain instead of (39):

$$\left(\frac{\partial \varphi}{\partial t}\right)_{z=Z} + S\int_{z_1}^{Z} \frac{\partial p}{\partial x} dz - S'\int_{z_1}^{Z} \frac{\partial p'}{\partial x} dz = \frac{\partial \varphi_o}{\partial t},$$

where, as before, $\varphi = \psi - \psi'$ and φ_o is the value of this difference along the bottom and along the discontinuity surface. Substituting herein the coordinates x and $Z = -x \operatorname{tg} \alpha$ of a point at the bottom, we find:

$$S \int_{z_1}^{-x\, tg\alpha} \frac{\partial p}{\partial x} dz - S' \int_{z_1}^{-x\, tg\alpha} \frac{\partial p'}{\partial x} dz = 0,$$

or, according to (8) and (8′):

$$\int_{z_1}^{-x\, tg\alpha} \frac{\partial u}{\partial t} dz = \int_{z_1}^{-x\, tg\alpha} \frac{\partial u'}{\partial t} dz. \qquad (40)$$

If now, as implied by the quasistatic approximation, the horizontal velocity component were independent of z, it would follow from (40), that the two horizontal acceleration [1]) components are equal, and the same would be true for the two horizontal velocity components, since an additive stationary velocity field as part of the solution of the perturbation equations is not permitted. In reality, as we see from (40), the *mean values* of the horizontal acceleration components across the lower medium, taken along a vertical line from the discontinuity surface to the bottom, are equal for the two velocity fields (viz. that of the lower fluid and that of the upper one, extrapolated into the region of the lower one — compare the dashed lines in fig. 4).

For that reason the two horizontal acceleration components are, in general, not equal in a point of the plane $z = z_1$, neither are the horizontal velocities. The difference will, however, be small if the integration interval from z_1 to $-x \operatorname{tg} \alpha$ is small enough; this will be the case *near the point* of the wedge. *There the refraction of the streamlines will be small, in all cases* (even for an internal wave).

Now we pass on to the problem with which we are concerned, viz. that of two fluids, separated by a discontinuity surface and streaming with velocities V, V' in the y-direction of a *rotating system of reference*. Here the quasistatic approximation starts from the simplified equations (cf. (24)):

$$\left(\frac{\partial^2}{\partial t^2} + 4\omega_z^2\right)\frac{\partial \psi}{\partial z} + S\frac{\partial^2 p}{\partial x \partial t} = 0, \qquad (41a)$$

$$\frac{\partial p}{\partial z} = 0. \qquad (41b)$$

From (41) it follows, that here again, for small perturbations, p and u are independent of z. Assuming a horizontal bottom at $z = 0$ (the surface of the earth) and a discontinuity surface, the equilibrium position of which is represented by $z = x \operatorname{tg} \alpha$ (according to M a r g u l e s' formula), we can again, in the same manner as above, prove that in order to satisfy the kinematical boundary-surface conditions, the two solutions ψ and ψ', obtained by this approximation, must be identical, so that in any point of the boundary-surface the two velocity vectors would be equal.

---

[1]) We have already seen, that in treating small perturbations, the accelerations may be equalized to the partial derivatives of velocity with resect to time.

For an external wave this may be an approximation, but not for an internal one. Here, also, it is impossible to obtain solutions representing internal waves by the quasistatic approximation.

From the exact equations (24) it follows, in a manner analogous to that used in deriving (40), that:

$$S \int_0^{x\,tg\alpha} \frac{\partial^2 p}{\partial x \partial t} dz - S' \int_0^{x\,tg\alpha} \frac{\partial^2 p'}{\partial x \partial t} = 0,$$

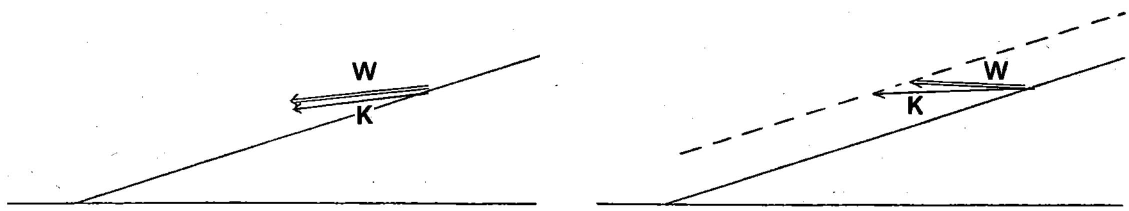
or:

$$\int_0^{x\,tg\alpha} \left(\frac{\partial^2}{\partial t^2} + 4\omega_z^2\right) u dz = \int_0^{x\,tg\alpha} \left(\frac{\partial^2}{\partial t^2} + 4\omega_z^2\right) u' dz. \tag{42}$$

Now for a basic solution of (41), i. e. for a simple wave, $\frac{\partial}{\partial t} = i\nu$, so that for such solutions (42) is equivalent to stating the equality of the *mean values* of u and u' across the lower fluid (along a vertical line) [1]). If we have to deal with a general superposition of such solutions, each of its component simple waves will satisfy (42); for that reason each of these simple waves has the same mean values of u_n and u'_n across the lower fluid and consequently the same is true for the superposition.

If the vertical distance x tg α, over which this mean value has to be taken, is small enough, the values of u and u' at the discontinuity surface will not differ much either; this will, therefore, be the case *near the point of the wedge*. For points, situated elsewhere, however, this conclusion does not generally hold.

Considering the situation at the surface of discontinuity somewhat more closely, we see that according to the quasistatic approximation the two velocity vectors in a point of this surface would be exactly equal, *either* both ascending *or* both descending; the same is true for the accelerations. When, for example, a discontinuity surface is in the course of assuming a smaller inclination, the heavier fluid will in general have a descending motion; this will surely be true if, as well as by the discontinuity surface and the horizontal bottom, it is bounded by a vertical rigid wall, see fig. 12. In atmospheric conditions the surface of symmetry of a symmetric thermic anticyclone in the cold air may serve as such a rigid wall; we might for instance have to deal with a "drop" or tongue of cold polar air, that flows out (compare figures 16 and 17). According to the quasistatic method the lighter fluid along the boundary-surface would then also be in a descending motion, see fig. 11*a*. In reality, however, this is not necessary, even if the difference between the two horizontal components is small, as is the case close to the earth's surface. This can be seen in fig. 11*b*. Here also the heavier fluid is descending; the vertical component is but small, however, owing to the proximity of the earth's surface. When the lighter fluid has a smaller tangential component to the left than the heavier fluid, their normal components being the same, it can have an upward component of its motion, in contrast to the heavier fluid.



a Figure 11 *b*

W denotes the motion of the lighter fluid, *K* that of the heavier one.

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1) Except for the case $\nu^2 = 4\omega_z$; this would be the case, already mentioned in the foregoing section, where all particles describe circles of inertia.

We assumed here a situation, with respect to the tangential components, which, according to the conclusions obtained in sections 2 and 3 for simpler models, is typical of an internal wave. Proceeding in this direction, we obtain for such a wave, in our case (rotating earth) a streamline pattern as represented by fig. 12. This figure and the fillowing ones are qualitative sketches; in planning them the analogous figures in the foregoing sections have been used in combination with fig. 10. As a general basic rule for the streamline patterns, we have the statements made at the end of section 1.

As to fig. 12, we have *calculated* [1]) the stream function fields in the two media by a numerical approximation method after having *assumed* the value of ψ along the line of discontinuity, making these values vary along this line in a manner analogous to the variation of ψ with x in fig. 10. Along the rectangular frame we have a constant value of ψ, which we may put equal to *zero*. The numerical calculation process follows a well-known algorithm of successive approximation, which is based on the relation

$$\frac{\triangle^2 \psi}{(\triangle x)^2} + \frac{\triangle^2 \psi}{(\triangle y)^2} = 0,$$

or

$$2\left[1 + \left(\frac{\triangle y}{\triangle x}\right)^2\right] \psi(x, y) = \left(\frac{\triangle y}{\triangle x}\right)^2 [\psi(x + \triangle x, y) + \psi(x - \triangle x, y)] + \psi(x, y + \triangle y) + \psi(x, y - \triangle y),$$

which is obtained from the relation $\nabla^2 \psi = 0$ by taking finite instead of infinitely small differences and which has to be applied to a lattice of points within the region, where the latter relation holds.

From such calculations it appears, that only few streamlines enter into the region to the left of the point A in fig. 12, even if the left vertical wall be placed at a much larger or at an infinite distance.

The closed streamlines passing the boundary-surface are characteristic of the situation in fig. 12. We see that in a certain point of the discontinuity surface the two velocities are purely tangential and directed oppositely. As to the direction of the arrows, fig. 12 applies to the case we have constantly taken for our example, viz. that of a discontinuity surface in the course of diminishing its inclination. In the opposite case the arrows in fig. 11*b* and 12 should be reversed.

Finally, these figures may also apply to accelerations instead of velocities; in that case the direction of the arrows does not depend upon the momentary state of motion of the boundary-surface but upon the momentary position of this surface with reference to its equilibrium position; fig. 11*b* and 12 e.g. would then (viz. if the arrows would design accelerations) be applicable to a boundary-surface being too steep, no matter whether it is in the course of tilting downwards or is erecting itself still more.

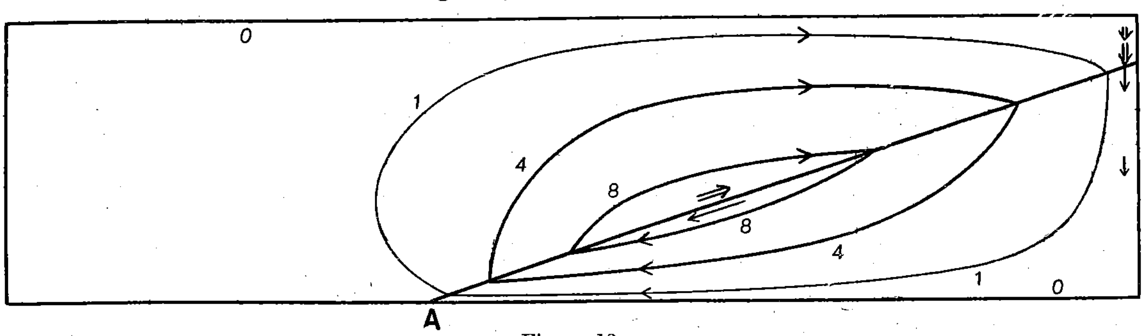

Figure 12

In the foregoing exposition we have considered an *internal* wave motion (closed streamlines exist, passing through the two fluids). An *external* wave will show quite a different picture. (We confine ourselves here to waves showing the features of *simple waves*).

[1]) From such calculations we find momentary flow patterns, which are compatible with the fundamental equations, but which in general do not represent fundamental solutions of these equations; or, in other words: we do not know the development in time of these situations, for they do not, in general, belong to simple harmonic solutions with respect to time.

Here the streamlines coming from the lower medium and cutting the discontinuity surface do not go round through the upper medium back to the lower one; for that reason oppositely directed motions are nowhere found on the boundary-surface. Whereas

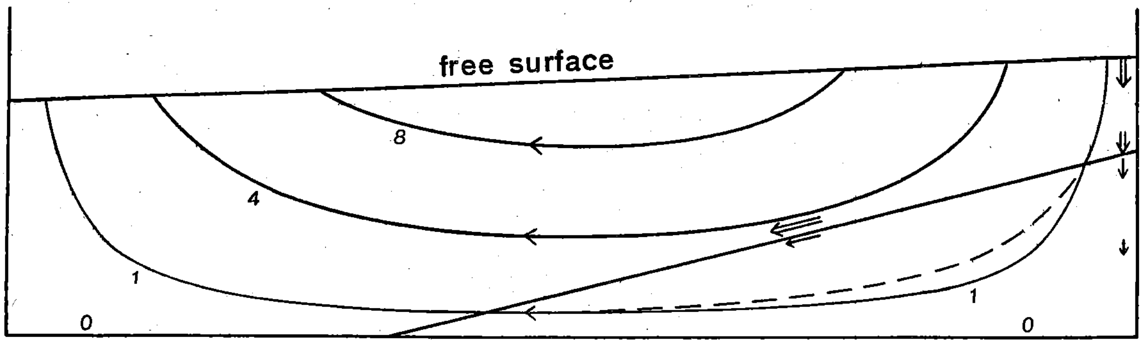

Figure 13

for an internal wave the refraction of the streamlines is "inward", here, according to the conclusions of sections 2 and 3 it may be "outward", see fig. 13; this refraction is small, however, provided the difference between the densities of the fluids be relatively small. In this case, the rules formulated for the internal waves concerning the motions on either side of the boundary-surface do not hold and the whole presents an opposite behaviour, which for the rest, is sufficiently illustrated by fig. 13; it should be borne in mind here, that all the arrows in this figure may again be reversed and that they may represent accelerations as well as velocities.

Figures 14 and 15 present schematically the velocity (or acceleration) distributions, corresponding to the situations of figures 12 and 13 and occurring when the discontinuity surface tilts downwards (resp. when it is too steep).

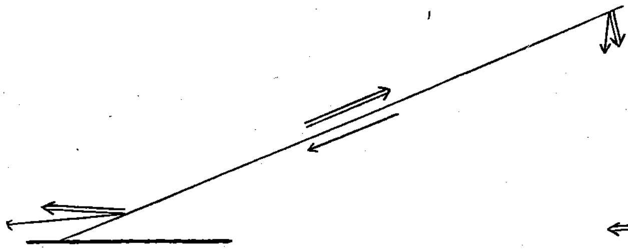

Figure 14

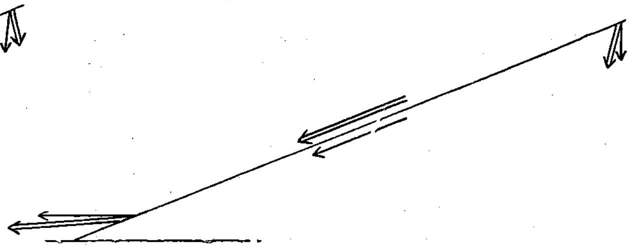

Figure 15

Now we come to the question as to when such a system of two fluids will assume an internal wave motion, and when an external wave motion.

To begin with, an internal wave is necessary if the whole system is enclosed in an invariable region, forcing the two fluids to perform, roughly speaking, oppositely directed motions.

If we have to deal with a system having a free surface, such a *determining* factor is less easy to indicate. What we can do here is to give a *characteristic* of an internal or of an external wave: If the free surface of the lighter medium undergoes deformations and displacements of a much smaller amplitude than the discontinuity surface in the interior of the system, we have to deal with an internal wave (the motion of the free surface has then even opposite phase, compare fig. 4*a*). If, however, these changes are larger than those taking place at the latter surface, we have an external wave; in that case the system of the two fluids oscillates more or less as a whole.

Instead of the free surface we may also consider some intermediate layer, between the surface of discontinuity and the free surface, and its deviation from equilibrium.

As to more quantitative statements with respect to the accelerations we may refer to sections 1, 2 and 3.

In the foregoing, the fluids were supposed to have rigid lateral boundaries. The rôle of such boundaries is to decide about the occurrence of standing waves and to determine a set of permitted wavelengths, the longest of which belongs to the fundamental mode of oscillation of the system.

If we now try to apply our considerations more or less qualitatively to the atmosphere, we are faced with the difficulty, that here such rigid lateral boundaries are absent. We have already seen that in the cold air on one side of the frontal surface a thermic high might take over the rôle of such a boundary, especially when we are concerned with a more or less symmetric ridge of cold air; compare figures 16 and 17, where in the plane of symmetry of the cold air a rigid wall may be conceived to exist. As to a possible boundary towards the other side, in the warm air, one might remark, that because of the finite circumference of the earth, the "film" of air surrounding it which we have to deal with may also be treated as finite. However, if this argument is to be valid, the wave motion should extend all around the earth, whereas we have confined ourselves to motions on a relatively limited scale. We will rather assume our wave-motion to be bounded towards the side of the warm air by some stable circulation system; in this system a dynamic *lateral* stability acts as a factor, which at a sufficient distance from the front counteracts the occurrence of horizontal perturbation motions. We might call to mind here for instance a strong anticyclone in the warm air, extending up to great heights. In the case of internal waves, as we have seen above, the wall may also be conceived at an infinite distance

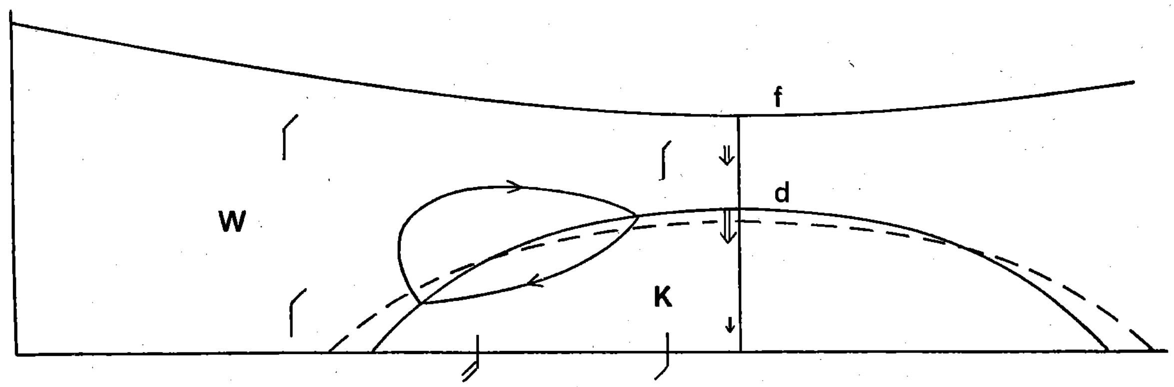

Figure 16

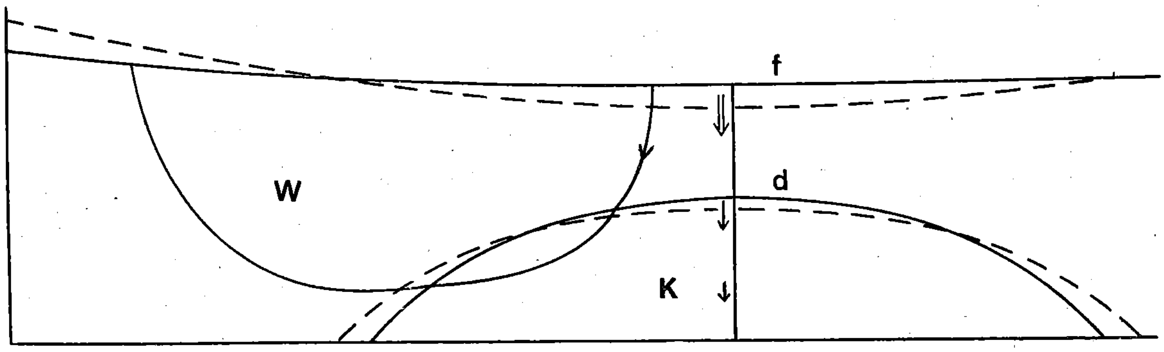

Figure 17.

Explanation to figures 16 and 17. Two homogeneous „air"masses W and K of different densities, streaming in directions perpendicular to the plane of the figures, are separated by a discontinuity surface d; on the upper side W is bounded by a free surface f. The broken lines represent the equilibrium positions of d and (in fig. 17) of f; in fig. 16 the perturbation of the free surface is unimportant. Both figures are purely qualitative and so are the wind arrows in fig. 16, which are drawn in the plane of the figure but in reality should point in directions normal to it, K streaming away and W towards the spectator; these wind arrows apply to the equilibrium state of the system.

without making many streamlines enter into the region to the left of the tongue of heavier fluid. For reasons of simplicity we have drawn a vertical streamline on the left hand side in figures 16 and 17.

Now, coming to the identification of the different types of flow patterns in the atmosphere, we may say, in accordance with the statements made above: if the higher layers of the air above the frontal surface, or for example the tropopause, which presents itself a characteristic surface (like the free surface in our fluid models), show but *small* deviations from equilibrium, as compared with that of the frontal surface, we may conclude that an internal wave motion exists, with all the consequences connected herewith. See fig. 16; the two air masses are represented here by homogeneous fluids (K and W), the upper one of which (W) has a free surface.

If, however, these upper layers (or the tropopause) show larger deviations from equilibrium and consequently also larger vertical accelerations than the frontal surface and the adjacent layers (a case, which *a priori* is possible as well as the former case), we have to deal with an external wave motion, with the consequences discussed above: the motion of the cold air and that of the warm air, taken as a whole, are of about the same direction, especially also so far as the vertical components are concerned [1]). See fig. 17.

The foregoing discussion may be called "symptomatical". A more "causal" point of view is the following. If we trace the causes of the fact, that at a certain moment a frontal system is not in equilibrium, we shall often find this cause not so much to lie in a displacment of the frontal surface as in a displacement of its equilibrium position by the change of atmospheric conditions. Now suppose the latter change to be confined more or less to the lower layers, or especially to the cold air, then there will appear no appreciable deviation of the upper layers of the warm air from equilibrium and according to the foregoing explanations we may expect an internal wave motion.

A similar reasoning might be applied to other cases, but we shall not enter further upon the matter here.

5. Complications. In the foregoing pages a rather ideal problem was considered. The motions studied were treated as free, frictionless oscillations taking place along discontinuity surfaces, which in the equilibrium position would be stationary. In the atmosphere the following complicating effects arise:

a. Translations of the system in a direction perpendicular to the front. If this translation were a mathematical one it would not exert any influence upon the motions. However, a translation caused by the pressure field gives rise in general to a difference of horizontal velocity (normal to the front) between the warm air and the cold air, *for instance* because of the density difference causing the warm air to have a larger geostrophic wind component normal to the front than the cold air (the tangential components of the pressure gradient being the same on either side). Now this effect is not very important as a rule; if non-geostrophic wind velocities occur, however, much larger differences may arise, causing the air masses to assume vertical velocity components relatively to each other.

b. In our considerations, homogeneous incompressible fluids, i. e. fluids having an indifferent stratification, were studied. Now, the stable stratification, which is mostly to be found in the atmosphere, has a "stabilizing" influence upon internal wave motions in general; that is to say: if a frontal surface is not in the equilibrium position but moves towards it, the temperature changes caused by the air motions accompanying the process cause the equilibrium position to advance, so to say, towards the actual position

[1]) In former discussions of this matter, attention has rather onesidedly been directed to fields of motion of the first type („internal wave"). An instance of frontal perturbations, where the vertical air motions on either side of the frontal surface have similar directions, is given by the cyclonic wave studied bij Bjerknes et al. 1933, page 520, fig. 78.

of the frontal surface; if, however, the frontal surface moves away from the equilibrium position, the latter will, in addition, displace itself in the opposite direction. Here we have supposed the frontal surface to be non-isentropic but to have potential temperatures increasing with height.[1]) Cf. G. Stüve, 1925. Stüve, following Exner, assumed a field of motion (near the frontal surface), which qualitatively corresponds to our internal wave. For an external wave motion, everything depends upon whether the differences between the tangential velocity components on either side of the frontal surface have the same sign as for an internal wave or not.

c. The vertical component of the Coriolis acceleration.

d. Friction at the earth's surface.

e. Shearing stresses between the two air masses at the boundary-surface. In the atmosphere a kinematical transition layer will be formed in the field of motion instead of a sharp discontinuity. V. Bjerknes c.s., l. c. page 499, state that friction will cause an upsliding motion of the warmer air and a downsliding motion of the colder air within this transition layer. It can, however, be proved [2]) that this is *not necessary*, but that a general state of things is possible, where gradient force, Coriolis acceleration and shearing stresses cancel each other so as to maintain an equilibrium velocity distribution with respect to z without the wind veering with height (apart from surface friction influences).

Even apart from these complications, however, atmospheric air motions near frontal surfaces will often have a forced and quasistationary character rather than an oscillatory one. In this connection we may refer to the remark at the end of section 3.

However, the conclusions and flow patterns we arrived at in the foregoing section are such, that they will retain their validity for the greater part even beyond the narrow and academic scheme of the perturbation equations from which we started and will therefore be applicable to a large extent to real atmospheric situations.

Concluding remarks

The investigation which was the object of this paper may be continued and extended in various directions.

First, the problem of motions occurring in and near wedge-shaped fluid layers (section 4) calls for a further exact treatment, where the quasistatic approximation has to be abandoned [3]). To begin with, the problem of one fluid layer over a sloping bottom ("rotationless" system) may be tackled. When here the exact solution will have been found, it may serve as starting point for the problems of two layers.

Secondly, it will be necessary to pursue the meteorological application of our hydrodynamical investigation. In the preceding pages we have only superficial touched upon this subject and such drawings as fig. 16 and fig. 17 do not yet tally with atmospheric conditions (even apart from the complications *a—e* mentioned above) especially as regards the upper boundaries. It will be necessary to acquaint ourselves better with the character of the three-dimensional fields of motion near frontal surfaces or frontal zones. This may be done in two, or perhaps three different ways:

first by means of a detailed and very complete analysis of aerological observations, especially as regards upper air winds;

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[1]) If the air is wet, as it often is near frontal surfaces, we should consider the „saturated-potential" temperature instead. Now, in each air mass, the latter is, in general, fairly constant along the frontal surface, especially along the upper boundary of the cold air (the lower boundary of the transition layer).

[2]) A study on this subject is soon to be published in this series (Groen, 1946).

[3]) In this connection one might for a moment consider the integration of atmospheric perturbation equations performed by Solberg, 1928, but his special solutions are not applicable to our case.

secondly one might attempt to pursue the development of concrete initial situations by means of a numerical attack upon the problem of solving the fundamental differential equations, on the basis of an approximative integration by finite steps — after the manner of Richardson's remarkable "Weather prediction by numerical process". Finally we might consider the possibility of experiments with fluid models.

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