KONINKLIJK NEDERLANDSCH METEOROLOGISCH INSTITUUT

DE BILT (NEDERLAND)

No. 125

MEDEDEELINGEN EN VERHANDELINGEN

SERIE B

DEEL I, No. 4

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ON THE CAUSES OF PRESSURE VARIATIONS AT THE GROUND

INDEX DECIMALIS: 551.543

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TE VERKRIJGEN BIJ DE: TO BE PURCHASED FROM:

RIJKSUITGEVERIJ / 'S-GRAVENHAGE

1946

PRIJS F 3,—* PRICE F 3,—*

Introduction and summary

Pressure distribution at the ground is still one of the most important data we can find on the weather-map. The changes in the pressure distribution, therefore, are also of great importance to the forecaster. So it is easy to understand that meteorologists always have tried to find rules which might aid in foretelling these pressure changes.

As vertical accelerations are generally very small compared with the acceleration of gravity and as they never lead to pressure variations of more than about 1 mb, we can consider pressure to be merely of static origin.

So changes of pressure must be a consequence of the motions and the structural variations of the air above the place in which the pressure is measured. Examples of such motions and structural variations of the air we find in the depressions of middle latitudes, for instance. These depressions for their part are connected with the general circulation and depend on energy supplied by radiation, by vertical and lateral or isentropic mixing, on orographic disturbances, etc. It is not known what this connection is, and so the exact physical cause of the pressure variations in the atmosphere is generally also unknown.

However, if we are aware of the density variations in each level above the place at the ground where we want to know the pressure variation it is possible to compute the latter. Now these density variations can be deduced from the flow pattern in each level as this is connected with the field of pressure according to the equations of motion.

Now it is not possible to solve the equations of motion exactly for a general case. We can only do this according to the method developed by Richardson (33), which will involve much work.

Fortunately it is possible to find an approximate solution for the equations of motion, which was indicated for the first time by Hesselberg (23) in 1915 and which was worked out in somewhat more detail by Philipps (31). This approximate solution enables us to compute the density variations which occur in the atmosphere and to calculate pressure variations at the ground qualitatively or even quantitatively. Some causes of pressure variations at the ground which were mentioned by other authors are contained in the more general results we will achieve.

If we neglect motions of the air involving great variations of geographical latitude, the local pressure variations at the ground may be a consequence of five processes in the atmosphere:

1. *Friction* generally alters the mass-content of a vertical column and so leads to pressure variations at the ground. Although it is possible to introduce friction formally, we neglect it as its exact influence is not yet known. There exist some estimates about the air transport through the isobars as a consequence of friction, but these estimates show rather different results. Owing to this neglect it is always possible that our results may have to be modified. It will appear, however, that these results confirm empirical outcome as far as this is known. The fact that the frictional layer only contains about 6 % of the total mass of the atmosphere seems to justify the neglect.

The four remaining processes which we are to consider in some detail are:

2. *Isallobaric effects* in the free atmosphere. They are usually due to temperature variations near the surface and are responsible for the formation of thermal cyclones and anticyclones and the monsoon circulations connected with them. Wexler (52) attempted to calculate the development of a polar anticyclone. It will appear to be possible to extend and improve his considerations.

3. *Divergent or convergent isobars or contourlines* in the free atmosphere may also lead to pressure variations at the ground. It will be seen that Scherhag's general rule,

stating that surface pressure falls where the high-level isobars show divergence, needs an extension. This has already been shown by some other authors. It will also appear, however, that Scherhag's simple rule will usually satisfy in the neighbourhood of a depression centre. The influence of variations in isobaric curvature on the surface pressure, which was indicated by Boyden (5) also follows from our more general considerations.

Atmospheric *steering* can also be better understood when the exact course of the isobars near the steered system is investigated.

Finally, it is possible to prove Rodewald's suggestion about the deepening of cyclones in the "delta" of an upper air-current theoretically by calculating from a certain isobaric pattern and comparing the result with a real case.

4. *Vertical motions* of the air may also lead to variations in surface pressure. Durst and Sutcliffe (9) tried to explain the deepening of tropical cyclones in this way. We shall see that their calculations are somewhat rough in some respects and that it will be necessary to take account of the exact distribution of the vertical velocities. This makes exact knowledge of surface friction necessary.

5. *Mass advection* was always considered as the most effective of the processes which can lead to surface pressure variations. Exner (19, 20, 21), for instance, published a theory of surface pressure variations which was based on advection. It appears, however, that advection of air belonging to one air-mass is usually not able to cause pressure variations of importance.

In a discontinuous density field mass advection may lead to notable pressure changes, however. We have two possibilities to distinguish in that case:

α. pure advection,

and

β. slope variations of frontal surfaces.

In the relevant chapter it will be shown, moreover, that the considerations of Ertel and others on "*singular advection*" are wrong.

Generally speaking, we may say that it appears from the following work that the most important causes for the variations of surface pressure are to be sought in the troposphere. This result is in accordance with statements of Bergeron (2) and others. It is a simple consequence of the fact that the troposphere contains about $^4/_5$ of the whole mass of the atmosphere.

Of course it is quite possible that some causes of pressure variations are not contained in the approximated equations. For the moment it looks, however, as if this is not so.

PART I

The general equation

1. *Introduction*

The attempts made to explain pressure variations at the ground with the aid of the existing flow pattern are numerous. Modern theoretical considerations on the subject start from the relation:

$$\frac{\partial p_o}{\partial t} = \int_o^\infty g \frac{\partial \varrho}{\partial t} dz \tag{1.1}$$

Substituting the equation of continuity:

$$\frac{\partial \varrho}{\partial t} = -\left\{\frac{\partial}{\partial x}(\varrho v_x) + \frac{\partial}{\partial y}(\varrho v_y) + \frac{\partial}{\partial z}(\varrho v_z)\right\} \tag{1.2}$$

into (1.1), we get:

$$\frac{\partial p_o}{\partial t} = -\int_o^\infty g\left\{\frac{\partial}{\partial x}(\varrho v_x) + \frac{\partial}{\partial y}(\varrho v_y) + \frac{\partial}{\partial z}(\varrho v_z)\right\}dz = -\int_o^\infty g\left\{\frac{\partial}{\partial x}(\varrho v_x) + \frac{\partial}{\partial y}(\varrho v_y)\right\}dz - [g\varrho v_z]_o^\infty \tag{1.3}$$

In these relations the symbols have the following meaning:

p_o	the surface pressure,
ϱ	the density,
g	the acceleration of gravity,
x, y, z	the three Cartesian coordinates,
v_x, v_y, v_z	the three Cartesian components of the velocity $\vec{v}$,
t	the time.

The introduction of $[g\varrho v_z]_o^\infty$ in (1.3) is allowed only when $\varrho\vec{v}$ does not show any discontinuity. As even the sharpest frontal surface is nothing else than a transition layer we shall suppose that this condition is fulfilled. In the last chapter we will return to the question of discontinuities in some detail.

As $[g\varrho v_z]_o^\infty$ is equal to zero and we are allowed to introduce some mean value of g, which will differ only slightly from the value it has at sea-level, (1.3) changes into the fundamental relation, connecting the variations of surface pressure with the divergence of the horizontal component of momentum:

$$\frac{\partial p_o}{\partial t} = -g\int_o^\infty \left\{\frac{\partial}{\partial x}(\varrho v_x) + \frac{\partial}{\partial y}(\varrho v_y)\right\}dz \tag{1.4}$$

The most recent attempt to solve the problem of pressure variations applying (1.4) was made by D u r s t (8). He determined the deviation of the real wind from the geostrophic wind with the aid of maps and computed the divergence of this deviation. It appeared that good results are to be expected, especially when pilot balloon observations become more accurate and by application of radio methods more independent of meteorological conditions.

Older than this method in which the wind field is considered are those in which it was tried to calculate $\frac{\partial p_o}{\partial t}$ or at least to determine its sign by investigation of the field of pressure. For at each moment the wind field is connected with the field of pressure and usually the latter can be constructed easily from the aerological observations.

Some of the oldest weather rules are based on the appreciation of certain isobaric configurations at the ground. In the last decades before the war the method has been applied to high-level isobars. The several investigations find their height in the empirical rule of Scherhag (44, 45, 46, 47, 48, 49):

Surface pressure falls below diverging high-level isobars (or isohypses of an isobaric surface).

This important result was confirmed and extended by Rodewald (34, 35, 36, 37, 38, 39, 40, 41, 42).

The many attempts to prove Scherhag's divergence-theory theoretically failed however. We mention the publications of Sieber (51), Ertel (10, 18) and Baur and Philipps (1).

This failure must be ascribed partly to a wrong interpretation of Scherhag's problem. The essential difficulties which presented themselves lay, however, in proving the equivalence of isobaric divergence and the divergence of momentum given by $\frac{\partial}{\partial x}(\varrho v_x) + \frac{\partial}{\partial y}(\varrho v_y)$.

It is the purpose of the following work to connect the course of the isobars or level lines in the free atmosphere and its variations with a divergence of the momentum and so with the aid of (1.4) with variations in surface pressure.

2. *The approximate value of the wind velocity in the free atmosphere*

In order to find the general connection between wind velocity and field of pressure we start from the simplified equations of motion:

$$\frac{dv_x}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + lv_y + \frac{\partial}{\partial z}\left(\frac{A}{\varrho}\frac{\partial v_x}{\partial z}\right) \qquad \frac{dv_y}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial y} - lv_x + \frac{\partial}{\partial z}\left(\frac{A}{\varrho}\frac{\partial v_y}{\partial z}\right) \qquad \frac{dv_z}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} - g \tag{2.1}$$

In these equations the new symbol l denotes $2\omega \sin \varphi$, ω being the angular velocity of the earth's rotation and φ the geographical latitude. A is the eddy convectivity.

Henceforth we neglect the influence of friction, which is permitted as we shall mainly consider the field of motion in the free atmosphere, that is above say 500 m. As pressure is almost completely of static origin, we also neglect the vertical acceleration Equations (2.1) then reduce to:

$$\frac{dv_x}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + lv_y \qquad \frac{dv_y}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial y} - lv_x \qquad 0 = -\frac{1}{\varrho}\frac{\partial p}{\partial z} - g \tag{2.2}$$

The solution of the first two of these equations was given by Hesselberg (23) and afterwards by Philipps (31):

$$v_x = -\frac{1}{\varrho l}\frac{\partial p}{\partial y} - \frac{1}{l^2}\frac{d}{dt}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right) + \frac{1}{l^3}\frac{d^2}{dt^2}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right) - \cdots\cdots\cdots$$

$$v_y = +\frac{1}{\varrho l}\frac{\partial p}{\partial x} - \frac{1}{l^2}\frac{d}{dt}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right) - \frac{1}{l^3}\frac{d^2}{dt^2}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right) + \cdots\cdots\cdots \tag{2.3}$$

so that the velocity is give by infinite series which are usually supposed to converge rapidly. Of course it is always possible, that in certain circumstances the series do not converge rapidly enough for us to ignore them beyond the second or third term. As it is generally impossible to compute $\frac{d^n}{dt^n}\left(\frac{1}{\varrho}\nabla p\right)$ from the weather map for $n > 2$, whereas afterwards we still have to take $div\,(\varrho\vec{v})$ in order to compute pressure variations, we ignore the series beyond the second term:

$$v_x = -\frac{1}{\varrho l}\frac{\partial p}{\partial y} - \frac{1}{l^2}\frac{d}{dt}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right) \qquad v_y = +\frac{1}{\varrho l}\frac{\partial p}{\partial x} - \frac{1}{l^2}\frac{d}{dt}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right) \tag{2.4}$$

It may be also clear, that in adopting (2.4) as a reasonable approximation, we have ignored accidental parts of the velocity which are not connected with the field of pressure. It is probable, however, that this neglect is permissible.

Working out (2.4) we get:

$$v_x = -\frac{1}{\varrho l}\frac{\partial p}{\partial y} - \frac{1}{l^2}\left\{\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right\}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)$$

$$v_y = +\frac{1}{\varrho l}\frac{\partial p}{\partial x} - \frac{1}{l^2}\left\{\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right\}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right) \qquad (2.5)$$

In these equations v_x and v_y appear on both sides. This means, that these quantities can be found as the quotient of two determinants. In further developing the calculations these expressions give rise to great difficulties. It is for this reason that it forms a sufficient approximation if we substitute for v_x and v_y on the right hand side of (2.5) the geostrophic part of the real wind:

$$v_{xg} = -\frac{1}{\varrho l}\frac{\partial p}{\partial y} \text{ and } v_{yg} = \frac{1}{\varrho l}\frac{\partial p}{\partial x}$$

According to Ertel (16) this substitution is usually allowed. The errors due to it are of the same order of magnitude as those we make neglecting (2.3) beyond the second terms.

So finally the equations from which we are to start our considerations get the following form:

$$v_x = -\frac{1}{\varrho l}\frac{\partial p}{\partial y} - \frac{1}{l^2}\left\{\frac{\partial}{\partial t} - \frac{1}{\varrho l}\frac{\partial p}{\partial y}\frac{\partial}{\partial x} + \frac{1}{\varrho l}\frac{\partial p}{\partial x}\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right\}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)$$

$$v_y = \frac{1}{\varrho l}\frac{\partial p}{\partial x} - \frac{1}{l^2}\left\{\frac{\partial}{\partial t} - \frac{1}{\varrho l}\frac{\partial p}{\partial y}\frac{\partial}{\partial x} + \frac{1}{\varrho l}\frac{\partial p}{\partial x}\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right\}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right) \qquad (2.6)$$

These equations were also deduced by van Mieghem (30). The isallobaric wind-component according to Brunt and Douglas (7) is contained in them.

Finally we can formulate the problem we want to solve in the following way: Is it possible to draw conclusions with respect to surface pressure variations from the form and the variations of the field of pressure in the free atmosphere as far as these are contained in the approximated equations (2.6)? Are the values of $\frac{\partial p_o}{\partial t}$ calculated in this way in accordance with the observed values?

3. *The divergence of momentum*

As we have seen already, the connection between the velocity components (2.6) and the pressure variations at the ground are given by

$$\frac{\partial p_o}{\partial t} = -g\int_o^\infty \left\{\frac{\partial}{\partial x}(\varrho v_x) + \frac{\partial}{\partial y}(\varrho v_y)\right\} dz \qquad (3.1)$$

In integrating from the ground to the upper limit of the atmosphere we neglect friction in the planetary boundary layer. Substituting (2.6) into (3.1) we make an error of 15 to 20 % at most according to the estimation made by Lotz (26).

Taking l independent of x and y the integrand of (3.1) becomes

$$I=-\left[\frac{1}{l^2}\frac{\partial}{\partial x}\left\{\left(\varrho\frac{\partial}{\partial t}-\frac{1}{l}\frac{\partial p}{\partial y}\frac{\partial}{\partial x}+\frac{1}{l}\frac{\partial p}{\partial x}\frac{\partial}{\partial y}+\varrho v_z\frac{\partial}{\partial z}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)\right\}+\right.$$

$$\left.+\frac{1}{l^2}\frac{\partial}{\partial y}\left\{\left(\varrho\frac{\partial}{\partial t}-\frac{1}{l}\frac{\partial p}{\partial y}\frac{\partial}{\partial x}+\frac{1}{l}\frac{\partial p}{\partial x}\frac{\partial}{\partial y}+\varrho v_z\frac{\partial}{\partial z}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right)\right\}\right] \quad . \tag{3.2}$$

If I is known in each level, the pressure variations at the ground can be computed. It appears from (3.2) that these pressure variations can be due to three different kinds of process:

1. Isallobaric convergence or divergence in the free atmosphere.
2. Convergence or divergence owing to horizontal motions.
3. Convergence or divergence owing to vertical motions.

The three corresponding parts of I are:

$$I_1=-\frac{1}{l^2}\left[\frac{\partial}{\partial x}\left\{\varrho\frac{\partial}{\partial t}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{\varrho\frac{\partial}{\partial t}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right)\right\}\right] \tag{3.3}$$

$$I_2=-\frac{1}{l^3}\left[\frac{\partial}{\partial x}\left\{\left(-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}+\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{\left(-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}+\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right)\right\}\right] \tag{3.4}$$

$$I_3=-\frac{1}{l^2}\left[\frac{\partial}{\partial x}\left\{\varrho v_z\frac{\partial}{\partial z}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{\varrho v_z\frac{\partial}{\partial z}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right)\right\}\right] \tag{3.5}$$

It will be understood that these three effects will never occur separately, but always in combination with each other, a fact which complicates the calculations considerably.

Nevertheless in spite of all approximation and calculation difficulties it appears to be possible to obtain some qualitative or even quantitative results.

PART II

Isallobaric processes

4. *Transformation of I_1*

It can be seen from (3.3) that I_1 depends on local variations of density as well as on pure isallobaric processes. Both effects cooperate in the case of the formation of an anticyclone owing to cooling, and the deepening of a depression owing to radiative heating or to heating due to conditional instability.

In evaluating (3.3) we find:

$$I_1 = I_1' + I_1'' = -\frac{1}{l^2}\left[\frac{1}{\varrho^2}\left(\frac{\partial\varrho}{\partial x}\frac{\partial\varrho}{\partial t}\frac{\partial p}{\partial x}+\frac{\partial\varrho}{\partial y}\frac{\partial\varrho}{\partial t}\frac{\partial p}{\partial y}\right)-\frac{1}{\varrho}\left(\frac{\partial^2\varrho}{\partial x dt}\frac{\partial p}{\partial x}+\frac{\partial^2\varrho}{\partial y\partial t}\frac{\partial p}{\partial y}\right)-\frac{1}{\varrho}\left\{\frac{\partial\varrho}{\partial t}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right\}\right]-$$

$$-\frac{1}{l^2}\left[\frac{\partial^3 p}{\partial x^2\partial t}+\frac{\partial^3 p}{\partial y^2\partial t}\right] \qquad (4.1)$$

According to Hesselberg and Friedmann (24) the term containing $\frac{\partial}{\partial t}\varrho$ can be neglected with respect to those containing $\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x}\right)$ and $\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial y}\right)$. This would mean that I_1' is only small compared with I_1''.

Although the results of Hesselberg and Friedmann are not correct in every respect as they, in 1914, could not take into account the existence of discontinuities in the atmosphere, we will accept their view on the comparability of I_1' and I_1'' and restrict ourselves to considering I_1''. In his article on the formation of radiative anticyclones Wexler (52) also found that terms depending on $\frac{\partial\varrho}{\partial t}$ had only a slight influence on the final result.

We shall return to I_1'' when we consider the influence of advection of air of different density on surface pressure.

5. *Thermal and monsoonal pressure variations*

Wexler (52) was the first who connected the formation of an anticyclone with isallobaric effects due to radiative cooling. In doing so he started from a vertical temperature distribution in which a thick isothermal layer caused by radiation represented the polar air.

Although he succeeded, at least qualitatively, in connecting the pressure rise at the ground with isallobaric effects in the free atmosphere due to the shrinking of the cooled air, there are, however, some shortcomings in his calculations.

In the first place, Wexler did not mention the influence of the horizontal dimensions of the region where the cooling takes place. It is evident, that these dimensions must be of some importance as the isallobaric convergence will be greater when the region is small than when it is large, provided that the rate of cooling is the same in both cases.

Secondly, Wexler was not able to calculate the exact pressure variation as he only considered to a first approximation the fact that there exists a divergent motion of air in the lower levels owing to the mass supply in the higher ones.

It appears to be possible to extend the dynamical part of his calculations into both directions. In order to adapt our theoretical considerations to reality we accept some bell-shaped distribution of the radiative temperature variation over the considered area. Perhaps an error-curve would suit best. In order to facilitate the further calculations, however, we shall represent the distribution of the temperature variations by means of harmonic functions. The fact that these functions are periodic is a disadvantage; the alternation of land and sea as found on the earth may, however, form to some extent a justification of their introduction.

Doing so, we introduce the following function for the temperature variation in a point (x, y, z):

$$\frac{\partial T}{\partial t} = \tau_o e^{-k_T z} \cdot \tfrac{1}{4} \left[\left(\cos\frac{\pi x}{\lambda_x} + 1 \right) \left(\cos\frac{\pi y}{\lambda_y} + 1 \right) \right] \tag{5.1}$$

In this equation the symbols have the following meanings:

τ_o is the temperature variation in unit time on the ground in the centre of the considered region, that is in the point $x = y = z = 0$. It may be a function of time.

k_T measures the rate in which the radiative temperature variation diminishes with growing height. This quantity may also be a function of time depending on the stability of the atmosphere.

λ_x and λ_y indicate half the dimensions in the x- and y-directions of the region where the radiative temperature variation takes place. We see that $\frac{\partial T}{\partial t}$ is zero at the borders of this region.

If no vertical displacement of the air should take place owing to the variation of temperature, the local radiative density variation would be given by:

$$\frac{\partial \varrho^*}{\partial t} = -\frac{\varrho}{T}\frac{\partial T}{\partial t} = -\frac{\varrho_o}{T_o} e^{-kz} \frac{\partial T}{\partial t} = \frac{\delta_o}{\tau_o}\frac{\partial T}{\partial t} e^{-kz} = \tfrac{1}{4}\, \delta_o\, e^{-(k_T+k)z}. \left[\left(\cos\frac{\pi_x}{\lambda_x} + 1 \right) \left(\cos\frac{\pi_y}{\lambda_y} + 1 \right) \right] \tag{5.2}$$

In this equation the factor e^{-kz} originates from the fact that $\frac{\varrho}{T}$ decreases with increasing height according to $\frac{\varrho}{T} = \frac{\varrho_o}{T_o} e^{-kz}$ where k depends on the mean temperature and is about 10^{-4} if z is expressed in meters. $\delta_o = -\frac{\varrho_o}{T_o}\tau_o$ denotes the radiative density variation in the point $x = y = z = 0$. A positive τ_o corresponds with a negative δ_o and vice versa.

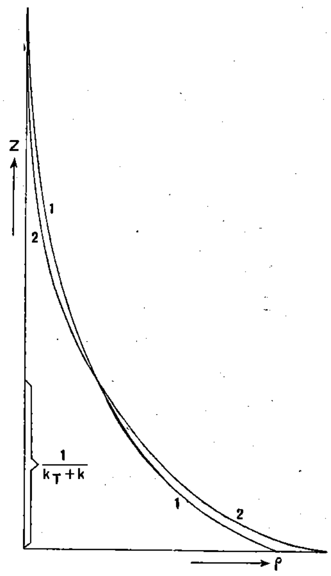

Figure 1.

In reality (5.2) does not give the right local density variation as the air is displaced vertically when it becomes denser or lighter. This vertical displacement will be such, that $\int_0^\infty \frac{\partial \varrho}{\partial t} dz = 0$, $\frac{\partial \varrho}{\partial t}$ representing the local density variation during the displacement. For it is obvious that density variations in a vertical column in themselves cannot lead to pressure variations at the ground as the total mass in the column is not effected by this process.

These pressure variations only occur then, when air leaves the column or flows into by horizontal advection. If the air were to be heated or cooled everywhere to the same degree, no pressure variations at the surface of the earth would take place at all.

It is easy to see, that the condition $\int_0^\infty \frac{\partial \varrho}{\partial t} dz = 0$ is fulfilled by the following expression for $\frac{\partial \varrho}{\partial t}$ of which we suppose that it gives a rather good account of the density variations occurring in reality:

$$\frac{\partial \varrho}{\partial t} = \tfrac{1}{4}\, \delta_o e^{-(k_T+k)z} \left[\left(\cos\frac{\pi x}{\lambda_x} + 1 \right) \left(\cos\frac{\pi y}{\lambda_y} + 1 \right) \right] [1 - (k_T + k)\, z] \tag{5.3}$$

The physical meaning of the factor $[1 - (k_T + k)\,z]$ is that radiation cooling leads to a local increase of density up to a height of $z = \frac{1}{k_T + k}$ meters whereas above this level density decreases locally owing to the descent of the air. In fig. 1 this behaviour of the air is represented in a density-height-diagram. Curve number 1 shows the distribution of density before the cooling, number 2 after the cooling has taken place for some time. In the same way radiation heating leads to density fall below and density increase above the level $z = \frac{1}{k_T + k}$.

We can now distinguish three successive approximations according to which we can calculate the pressure variation due to the density variation.

a. Simple integration.

If we suppose that the surface pressure remains constant at first, which is not true of course, and which is the same approximation as Wexler applied, we find for the pressure variation in the level z the following expression:

$$\frac{\partial p_z}{\partial t} = g \int_z^{\infty} \frac{\partial \varrho}{\partial t}\, dz = -g \int_0^{z} \frac{\partial \varrho}{\partial t}\, dz = -\tfrac{1}{4}\, g\, \delta_o\, z\, e^{-(k_T+k)\,z} \left[\left(\cos\frac{\pi x}{\lambda_x} + 1\right)\left(\cos\frac{\pi y}{\lambda_y} + 1\right) \right] \tag{5.4}$$

The isallobaric divergence in z is equal to

$$I_1'' = -\frac{1}{l^2}\left[\frac{\partial^3 p_z}{\partial x^2 dt} + \frac{\partial^3 p_z}{\partial y^2 \partial t}\right] = \tfrac{1}{4}\frac{g}{l^2}\delta_o\, z\, e^{-(k_T+k)z} \cdot \left[-\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right)\cos\frac{\pi x}{\lambda_x}\cos\frac{\pi y}{\lambda_y} - \frac{\pi^2}{\lambda_x^2}\cos\frac{\pi x}{\lambda_x} - \frac{\pi^2}{\lambda_y^2}\cos\frac{\pi y}{\lambda_y}\right] \tag{5.5}$$

Now neglecting the fact that we started with taking surface pressure constant we find from (5.5) the following value of the variation of surface pressure:

$$\frac{\partial p_o}{\partial t} = -g \int_0^{\infty} I_1''(z)\, dz = \tfrac{1}{4}\frac{g^2}{l^2}\delta_o \frac{1}{(k_T + k)^2}\left[\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right)\cos\frac{\pi x}{\lambda_x}\cos\frac{\pi y}{\lambda_y} + \frac{\pi^2}{\lambda_x^2}\cos\frac{\pi x}{\lambda_x} + \frac{\pi^2}{\lambda_y^2}\cos\frac{\pi y}{\lambda_y}\right] \tag{5.6}$$

As was said already this method of calculating $\frac{dp_o}{\partial t}$ is the same one as Wexler applied. In starting from the expression for I_1'' of (4.1) we were able, however, to introduce the horizontal dimensions λ_x and λ_y of the considered region, which means a first improvement of Wexler's results.

Nevertheless, it just appears from the introduction of λ_x and λ_y how insufficient this first approximation is. For it follows from (5.6) that $\frac{\partial p_o}{\partial t}$ increases when λ_x and λ_y decrease, the other quantities remaining constant. Even with a small temperature variation, $\frac{\partial p_o}{\partial t}$ would be very substantial if only the region in which the temperature variation occurs is small enough. Small islands for instance would show very great daily pressure variations according to (5.6). This result is very unsatisfying. It is a consequence of the fact that we started our calculations from a constant surface pressure and doing so ignored the isallobaric effects in the lower levels which are of opposite sign than those in higher levels. A radiative anticyclone, for instance, is formed by *isallobaric convergence* in the free atmosphere but it is partly annulled by the *isallobaric divergence* in the lower parts of the atmosphere, a divergence which is caused by the anticyclogenesis itself.

Wexler already pointed to this imperfection of the calculations, but he supposed that the problem could not be solved exactly owing to the appearance of differential equations, the solution of which was unknown.

b. Differential equation.

Now the introduction of the harmonic functions enables us to calculate the problem more exactly, taking also account of the isallobaric divergence in the lower levels. Equation (5.4) gives only one part of the pressure variation in level z, namely the part that depends on the radiative density variations only. Another part is due to isallobaric effects in the levels above z. Taking also account of this effect we have to solve the following equation:

$$\frac{\partial p_z}{\partial t} = -g \int_o^z \frac{\partial \varrho}{\partial t} dz + \frac{g}{l^2} \int_z^\infty \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial p_z}{\partial t} dz \tag{5.7}$$

This equation can be transformed by differentiating it with respect to z into the following partial differential equation of the second order for $q_z = \frac{\partial p_z}{\partial t}$:

$$\frac{\partial q_z}{\partial z} = -\tfrac{1}{4} g\, \delta_o \left[1 - (k_T + k)\, z\right] e^{-(k_T+k)\, z} \left[\left(\cos\frac{\pi x}{\lambda_x} + 1\right)\left(\cos\frac{\pi y}{\lambda_y} + 1\right)\right] - \frac{g}{l^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q_z \tag{5.8}$$

In order to solve this equation we now substitute

$$q_z = q_{xy} \cos\frac{\pi x}{\lambda_x} \cos\frac{\pi y}{\lambda_y} + q_x \cos\frac{\pi x}{\lambda_x} + q_y \cos\frac{\pi y}{\lambda_y} + q_o \tag{5.9}$$

where q_{xy}, q_x, q_y and q_o are merely functions of z.

Introducing this expression for q_z into (5.8) the last term of his equation becomes:

$$-\frac{g}{l^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q_z = \frac{g}{l^2}\left\{ q_{xy}\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right) \cos\frac{\pi x}{\lambda_x} \cos\frac{\pi y}{\lambda_y} + q_x \frac{\pi^2}{\lambda_x^2} \cos\frac{\pi x}{\lambda_x} + q_y \frac{\pi^2}{\lambda_y^2} \cos\frac{\pi y}{\lambda_y}\right\} \tag{5.10}$$

Introducing (5.9) and (5.10) into (5.8) and realising that this equation must be satisfied for all values of x and y, we can reduce (5.8) to four linear differential equations of the first order:

$$\frac{\partial q_o}{\partial z} = -\tfrac{1}{4} g\, \delta_o \left[1 - (k_T + k)\, z\right] e^{-(k_T+k)\, z} \tag{5.11}$$

$$\frac{\partial q_x}{\partial z} = -\tfrac{1}{4} g\, \delta_o \left[1 - (k_T + k)\, z\right] e^{-(k_T+k)\, z} + \frac{g}{l^2}\frac{\pi^2}{\lambda_x^2} q_x \tag{5.12}$$

$$\frac{\partial q_y}{\partial z} = -\tfrac{1}{4} g\, \delta_o \left[1 - (k_T + k)\, z\right] e^{-(k_T+k)\, z} + \frac{g}{l^2}\frac{\pi^2}{\lambda_y^2} q_y \tag{5.13}$$

$$\frac{\partial q_{xy}}{\partial z} = -\tfrac{1}{4} g\, \delta_o \left[1 - (k_T + k)\, z\right] e^{-(k_T+k)\, z} + \frac{g}{l^2}\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right) q_{xy} \tag{5.14}$$

Taking account of the fact that $\frac{\partial p_\infty}{\partial t}$ must be zero, we get the following expression for $q_z = \frac{\partial p_z}{\partial t}$:

$$\frac{\partial p_z}{\partial t} = \tfrac{1}{4} g\, \delta_o\, e^{-(k_T+k)\, z} \left[\left\{ \frac{\frac{g}{l^2}\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right)}{\left[k_T + k + \frac{g}{l^2}\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right)\right]^2} - \frac{(k_T + k)\, z}{k_T + k + \frac{g}{l^2}\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right)} \right\} \cos\frac{\pi x}{\lambda_x} \cos\frac{\pi y}{\lambda_y} + \right.$$

$$\left. + \left\{ \frac{\frac{g}{l^2}\frac{\pi^2}{\lambda_x^2}}{\left(k_T + k + \frac{g}{l^2}\frac{\pi^2}{\lambda_x^2}\right)^2} - \frac{(k_T + k)\, z}{k_T + k + \frac{g}{l^2}\frac{\pi^2}{\lambda_x^2}} \right\} \cos\frac{\pi x}{\lambda_x} + \left\{ \frac{\frac{g}{l^2}\frac{\pi^2}{\lambda_y^2}}{\left(k_T + k + \frac{g}{l^2}\frac{\pi^2}{\lambda_y^2}\right)^2} - \frac{(k_T + k)\, z}{k_T + k + \frac{g}{l^2}\frac{\pi^2}{\lambda_y^2}} \right\} \cos\frac{\pi y}{\lambda_y} - z \right] \tag{5.15}$$

From this general relation we find the pressure variation at the ground by putting $z = 0$. It appears that $\frac{\partial p_o}{\partial t} = 0$ if $\lambda_x = \lambda_y = \infty$ as well as if $\lambda_x = \lambda_y = 0$, so that the unsatisfying result we found in disregarding isallobaric divergence near the ground does not come into existence any longer. There exists a maximum value of $\frac{\partial p_o}{\partial t}$ for some finite dimensions of the considered region. If we suppose the region to be a square one we find $\frac{\partial p_o}{\partial t}$ to be a maximum for $\lambda_x = \lambda_y = \lambda_m$, λ_m being determined by the equation $\frac{g}{l^2} \frac{\pi^2}{\lambda_m^2} = 0{,}834\ (k_T + k)$.

c. Integral equation.

It can be easily understood that the differential equations (5.7) or (5.11) to (5.14) do not yet describe the phenomena we are studying in an adequate way. For the pressure variation $\frac{\partial p(z)}{\partial t}$ which occurs in a level z does not only depend on the density variation and the mass advection above that level, but also on the vertical motions of the air in that level.

Rossby (43) may be considered the first who developed a consequent advection theory. He based his considerations on some propositions, which are not valid in our case. Nevertheless it will appear possible to apply his theory to the problem treated here.

According to Rossby the local pressure variation $\frac{\partial p(z)}{\partial t}$ is given by

$$\frac{\partial p(z)}{\partial t} = \pi(z) - \frac{mg}{R} \frac{c_v}{c_p} \frac{p(z)}{T(z)} \int_0^z \frac{\pi(\xi)}{p(\xi)} d\xi \tag{5.16}$$

In this equation π (z) signifies the total mass advection above z, that is $\int_z^\infty \frac{\partial \varrho}{\partial t} dz$, m signifies the molecular weight of air, c_v signifies the specific heat of air at constant volume, c_p at constant pressure. In an incompressible atmosphere we should find $\frac{\partial p(z)}{\partial t} = \pi(z)$. The last term of the right hand side of (5.16) is a consequence of the compressibility of the atmosphere. In order to apply (5.16) to our problem we must know the value of π (z) for each level. We suppose, therefore, for the moment that the atmosphere is incompressible. Then $\pi(z)$ is composed of two parts, namely first of the massincrease due to the divergence above the level considered, $\int_z^\infty \frac{g}{l^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial p(\xi)}{\partial t} d\xi$ and secondly of the mass increase due to the divergence beneath the level considered. This divergence leads to vertical motions in the level z, giving rise to mass changes above z which owing to the supposed incompressibility of the atmosphere are equal to $g\varrho v_z = -g \int_0^z \left(\frac{\partial \varrho v_x}{\partial x} + \frac{\partial \varrho v_y}{\partial y}\right) dz = \int_0^z \frac{g}{l^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial p(\xi)}{\partial t} d\xi$. We find, therefore, π to be a constant in our special case, namely

$$\pi(z) = \frac{g}{l^2} \int_0^\infty \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial p(\xi)}{\partial t} d\xi \tag{5.17}$$

This means, that returning to a compressible atmosphere π (ξ) can be put before the integral in (5.16) so that we get

$$\frac{\partial p(z)}{\partial t} = -g \int_o^z \frac{\partial \varrho}{\partial t} dz + \frac{g}{l^2} \int_o^\infty \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial p\ (\xi)}{\partial t} d\xi . \left\{ 1 - \frac{mg}{R} \frac{c_v}{c_p} \frac{p(z)}{T(z)} \int_o^z \frac{d\xi}{p(\xi)} \right\} \tag{5.18}$$

The expression between the brackets can be represented by a function of z, $r(z)$, a function which was introduced by Rossby for the case of so-called special advection. This function can be approximated by the exponential expression $e^{-k_\pi z}$ where k_π has a value of about $\frac{1}{11\,000}$ when z is expressed in meters.

Introducing the notation (5.9), for $q(z) = \frac{\partial p(z)}{\partial t}$ we find the following four equations to solve:

$$q_o(z) = -\tfrac{1}{4} g\, \delta_o\, z\, e^{-(k_T + k)\, z} \tag{5.19}$$

$$q_x(z) = -\tfrac{1}{4} g\, \delta_o\, z\, e^{-(k_T + k)\, z} - \frac{g}{l^2} \frac{\pi^2}{\lambda_x^2} \int_o^\infty e^{-k_\pi z} q_x(\xi)\, d\xi \tag{5.20}$$

$$q_y(z) = -\tfrac{1}{4} g\, \delta_o\, z\, e^{-(k_T + k)\, z} - \frac{g}{l^2} \frac{\pi^2}{\lambda_y^2} \int_o^\infty e^{-k_\pi z} q_y(\xi)\, d\xi \tag{5.21}$$

$$q_{xy}(z) = -\tfrac{1}{4} g\, \delta_o z\, e^{-(k_T + k)\, z} - \frac{g}{l^2} \left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right) \int_o^\infty e^{-k_\pi z} q_{xy}(\xi)\, d\xi \tag{5.22}$$

Applying the simple solution method, valid when the nucleus of an integral equation can be written as a polynomial consisting of terms of the form $s_v(z)\, t_v(\xi)$ (see Frank-v. Mises, Die Differential und Integralgleichungen der Mechanik und Physik, Part I, 1930, page 485), we easily find the following expression for the pressure variation due to radiation density variations:

$$\begin{aligned} q(z) = {} & -\tfrac{1}{4} g\, \delta_o\, z\, e^{-(k_T + k) z} \left[1 + \cos\frac{\pi x}{\lambda_x} + \cos\frac{\pi y}{\lambda_y} + \cos\frac{\pi x}{\lambda_x} \cos\frac{\pi y}{\lambda_y} \right] + \\ & + \frac{\tfrac{1}{4} g^2\, l^2 .\, \delta_o\, e^{-k_\pi z}}{(k_T + k)^2} \left[\frac{\pi^2/\lambda_x^2}{\left(1 + \frac{g}{l^2} \frac{\pi^2}{\lambda_x^2} \frac{1}{k_\pi}\right)} \cos\frac{\pi x}{\lambda_x} + \frac{\pi^2/\lambda_y^2}{\left(1 + \frac{g}{l^2} \frac{\pi^2}{\lambda_y^2} \frac{1}{k_\pi}\right)} \cos\frac{\pi y}{\lambda_y} + \right. \\ & \left. + \frac{\pi^2/\lambda_x^2 + \pi^2/\lambda_y^2}{\left\{ 1 + \frac{g}{l^2} \left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right) \frac{1}{k_\pi} \right\}} \cos\frac{\pi x}{\lambda_x} \cos\frac{\pi y}{\lambda_y} \right] \end{aligned} \tag{5.23}$$

We see, that according to this relation surface pressure does not change when the area where the temperature variation takes place has infinite dimensions. On the other side the pressure variation remains finite when the dimensions of the considered area become zero. This does not mean an imperfection of the theory as δ_o generally will be zero when the area becomes smaller. Moreover a density variation will only be perceptible to a small height when the horizontal dimensions of the area are small. This means that k_T will then be very large, another factor to reduce $q\,(z = o)$ to a very small value.

We will now test the foregoing theories by comparing the numerical results to which they lead with pressure variations such as are found in reality.

In the first place we can compare the theoretical results with the monsoonal pressure variations over the vast continents, Eurasia, North America and Australia. Doing so, we suppose that these pressure variations are principally a consequence of radiative cooling and heating.

Then δ_o indicates the yearly density range of the air at the ground in the centre of the considered continent.

As has been said already, k can be taken equal to 10^{-4}.

The value of k_T must be determined from empirical data. This can only be done in a rather arbitrary way, as the variation of temperature amplitude does not show an exact exponential course with height. From the obtainable data of North America and Siberia it follows, that in the lowest two kilometers k_T is about $\frac{1}{5000}$ if z is expressed in meters, where it must be taken equal to $\frac{1}{8000}$ if we consider the lowest 4 kilometers. If we also consider the higher troposphere and the lower stratosphere k_T must be taken equal to $\frac{1}{11000}$. In still higher levels the yearly temperature amplitude decreases quickly. As the yearly temperature variation in the higher levels does not depend exclusively on the heating or cooling from below, but also on vertical motions, due to shrinking and stretching, we will mainly consider the lowest levels and therefore, we will take k_T equal to $\frac{1}{8000} = 1.25.10^{-4}$ and combine it with the density variation at the ground. So $k + k_T$ is equal to $2.25.10^{-4}$.

Further we take g equal to 9,8, k_π to $\frac{1}{11000}$, l for Siberia and North America equal to 10^{-4} and for Australia equal to $0{,}7.10^{-4}$.

E u r a s i a. $\tau_o = 60°$ C, the yearly mean temperature amounts to 260°K this leading to a value of $3{,}0.10^{-4}\ \frac{\text{ton}}{\text{m}^3}$ for δ_o . $\lambda_x = 6000$ km, $\lambda_y = 3750$ km. The yearly pressure range over Siberia can be found by reducing the values given in the H a n d b u c h d e r K l i m a t o l o g i e Vol. III, part N_2 to sea level. It is a well-known fact that the way in which this reduction is made generally, namely by extrapolating the surface temperature to sea-level, leads to false results as the pressure values found in this way turn out to be too small in summer and too large in winter. We find a yearly pressure range of about 40 mb by that process. A correct extrapolation is only possible when the exact temperature height curve is known. If we reduce by multiplying the yearly pressure amplitude at station level by $\frac{1015}{p}$, where p represents the mean yearly pressure in the station level expressed in mb, we find a value for the yearly pressure range which will be somewhat too small. It turns out to be 21 or 22 mb in the centre of the continent. It will, therefore, not be far from the truth if we suppose the yearly pressure range to be about 30 mb.

The theoretical pressure variation in the point $x = y = z = 0$ according to the three approximations is found to be:

a. S i m p l e i n t e g r a t i o n 280 mb.

b. D i f f e r e n t i a l e q u a t i o n 19 mb.

c. I n t e g r a l e q u a t i o n 32 mb.

N o r t h A m e r i c a. $\tau_o = 35°$ C in the centre of the continent, the yearly mean temperature $= 270°$ K and $\delta_o = 1{,}6.10^{-4}\ \frac{\text{ton}}{\text{m}^3}$; $\lambda_x = \lambda_y = 3000$ km. Yearly pressure range in the centre of the continent 8 mb (Part J of the H a n d b u c h d e r K l i m a t o l o g i e).

Theoretical results:

a. S i m p l e i n t e g r a t i o n 330 mb.

b. Differential equation 9 mb.

c. Integral equation 18 mb.

Australia. $\tau_o = 18°$ C, the mean yearly temperature is equal to 290° K this leading to a value of δ_o of $0{,}8.10^{-4} \frac{\text{ton}}{\text{m}^3}$. $l = 0{,}7.10^{-4}$, $\lambda_x = 2500$ km and $\lambda_y = 1500$ km. The observed yearly pressure range in the centre of the continent amounts to about 13 mb. It should be borne in mind, however, that part of this pressure variation is due to the seasonal latitude variation of the subtropical high-pressure belt. This follows from the fact that coastal stations also show a yearly pressure variation, up to 7 or 8 mb. We consider, therefore, the real monsoonal pressure variation over Australia to be about 6 mb.

The theoretical results are:

a. Simple integration 900 mb.

b. Differential equation 1 mb.

c. Integral equation 10 mb.

It follows from these results that the third approximation is the best one. This is especially clear if we realise that the yearly pressure range is partly reduced by friction which is not implicated in our considerations. An exception seems to be formed by North America. However, the small pressure variation which is found here is for the greater part due to dynamical processes as many depressions cross the continent in winter. The first approximation which was used by Wexler is insufficient.

We can see from (5.23) that near the borders of the region considered the yearly pressure range has the opposite sign from the centre. This effect is well known, as in Western Europe for instance the yearly temperature and pressure range go parallel. The effect is not found so clearly near the borders of the other continents though it exists in East Asia and in North America.

As for Australia, the effect may be blurred here by the moving subtropical high pressure belt. It only appears that the calculated region with winter pressure rise is somewhat smaller than the observed one. In the case of Eurasia the maximum east-west dimension of the calculated region amounts to 9600 km, the maximum north-south dimension to 5000 km. The real dimensions are about 10 000 km and 7000 km.

It is also clear from (5.23) that the winter pressure rise changes into a fall of pressure at some height while the summer pressure fall becomes a pressure rise in the free atmosphere. Using the same constants as before we find that in the centre of Eurasia the level in which this transition takes place lies about 1500 m above sea-level. It follows from (5.23) that this value decreases with growing distance from the centre. Flohn (22) found the transition level in Jakutsk to lie at about 1000 meters, so that the accordance between theory and reality is also very good in this respect.

The very high pressures which occur frequently in winter over Siberia, Russia and North America may be due to local very strong radiation cooling. It is also possible that we have to do here with the advection of cold air of arctic origin or with processes in higher levels. Sometimes, for example, the Azores anticyclone moves towards the north-east to become stationary over European Russia. On the other hand, the very low pressures which may occur in summer are generally due to normal atmospheric depressions.

We see, therefore, that it is possible to give a satisfactory explanation of the yearly pressure variation over a continent by means of the foregoing approximate calculations.

We may conclude this section by mentioning Jeffreys' classical paper on the general theory of the monsoons (25).

Comparing his results with ours, we see that Jeffreys also starts from the approximation according to which the real wind is equal to the sum of a geostrophic part and an isallobaric one (equations (3) and (4) of section 3.1; γ^2 may be neglected with respect to $4\Omega^2$ as the monsoonal period amounts to a year).

Due to the fact that Jeffreys treats the problem in a rather special way, namely by considering the variations of the quantities $U = \int_o^\infty \varrho u \, dz$, $V = \int_o^\infty \varrho v \, dz$ and $P = \int_o^\infty p \, dz$, he arrives directly at his equation (7) of section (3.1) which equation corresponds with (5.18) of our paper if we integrate the latter to z from o to ∞. We only find a height h of the equivalent ocean of 11 km instead of the 7,3 km Jeffreys found.

This difference is a consequence of the fact that Jeffreys divides the pressure variations that occur in the free atmosphere in isothermal pressure variations and such variations which are connected with the entire temperature variations, that is also with the variations due to vertical motions which we neglected in the first instance. With this method corresponds a smaller value of k_T, so that the final numerical results of both theories are the same. It appears, therefore, that Jeffreys' value of h is equal to $\int_o^z r_1(z) \, dz$ where $r_1(z)$ is a reduction factor analogous to $r(z)$ but referring not to adiabatic but to isothermal pressure variations, so that there is a close connection between his theory and Rossby's advection theory.

Concluding we may say that both monsoon theories lead to the same result. We believe, however, that ours illustrates the meteorological processes which occur in monsoon countries somewhat more clearly. Incidentally we may notice that a shortcoming of both theories is the supposition that vast continents like Eurasia can be considerd as being flat.

6. *Isallobaric effects and cyclones.*

It has been mentioned by many authors that the deepening of cyclones, tropical as well as extra-tropical, may be largely caused by conditional instability. Shaw (50) for instance tried to explain the deepening of tropical cyclones by this mechanism, whereas Refsdal (32) also applied it to the deepening of depressions of moderate latitudes. It may easily be understood, that in the case of deepening by conditional instability the air becomes warmer and expands so that there arises isallobaric divergence in the higher levels.

We can apply formula (5.15) again if only we introduce the right values for the various constants. Temperature rise owing to conditional instability is general small; 5° C may be an average value or even a maximum one for the lowest levels of the atmosphere and the centre of the conditionally unstable region.

As the heating is transported to a very great height, we may take $k_T = \frac{1}{15000} m^{-1}$, a value which may be considered to be a maximum one.

If we introduce reasonable values for the various constants of (5.15) we see that the fall of pressure turns out to be very small, at the low latitudes at which tropical cyclones arise even less than 1 mb in total. We, therefore, come to the conclusion that some other mechanism must be responsible for the deepening of tropical cyclones.

There exist also some other indications that conditional instability does not play a predominant part in the deepening of these disturbances.

First, it is obvious, that the maximum deepening which is possible owing to the decrease of density of the air is just equal to the decrease of the weight of a column of unit square diameter. This means that the maximum fall of pressure in the case of a temperature rise of 5° C taking k_T equal to $\frac{1}{15000}$ m^{-1} amounts to.

$$\Delta p = g \int_0^\infty \Delta \varrho \, dz = g \int_0^\infty - \frac{5}{300} \cdot 1{,}16 \cdot 10^{-3} e^{-\frac{z}{6000}} dz = 11 \, mbar$$

This is much less than the fall of pressure which is generally observed in the centre of tropical cyclones.

Secondly in the centre of a tropical cyclone where the fall of pressure is largest, there is a slow descent of air, causing the so called "eye" of the cyclone. From this we conclude that there exists no divergence but convergence of air in the higher levels above the centre and the shower theory of tropical cyclones does not hold there.

It will be necessary, therefore, to look for other mechanisms in order to explain tropical disturbances. It seems most probable for the moment, that Rodewald's theory (35) stating that they are due to strong divergence of an upper air current is correct. We shall return in some detail to this mechanism in the next chapter.

Perhaps the suggestions of Durst and Sutcliffe (9) indicate a direction in which we shall also find part of the solution. We shall come back to this opinion.

It wil also be difficult to ascribe other phenomena like the strongly deepening cyclones coming from Canada to the heating the air undergoes when the depression crosses the Atlantic, as we can not divide the isallobaric effects from others which may be due to the motion of the depressions. It is most probable that divergence of upper air-currents plays an important role here too. This is also the case with the obviously frontless depressions coming from Arctic regions and moving slowly Southward along the Norwegian coast. Sometimes we also find a shallow "warm water depression" over the North Sea or the Caspian Sea. These depressions may be partly due to false reductions of the surrounding colder inland stations, partly to coastal effects as small temperature differences occurring here over small distances give rise to rather big isallobaric gradients. It is the same effect that gives rise to the nightly landbreeze. Under these circumstances the function $\left(1 + \cos \frac{\pi x}{\lambda_x}\right)\left(1 + \cos \frac{\pi y}{\lambda_y}\right)$ does not form any longer a good approximation for the density profile in the depression.

In concluding we may say, that it appears to be difficult to explain the deepening of tropical or extratropical disturbances with the aid of conditional instability or ordinary heating from below.

PART III

Pressure variations under the influence of horizontal motions

7. *Advection of air of different density.*

The pressure variation at the ground owing to horizontal motions in the atmosphere is given by $\frac{\partial p_0}{\partial t} = -g \int\limits_0^\infty I_2 \, dz$ *with*

$$I_2 = -\frac{1}{l^3}\left[\frac{\partial}{\partial x}\left\{\left(-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}+\frac{\partial p}{\partial x}\frac{\partial}{dy}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{\left(-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}+\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right)\right\}\right] \tag{7.1}$$

As a first approximation we neglect induced effects for the moment. We will return to this question later on.

Working out (7.1) and writing α instead of $\frac{1}{\varrho}$, we get:

$$I_2 = -\frac{1}{l^3}\left[\left(-\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right)\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}+\frac{\partial^2\alpha}{\partial x\partial y}\left\{\left(\frac{\partial p}{\partial x}\right)^2-\left(\frac{\partial p}{\partial y}\right)^2\right\}-2\frac{\partial\alpha}{\partial x}\left(\frac{\partial p}{\partial y}\frac{\partial^2 p}{\partial x^2}+\frac{\partial p}{\partial y}\frac{\partial^2 p}{\partial y^2}\right)+\right.$$

$$\left.+2\frac{\partial\alpha}{\partial y}\left(\frac{\partial p}{\partial x}\frac{\partial^2 p}{\partial x^2}+\frac{\partial p}{\partial x}\frac{\partial^2 p}{\partial y^2}\right)+\alpha\left(-\frac{\partial^3 p}{\partial x^3}\frac{\partial p}{\partial y}+\frac{\partial^3 p}{\partial x^2\partial y}\frac{\partial p}{\partial x}-\frac{\partial^3 p}{\partial x\partial y^2}\frac{\partial p}{\partial y}+\frac{\partial^3 p}{\partial y^3}\frac{\partial p}{\partial x}\right)\right] \tag{7.2}$$

We can distinguish here between terms which depend on the inhomogeneity of the horizontal field of density:

$$I_2' = -\frac{1}{l^3}\left[\left(-\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right)\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}+\frac{\partial^2\alpha}{\partial x\partial y}\left\{\left(\frac{\partial p}{\partial x}\right)^2-\left(\frac{\partial p}{\partial y}\right)^2\right\}-2\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right] \tag{7.3}$$

and such terms which are independent of the horizontal density distribution:

$$I_2'' = -\frac{\alpha}{l^3}\left[-\frac{\partial^3 p}{\partial x^3}\frac{\partial p}{\partial y}+\frac{\partial^3 p}{\partial x^2\partial y}\frac{\partial p}{\partial x}-\frac{\partial^3 p}{\partial x\partial y^2}\frac{\partial p}{\partial y}+\frac{\partial^3 p}{\partial^3 y}\frac{\partial p}{\partial x}\right] \tag{7.4}$$

It is seen from (7.3) that the influence of the advection of air with different density is much more complicated even in our approximation than was proposed by Exner (19, 20, 21) in his famous articles on the subject. Moreover we should realize that the terms depending on $\frac{\partial\alpha}{\partial t}$ which were discussed in 4, are still to add if we want to calculate the entire pressure variation due to mass advection. So, taking account of (4.1) we have to consider the entire expression:

$$I_1' + I_2' = -\frac{1}{l^2}\left[-\frac{1}{\alpha^2}\left(\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial t}\frac{\partial p}{\partial x}+\frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial t}\frac{\partial p}{\partial y}\right)+\frac{1}{\alpha}\left(\frac{\partial^2\alpha}{\partial x\partial t}\frac{\partial p}{\partial x}+\frac{\partial^2\alpha}{\partial y\partial t}\frac{\partial p}{\partial y}\right)+\frac{1}{\alpha}\frac{\partial\alpha}{\partial t}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right]-$$

$$-\frac{1}{l^3}\left[\left(-\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}\right)\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}+\frac{\partial^2\alpha}{\partial x\partial y}\left\{\left(\frac{\partial p}{\partial x}\right)^2-\left(\frac{\partial p}{\partial y}\right)^2\right\}-2\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right] \tag{7.5}$$

It is immediately clear that owing to (7.5) a field with equidistant isosteres perpendicular to equidistant isobars, the case Exner considered, does not give rise to any pressure variation in our approximation. If isosteres nor isobars are equidistant and denser air is advected, both positive and negative pressure variations are possible. The sign which will occur, merely depends on the second derivatives of p. We see that I_1' depends on the local density variations which are caused by the advection, while I_2' is connected with the fact that the air moves in a field with variable density.

It will be a little strange at first sight, that no term is contained in (7.5) representing the pressure variation by advection as we normally imagine it, that is by geostrophical advection of air of different density (fig. 2). For if $\alpha_1 > \alpha_3$, that is $\varrho_1 < \varrho_3$, we should expect, and in weather practice we nearly always do expect, a rising barometer as a consequence of the advection of the heavier air. If no increase of pressure occurs we ascribe this fact to "processes in the upper atmosphere" a normal mode to explain unexpected occurrences in meteorology. We can easily show, however, that it will generally be difficult to draw conclusions as to pressure variations from the distribution of isobars and isodenses. For, $\frac{\partial \varrho}{\partial t} = -\left[\frac{\partial \varrho v_x}{\partial x} + \frac{\partial \varrho v_y}{\partial y}\right]$, or if for simplicity we take the isosteres perpendicular to the isobars $\frac{\partial \varrho}{\partial t} = - v_y \frac{\partial \varrho}{\partial y} - \varrho \frac{\partial v_y}{\partial y}$ and with $v_y = \frac{1}{\varrho l}\frac{\partial p}{\partial x}$ and $\frac{\partial^2 p}{\partial x \partial y} = 0$ we get $\frac{\partial \varrho}{\partial t} = -\frac{1}{\varrho l}\frac{\partial p}{\partial x}\frac{\partial \varrho}{\partial y} + \frac{1}{\varrho l}\frac{\partial p}{\partial x}\frac{\partial p}{\partial y} = 0.$

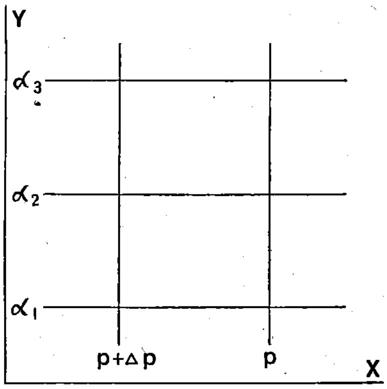

Figure 2.

This elementary example may serve to prove that things are not so simple with mass advection as they often are considered to be. If we have a field of density with equidistant isosteres perpendicular to the isobars and we put the axes of the coordinate system as indicated in fig. 2, we see that $\frac{\partial^2 p}{\partial y^2}$ must differ from zero if any pressure variation is to arise. The sign of $\frac{\partial p_0}{\partial t}$ depends on that of $\frac{\partial^2 p}{\partial y^2}$ then. Introducing reasonable values for the various quantities we find that pressure variations due to mass advection in an atmosphere without frontal surfaces to be very insignificant.

If for example we take a horizontal temperature gradient of 3° C per 1000 km, in such a way that colder air, 5000 m deep, is advected, a pressure gradient $-\frac{\partial p}{\partial y}$ equal to 2 mb per 100 km, that is 2.10^6 in the meter-ton-second system and $\frac{\partial^2 p}{\partial y^2}$ equal to 10^{-12} in the same system we find $\frac{\partial p_0}{\partial t} = -g \int_0^{5000} (I_1' + I_2')\, dz = 1$ mb per day. In this calculation we did not even take account of the compensating isallobaric effects which are always present when a pressure changing mechanism comes into being. With the opposite sign of $\frac{\partial^2 p}{\partial y^2}$ we would have found a fall of pressure of 1 mb per day.

We may say, therefore, that the pressure variations due to mass advection depend largely on the exact distribution of isobars, while generally speaking these pressure variations are insignificantly small.

8. *Divergence or convergence of isobars.*

Of much greater importance then the advection of air of different density is the contribution to $\frac{\partial p_0}{\partial t}$ owing to

$$I_2'' = -\frac{\alpha}{l^3}\left[-\frac{\partial^3 p}{\partial x^3}\frac{\partial p}{\partial y} + \frac{\partial^3 p}{\partial x^2 \partial y}\frac{\partial p}{\partial \alpha} - \frac{\partial^3 p}{\partial x \partial y^2}\frac{\partial p}{\partial y} + \frac{\partial^3 p}{\partial y^3}\frac{\partial p}{\partial x}\right] \tag{8.1}$$

It can be easily seen that this expression depends on the distribution of the isobars. In the case of rectilinear equidistant isobars, the pressure distribution with which geostrophic wind is possible, I_2'' is zero.

In order to recognize the exact meaning of (8.1) let us consider fig. 3. In this figure s_1, s_1', s_2 and s_2' are elements of two isobars; G_1, G_1', G_2 and G_2' denote negative pressure gradients in the midst of the isobaric elements s. These elements s are chosen in such a way that both areas F_1 and F_2 are equal to each other, say F. The line elements l, m and n are perpendicular to the isobars. In order to evaluate I_2 in P we

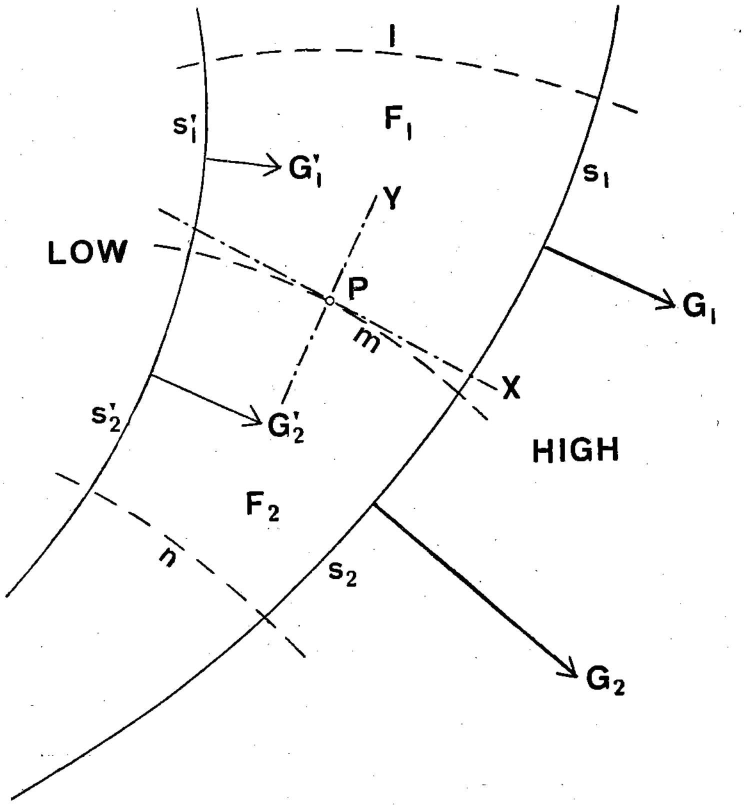

Figure 3.

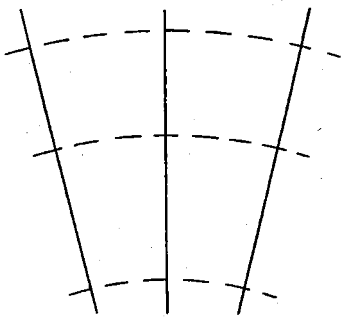

Figure 4.

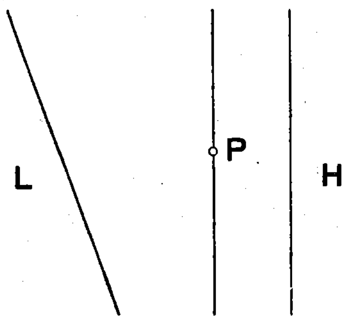

Figure 5.

choose the coordinates in such a way that the origin of the coordinate system coincides with P, the y-axis being the tangent of the isobar through P.

As in $P \frac{\partial p}{\partial y} = 0$, I_2'' reduces to $-\frac{\alpha}{l^3}\frac{\partial p}{\partial x}\left[\frac{\partial}{\partial y}\Delta p\right]$, Δp being $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \equiv div \nabla p \equiv div\, G$. Now from the figure it follows, that

$$\frac{\partial}{\partial y} div\, G = \frac{(div\, G)_1 - (div\, G)_2}{\frac{1}{2}(s_1 + s_2)} = \left[\frac{s_1 G_1 - s_1' G_1'}{F_1} - \frac{s_2 G_2 - s_2' G_2'}{F_2}\right]\frac{2}{s_1 + s_2}$$

or

$$I_2'' = -\frac{\alpha}{l^3}\frac{G_1 + G_2}{s_1 + s_2}\left[\frac{s_1 G_1 - s_1' G_1'}{F} - \frac{s_2 G_2 - s_2' G_2'}{F}\right] \tag{8.2}$$

We can use this relation to determine the sign of $\frac{\partial p_0}{\partial t}$ from the high level maps. It is immediately clear that (8.2) is not in accordance with Scherhag's rule saying that surface pressure falls when the isobars in the higher levels diverge.

We can see from our formula for instance that symmetrically diverging isobars do not lead to any pressure variation at all (fig. 4). For in that case $s_1 = s_1'$; $s_2 = s_2'$ and $G_1 = G_1'$; $G_2 = G_2'$. Another example, showing that Scherhag's simple rule is not quite correct is offered by fig. 5. In this situation, in which the lines represent upper isobars, surface pressure rises in P in spite of the divergence. This case was already mentioned by Sieber (51). A third example is offered by equidistant isobars showing variation of curvature (fig. 6). As G is constant and $s_2 > s_2'$, $s_1 > s_1'$, $s_1' < s_2'$ and $s_1 > s_2$ we have $[(s_1 - s_1')\, G - (s_2 - s_2')\, G] < 0$ so that I_2'' is negative and surface pressure rises. This result can also be found in Brunt (6) and goes back on

B o y d e n (5). If on the contrary the air is moving around an anticyclone, every increase of isobaric curvature leads to a fall of surface pressure there.

If we call the divergence *uniform* when $G_1 : G_1' = G_2 : G_2'$ and if the isobars can be considered to be straight lines, we can find a simple rule concerning the influence of uniform divergence on suface pressure, a rule which takes the place of S c h e r h a g's. For in that case $s_2G_2 > s_1G_1$ and $s_2'G_2' > s_1'G_1'$, so that according to (8.2) the sign of $\frac{\partial p_0}{\partial t}$ is determined by $G_2^2 - G'^2_2$ as s is approximately proportional to G (fig. 7). Then the rule says:

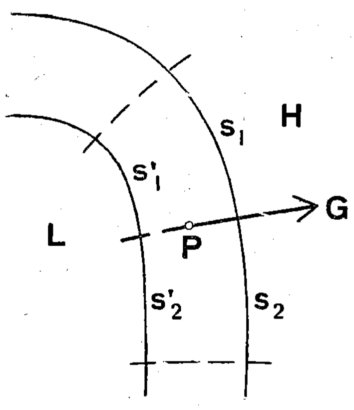

Figure 6.

a. If the high level isobars diverge uniformly, surface pressure falls (rises) if the largest gradients lie on the high (low) pressure side,

and inversely:

b. If the high level isobars converge uniformly, surface pressure rises (falls) if the largest gradients lie on the high (low) pressure side.

Nevertheless S c h e r h a g's simple rule often shows good results. This is a consequence of the fact that near the centre of a depression the gradient distribution is such that divergent isobars give rise to pressure fall indeed.

If we want to apply the divergence-theory to a map on which the isohypses of an isobaric surface are represented we have to transform (8.1) accordingly. As $dz = -\frac{\alpha}{g}dp$ and we neglect horizontal differentiations of α we get:

$$*I_2'' = \frac{g^3}{l^3}\frac{1}{\alpha}\left[-\frac{\partial^3 z}{\partial x^3}\frac{\partial z}{\partial y} + \frac{\partial^3 z}{\partial x^2 \partial y}\frac{\partial z}{\partial x} - \frac{\partial^3 z}{\partial x \partial y^2}\frac{\partial z}{\partial y} + \frac{\partial^3 z}{\partial y^3}\frac{\partial z}{\partial x}\right] \tag{8.3}$$

where the differential quotients $\frac{\partial^{n+m} z}{\partial x^n \partial y^m}$ are taken along the isobaric surface.

We must realise that it is not only the field of pressure in one level which determines the pressure variations at the ground but the sum of the influences of all such fields from the ground to the upper limit of the atmosphere. Some meteorologists suppose that the course of the isohypses of the 500 mb surface are characteristic for the stratosphere and they consider the good results they obtain by using this surface as a proof of the fact that the stratosphere is of prime importance in atmospheric dynamics. It is clear, however, that the influence of the troposphere which contains $\frac{4}{5}$ part of the total mass of the atmosphere must be greater. This also holds for I_2'' which decreases with decreasing p. On the contrary, the fact that the 500 mb suface, which marks approximately the baric middle of the atmosphere, appears to be representative can be considered as a proof of the fact that the whole atmosphere influences the pressure variations at the ground and, therefore, of the relative importance of the troposphere. Owing to this it is not likely, that an extension of our aerological information to higher levels by using radio meteorographs will enable us to use the divergence criterion with much more success. On the other hand it is clear, that the use of aerological data of higher levels will doubtlessly improve our insight in weather and inform us on the motion of the centres of action.

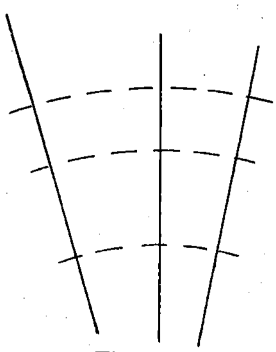
Figure 7.

There are several atmospheric phenomena that can be better understood qualitatively if we apply the foregoing reasonings. An example is offered by the displacement of troughs, in front of which the isobars diverge and in the back of which they converge so that according to our rule near the depression centre pressure will fall in front of the trough whereas it will rise in the back of it. It is clear that the trough must move more slowly than corresponds with the gradient wind velocity as the air must move through the isobaric field if the divergence and convergence are to come into action. This can explain the fact well known from the weather map, that troughs always move much more slowly than fronts, the pressure distribution being about the same.

If the distribution of the isobars is unsymmetrical the trough may deepen of fill.

Of course in order to obtain quantitative results it would be necessary to know the field of pressure in the whole atmosphere above the trough. In practice we can determine the velocity of a trough best by applying kinematical analysis. Finally it may be worth remarking that it is J. Bjerknes (3) who was the first to ascribe the displacement of troughs to divergence in their front and convergence in their back.

9. *The steering and deepening of disturbances.*

It appears also to be possible to explain the steering and the intensity variations of atmospheric disturbances with the aid of our divergence-convergence considerations. As was the case with the monsoonal pressure variations we can even arrive at quantitative results which do not differ too much from empirical data.

The most general form of steering is found when a depression moves anticyclonically around a warm "high". An almost classical example, though not always recognized as such, is offered by the motion of tropical cyclones around the subtropical high pressure cells. The disturbance happens to follow the high level isobars of the warm "high" almost exactly.

Under certain circomstances the steering upper current shows a strong divergence. Rodewald (42) was the first to extend Scherhag's rule for divergent, isobars, saying:

A depression arriving into a divergent upper current, the so called delta of the frontal zone, is going to deepen.

In a frontal zone we have to understand a region in which great wind velocities occur owing to the presence of a frontal surface, so that the frontal zone is much more extensive than the transition layer of which the frontal surface forms the idealized picture. Such frontal zones often occur near the South East coast of the United States, where the frontal zone broadens eastward, causing the upper current to diverge.

Again according to Rodewald (35) the divergence of the upper current which must be regarded as the cause of the deepening of tropical cyclones, is due to fronts. He supposes that three air masses of different origin are in contact with each other in the relative situations, causing strong divergence near the triple-point (Dreimasseneck).

Probably the depressions mentioned already which move from the Arctic Ocean southwards along the Norwegian coast also deepen as a consequence of a divergent upper current. If there lies a cold anticyclone over Scandinavia while the warmer Atlantic air is rather homogeneous it is easy to see how the upper current will give rise to southward moving disturbances and how it will be divergent.

The considerations of Rodewald on the effect of divergent upper isobars on the deepening of depressions were merely qualitative. If we want to make a quantitative statement on the displacement and the deepening of pressure systems we are not allowed to restrict ourselves to the rather qualitative view of the foregoing section. On the contrary, as in the case of the monsoonal pressure variations, we shall have to take account of induced isallobaric effects as divergence in the free atmosphere will give rise to vertical motions.

In order to be able to compute the combined effect we add two fields of pressure, namely the divergent upper current which by itself does not give rise to any pressure variation at the ground and the "disturbance" in the form of a cyclone. The first field can be represented by

$$p_1(z) = p_o\, e^{-k_o z} - \left[\left\{c_1 + c_1'\left(1 + \frac{x}{\lambda}\right)\right\} \sin\frac{\pi y}{2\lambda} \sin\frac{\pi z}{\zeta}\right] e^{-k_1 z} \tag{9.1}$$

whereas the disturbance is given by

$$p_2(z) = -\tfrac{1}{4}\, c_2 \left[\left(\cos\frac{\pi x}{\lambda} + 1\right)\left(\cos\frac{\pi y}{\lambda} + 1\right)\right] e^{-k_2 z} \tag{9.2}$$

In these two relations c_1 and c_2 are a measure for the pressure gradients occurring in the frontal zone and the disturbance respectively; c_1' is a measure for the rate of divergence in the upper current. λ marks the radius of the depression whereas k_o, k_1, k_2 denote the decrease of the various pressure systems with growing height. It is clear, that the introduction of a pressure distribution in the depression according to trigonometric functions forms only a rough approximation of the pressure distributions found in reality. It is necessary to introduce it, however, to make calculations possible. It is always possible to generalise the considerations by writing

$$p_2(z) = -\tfrac{1}{4}\, c_2\, [\varphi x - \varphi y]\, e^{-k_2 z} \tag{9.2'}$$

where φ and φ_y denote the Fourier series for the cyclonic cross-sections $y = 0$ and $x = 0$.

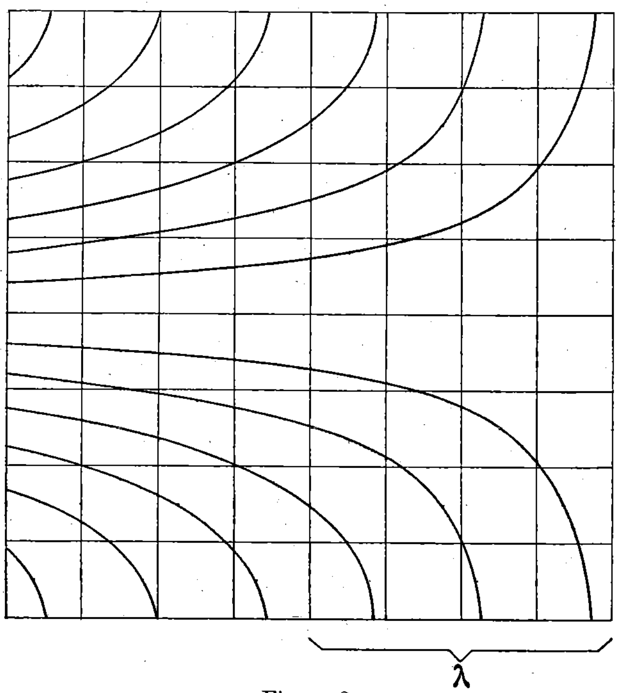

Figure 8.

The introduction of the factor $\sin\frac{\pi z}{\zeta}$ in the expression for the divergent field of pressure in the frontal zone is to indicate the fact that the pressure gradient being zero at the ground increases with growing height to a maximum value at an altitude $z_m = \frac{\zeta}{\pi}$ arc. tg $\frac{\pi k_1}{\zeta}$. Taking ζ equal to 20 000 m and $k_1 = 0{,}9 . 10^{-4}\text{m}^{-1}$ we find the maximum pressure gradient at 6666 m whereas its direction changes at 20 000 m. The second maxium is found at at height of 26 666 m, its absolute value being 0,16 of that at 6666 m, whereas the third maximum at 46 666 m has only a value of 0,03 times that at 6666 m. This means, that the variation of the pressure gradient with height as we find it in moderate latitudes (westerly winds in the troposphere and the lower part of the stratosphere and easterly winds above) is sufficiently correctly represented by the factor $\sin\frac{\pi z}{\zeta}\, e^{-k_1 z}$. Fig. 8 shows the course of the upper isobars according to (9.1).

The total field of pressure is given by the addition of the fields (9.1) and (9.2):

$$P(z) = p_o\, e^{-k_o z} - \left[\left\{c_1 + c_1'\left(1 + \frac{x}{\lambda}\right)\right\} \sin\frac{\pi y}{2\lambda} \sin\frac{\pi z}{\zeta}\right] e^{-k_1 z} - \tfrac{1}{4}\, c_2 \left[\left(\cos\frac{\pi x}{\lambda} + 1\right)\left(\cos\frac{\pi y}{\lambda} + 1\right)\right] e^{-k_2 z} \tag{9.3}$$

According to the developments made so far in this chapter, the pressure variation at the ground should be given by

$$\frac{\partial P_o}{\partial t} = \frac{g}{l^3} \int_o^\infty \alpha \left[-\frac{\partial^3 P}{\partial x^3}\frac{\partial P}{\partial y} + \frac{\partial^3 P}{\partial x^2 \partial y}\frac{\partial P}{\partial x} - \frac{\partial^3 P}{\partial x \partial y^2}\frac{\partial P}{\partial y} + \frac{\partial^3 P}{\partial y^3}\frac{\partial P}{\partial x}\right] dz \tag{9.4}$$

It appears, however, that in applying this relation, we find far too great values for the pressure variations in a diverging isobaric system. As was mentioned already, this comes from the fact, that we are not allowed to ignore the compensating isallobaric effects which occur owing to the divergence and convergence in the atmosphere. In analogy with equation (5.18) we shall have to solve the following more general equation in which $q(z)$ stands for $\frac{\partial P_z}{\partial t}$, the local pressure variation; and in which Rossby's advectiontheory is contained:

$$q(z) = \frac{g}{l^3}\int_0^\infty \alpha \left[-\frac{\partial^3 P}{\partial x^3}\frac{\partial P}{\partial y} + \frac{\partial^3 P}{\partial x^2 \partial y}\frac{\partial P}{\partial x} - \frac{\partial^3 P}{\partial x \partial y^2}\frac{\partial P}{\partial y} + \frac{\partial^3 P}{\partial y^3}\frac{\partial P}{\partial x} \right] e^{-k_\pi z}\, d\xi + \frac{g}{l^2}\int_0^\infty e^{-k_\pi z}\cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q_\xi d\xi \quad (9.5)$$

In this equation k_π is again equal to $\frac{1}{11\,000}$ if z is expressed in metres.

From the fact that (9.5) must be satisfied for all values of x and y, it follows that q_z must be composed of 12 parts q_i between which exists the relation:

$$q(z) = q_1 \sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\left(\cos\frac{\pi y}{\lambda} - \cos\frac{\pi x}{\lambda}\right) + q_2 \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} + q_3 \cos\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} + q_4 \sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} +$$

$$+ q_5 \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} + q_6 \cos\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} + q_7 \cos\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} + q_8\, x \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} +$$

$$+ q_9\, x \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} + q_{10}\, x \sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} + q_{11}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} + q_{12}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} \quad (9.6)$$

The quantities q_i must be determined from the following twelve equations:

$$q_1(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[\frac{1}{16}\frac{\pi^4}{\lambda^4}c_2^2 e^{-2k_2\xi}\right]d\xi + \frac{5g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_1(\xi)\, d\xi$$

$$q_2(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[-\frac{3}{32}\frac{\pi^4}{\lambda^4}(c_1 + c_1')\, c_2 \sin\frac{\pi\xi}{\zeta} e^{-(k_1+k_2)\xi}\right]d\xi + \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_2(\xi)\, d\xi$$

$$q_3(z) = \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_3(\xi)\, d\xi - \frac{2g}{l^2}\frac{\pi}{\lambda}\int_0^\infty e^{-k_\pi z} q_8(\xi)\, d\xi$$

$$q_4(z) = \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_4(\xi)\, d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_5(\xi)\, d\xi$$

$$q_5(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[-\frac{7}{32}\frac{\pi^4}{\lambda^4}(c_1 + c_1')\, c_2 \sin\frac{\pi\xi}{\zeta} e^{-(k_1+k_2)\xi}\right]d\xi + \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_5(\xi)\, d\xi -$$

$$- \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_4(\xi)\, d\xi$$

$$q_6(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[\frac{7}{16}\frac{\pi^3}{\lambda^3}\frac{c_1'}{\lambda} c_2 \sin\frac{\pi\xi}{\zeta} e^{-(k_1+k_2)\xi}\right]d\xi + \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_6(\xi)\, d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_7(\xi)\, d\xi -$$

$$- 2\frac{g}{l^2}\frac{\pi}{\lambda}\int_0^\infty e^{-k_\pi z} q_{10}(\xi)\, d\xi$$

$$q_7(z) = \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_7(\xi)\,d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_6(\xi)\,d\xi - 2\frac{g}{l^2}\frac{\pi}{\lambda}\int_0^\infty e^{-k_\pi z} q_9(\xi)\,d\xi$$

$$q_8(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[-\frac{3}{32}\frac{\pi^4}{\lambda^4}\frac{c_1'}{\lambda}c_2\sin\frac{\pi\xi}{\zeta}e^{-(k_1+k_2)\xi}\right]d\xi + \frac{5}{4}\frac{g}{l_2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_8(\xi)\,d\xi$$

$$q_9(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[-\frac{7}{32}\frac{\pi^4}{\lambda^4}\frac{c_1'}{\lambda}c_2\sin\frac{\pi\xi}{\zeta}e^{-(k_1+k_2)\xi}\right]d\xi + \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_9(\xi)\,d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_{10}(\xi)\,d\xi$$

$$q_{10}(z) = \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_{10}(\xi)\,d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_9(\xi)\,d\xi$$

$$q_{11}(z) = -\frac{g}{l^3}\int_0^\infty e^{-k_\pi z}\alpha\left[\frac{3}{16}\frac{\pi^3}{\lambda^3}\frac{c_1'}{\lambda}c_2\sin\frac{\pi\xi}{\zeta}e^{-(k_1+k_2)\xi}\right]d\xi + \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_{11}(\xi)\,d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_{12}(\xi)\,d\xi$$

$$q_{12}(z) = \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_{12}(\xi)\,d\xi - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\int_0^\infty e^{-k_\pi z} q_{11}(\xi)\,d\xi \tag{9.7}$$

If we put the specific volume α equal to $\alpha_o\, e^{+k\xi}$, α_o being the value of α at the ground and if we call $Q_i = \int_0^\infty q_i(\xi)\,d\xi$, we can easily write down twelve equations from which the 12 Q_i's can be computed:

$$Q_1 = -\frac{g}{l^3}\alpha_o\frac{1}{16}\frac{\pi^4}{\lambda^4}c_2^2\frac{1}{2k_2-k}\cdot\frac{1}{k_\pi} + \frac{5g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_1$$

$$Q_2 = \frac{g}{l^3}\alpha_o\frac{3}{32}\frac{\pi^4}{\lambda^4}(c_1+c_1')c_2\frac{\pi/\zeta}{(k_1+k_2-k)^2+\pi^2/\zeta^2}\frac{1}{k_\pi} + \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_2$$

$$Q_3 = \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_3 - \frac{2g}{l^2}\frac{\pi}{\lambda}\frac{1}{k_\pi}Q_8 \qquad Q_4 = \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_4 - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_5$$

$$Q_5 = \frac{g}{l^3}\alpha_o\frac{7}{32}\frac{\pi^4}{\lambda^4}(c_1+c_1')c_2\frac{\pi/\zeta}{(k_1+k_2-k)^2+\pi^2/\zeta^2}\cdot\frac{1}{k_\pi} + \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_5 - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_4$$

$$Q_6 = -\frac{g}{l^3}\alpha_o\frac{7}{16}\frac{\pi^3}{\lambda^3}\frac{c_1'}{\lambda}c_2\frac{\pi/\zeta}{(k_1+k_2-k)^2+\pi^2/\zeta^2}\cdot\frac{1}{k_\pi} + \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_6 - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_7 - 2\frac{g}{l^2}\frac{\pi}{\lambda}\frac{1}{k_\pi}Q_{10}$$

$$Q_7 = \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_7 - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_6 - 2\frac{g}{l^2}\frac{\pi}{\lambda}\frac{1}{k_\pi}Q_9$$

$$Q_8 = \frac{g}{l^3}\alpha_o\frac{3}{32}\frac{\pi^4}{\lambda^4}\frac{c_1'}{\lambda}c_2\frac{\pi/\zeta}{(k_1+k_2-k)^2+\pi^2/\zeta^2}\cdot\frac{1}{k_\pi} + \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_8$$

$$Q_9 = \frac{g}{l^3}\alpha_o\frac{7}{32}\frac{\pi^4}{\lambda^4}\frac{c_1'}{\lambda}c_2\frac{\pi/\zeta}{(k_1+k_2-k)^2+\pi^2/\zeta^2}\cdot\frac{1}{k_\pi} + \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_9 - \frac{9}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_{10}$$

$$Q_{10} = \frac{9}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_{10} - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_9$$

$$Q_{11} = -\frac{g}{l^3}\alpha_o\frac{3}{16}\frac{\pi^3}{\lambda^3}\frac{c_1'}{\lambda}c_2\frac{\pi/\zeta}{(k_1+k_2-k)^2+\pi^2/\zeta^2}\cdot\frac{1}{k_\pi} + \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_{11} - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_{12}$$

$$Q_{12} = \frac{5}{4}\frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_{12} - \frac{g}{l^2}\frac{\pi^2}{\lambda^2}\frac{1}{k_\pi}Q_{11} \tag{9.8}$$

In order to simplify the formulae we rewrite (9.8) using as abbrevations new symbols in the following way:

$$Q_1 = -\delta_1 + 5\,a\,Q_1 \qquad Q_5 = \delta_5 + {}^9/_4\,a\,Q_5 - a\,Q_4 \qquad Q_9 = \delta_9 + {}^9/_4\,a\,Q_9 - a\,Q_{10}$$

$$Q_2 = \delta_2 + {}^5/_4\,a\,Q_2 \qquad Q_6 = -\delta_6 + {}^9/_4\,a\,Q_6 - a\,Q_7 - b\,Q_{10} \qquad Q_{10} = {}^9/_4\,a\,Q_{10} - a\,Q_9$$

$$Q_3 = {}^5/_4\,a\,Q_3 - b\,Q_8 \qquad Q_7 = {}^9/_4\,a\,Q_7 - a\,Q_6 - b\,Q_9 \qquad Q_{11} = -\delta_{11} + {}^5/_4\,a\,Q_{11} - a\,Q_{12}$$

$$Q_4 = {}^9/_4\,a\,Q - a\,Q_5 \qquad Q_8 = \delta_8 + {}^5/_4\,a\,Q_8 \qquad Q_{12} = {}^5/_4\,a\,Q_{12} - a\,Q_{11} \tag{9.9}$$

We find the following values for Q_i:

$$Q_1 = \frac{\delta_1}{5a-1} \qquad Q_2 = \frac{-\delta_2}{{}^5/_4\,a-1} \qquad Q_3 = \frac{-b\,\delta_8}{({}^5/_4\,a-1)^2} \qquad Q_4 = \frac{-a\,\delta_5}{({}^9/_4\,a-1)^2-a^2} \qquad Q_5 = \frac{-({}^9/_4\,a-1)\,\delta_5}{({}^9/_4\,a-1)^2-a^2}$$

$$Q_6 = \frac{({}^9/_4\,a-1)\left\{\delta_6 - b\,\dfrac{a\,\delta_9}{({}^9/_4\,a-1)^2-a^2}\right\} - ab\,\dfrac{({}^9/_4\,a-1)\,\delta_9}{({}^9/_4\,a-1)^2-a^2}}{({}^9/_4\,a-1)^2-a^2}$$

$$Q_7 = \frac{a\left\{\delta_6 - b\,\dfrac{a\,\delta_9}{({}^9/_4\,a-1)^2-a^2}\right\} - ({}^9/_4\,a-1)\,b\,\dfrac{({}^9/_4\,a-1)\,\delta_9}{({}^9/_4\,a-1)^2-a^2}}{({}^9/_4\,a-1)^2-a^2}$$

$$Q_8 = -\frac{\delta_8}{{}^5/_4\,a-1} \qquad Q_9 = \frac{-({}^9/_4\,a-1)\,\delta_9}{({}^9/_4\,a-1)^2-a^2} \qquad Q_{10} = \frac{-a\,\delta_9}{({}^9/_4\,a-1)^2-a^2} \qquad Q_{11} = \frac{({}^5/_4\,a-1)\,\delta_{11}}{({}^5/_4\,a-1)^2-a^2}$$

$$Q_{12} = \frac{a\,\delta_{11}}{({}^5/_4\,a-1)^2-a^2} \tag{9.10}$$

Now the same relations hold for the expressions $\frac{1}{k_\pi}\,e^{k_\pi z}\,q_i(z)$ as can be easily understood by comparing equations (9.7) with (9.8).

We find, therefore, the following expression for $q\,(z)$ which forms the basis for numerical considerations. As a is always large compared with 1, we can simplify the final result as follows:

$$\frac{1}{k_\pi}\,e^{-k_\pi z}\,q(z) = \frac{1}{5}\frac{\delta_1}{a}\sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\left(\cos\frac{\pi y}{\lambda} - \cos\frac{\pi x}{\lambda}\right) - \frac{4}{5}\frac{\delta_2}{a}\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} - \frac{16}{25}\frac{b\,\delta_8}{a^2}\cos\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} -$$

$$-\frac{16}{65}\frac{\delta_5}{a}\sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} - \frac{36}{65}\frac{\delta_5}{a}\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} + \left\{\frac{36}{65}\frac{\delta_6}{a} - \frac{16.72}{65^2}\frac{b\,\delta_9}{a^2}\right\}\cos\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} +$$

$$+\left\{\frac{16}{65}\frac{\delta_6}{a} - \frac{16.97}{65^2}\frac{b\,\delta_9}{a^2}\right\}\cos\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} - \frac{4}{5}\frac{\delta_8}{a}\,x\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} - \frac{36}{65}\frac{\delta_9}{a}\,x\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} -$$

$$-\frac{16}{65}\frac{\delta_9}{a}\,x\sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} + \frac{20}{9}\frac{\delta_{11}}{a}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} + \frac{16}{9}\frac{\delta_{11}}{a}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} \tag{9.11}$$

In order to understand better the meaning of (9.11) we do best to compare the value of $\left(\frac{\delta P}{\delta t}\right)_{z=0}$ that follows from this expression with the pressure variation we find in an empirically investigated situation. In doing so we must, of course, always realise, that the pressure pattern we introduced just to make calculations possible will usually differ widely from the much more irregular pressure distributions which are found in reality on the weather map. For λ for instance we will have to take some mean value of the various distances at which the surrounding anticyclones are found from the depression centre. Moreover, the pressure distribution in a depression will usually not show the harmonic profile. Nevertheless we may expect that if our approximation is sufficient to a first degree we will find an explanation in (9.11) of things happening when a depression moves in a delta of a frontal zone.

We compare, therefore, the results of (9.11) with the deepening that occurred in the depression that passed cape Hatteras on January 16 and 17, 1939, and which was investigated by Rodewald (42). We adapt our theoretical pressure distribution best to the empirical one by introducing the following values for the various constants, which are mean values for the period Rodewald considered:

$g = 9{,}8 \text{ m sec}^{-1}$	$\lambda = 2 \,.\, 10^6 \text{ m}$	$k_2 = {}^3/_2 \,.\, 10^{-4} \text{ m}^{-1}$	$c_1' = -\,14 \text{ cb}$
$l = 0{,}85 \,.\, 10^{-4} \text{ sec}^{-1}$	$k = 1{,}1 \,.\, 10^{-4} \text{ m}^{-1}$	$\zeta = 2 \,.\, 10^{-4} \text{ m}$	$c_2 = 2{,}5 \text{ cb}$
$\alpha_0 = 0{,}82 \,.\, 10^3 \text{ m}^3 \,.\, \text{ton}^{-1}$	$k_1 = 0{,}9 \,.\, 10^{-4} \text{ m}^{-1}$	$c_1 = 30 \text{ cb}$	

The last four values especially are rather uncertain. As we indicated already more generally, the divergence that was present on January 16. is not of the shape represented in fig. 8. At a distance of about 1000 km ($\frac{1}{2}\lambda$) east of the depression centre the divergence comes to an end altogether, in striking contrast with fig. 8. Near the depression centre the degree of divergence is larger than corresponds with a value of $c_1' = -\,14$ cb. A value of $-\,20$ cb would do better there. Adopting the above values of c_1 and c_1' we have only described a mean divergence.

At $x = y = 0$ we find by differentiation of (9.1) at 5000 m a pressure gradient of $16 \,.\, \sin \frac{\pi \,.\, 5000}{\zeta} \,.\, e^{-5000 k_1} \,.\, \frac{\pi}{2\lambda} \,.\, 10^5 = 0{,}57$ cb per 100 km, this value agreeing well with the observed inclination of the 500 mb surface of about 70 gdm per 100 km.

c_2 is found by taking the mean value of the pressure differences between the depression and the surroudning anticyclones.

Introducing into (9.11) the values thus fixed for the various constants we get the following relation for $\left(\frac{\delta P}{\delta t}\right)_{z=0}$, expressed in 10^{-5} mb per second:

$$\left(\frac{\delta P}{\delta t}\right)_{z=0} = +\,0{,}91 \sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\left(\cos\frac{\pi y}{\lambda} - \cos\frac{\pi x}{\lambda}\right) - 23{,}6 \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} + 10{,}0 \cos\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} -$$
$$-\,16{,}4 \sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} - 37{,}3 \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} - 10{,}9 \cos\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} + 4{,}5 \cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} +$$
$$+\,19{,}1 \frac{x}{\lambda}\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{2\lambda} + 32{,}7 \frac{x}{\lambda}\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} + 14{,}5 \frac{x}{\lambda}\sin\frac{\pi x}{\lambda}\sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} - 35{,}4 \sin\frac{\pi y}{\lambda}\sin\frac{\pi y}{2\lambda} -$$
$$-\,28{,}2 \cos\frac{\pi y}{\lambda}\cos\frac{\pi y}{2\lambda} \qquad (9.8)$$

In fig. 9 the distribution of $\left(\frac{\delta P}{\delta t}\right)_{z=0}$ expressed in 10^{-4} mb per second over the considered region of 4000 km square is represented. It is seen, that in the centre of the depression, ($x = y = 0$), $\left(\frac{\delta P}{\delta t}\right)_o$ is equal to $-\,13{,}7 \,.\, 10^{-5}$ mb per second. Now $\left(\frac{\delta P}{\delta t}\right)_o$ is proportional to c_2. If we suppose, therefore, that the general divergent field of pressure $p_1(z)$ as well as the cross section through the depression $p_2(z)$ remain unchanged and that the depression does not move too quickly through $p_1(z)$, the following relation is valid:

$$\int_o^t \left(\frac{\delta P}{\delta t}\right)_o \delta t = -\,\mu \int_o^t \left\{ c_2 - \int_o^t \left(\frac{\delta P}{\delta t}\right)_o dt \right\} dt$$

where μ is equal to $\frac{13{,}7 \,.\, 10^{-5}}{c_2}$. Integrating from $t = 0$ to $t = 86\,400$ we find a deepening of 23 mb during a day. In reality the depression near cape Hatteras deepened about 40 mb in 24 hours. The difference must be ascribed to the inconsistencies we mentioned already such as the shape of the pressure curve representing the depression and the fact that the depression is moving through the upper field of pressure. Nevertheless, it appears that we are able to calculate the deepening effect which diverging frontal zones have on depressions moving along them at least qualitatively. This result may be considered as a theoretical proof of Rodewald's empirical rule.

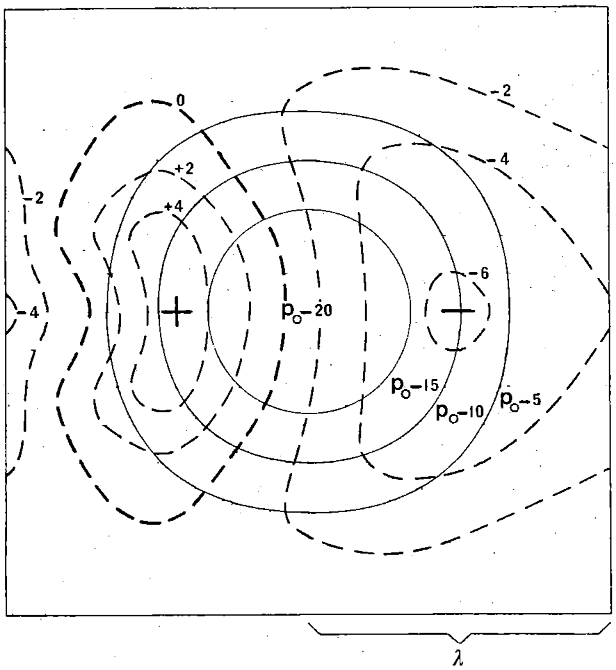

Figure 9.

We can see from fig. 9 that pressure falls more in advance of the depression than it does in the rear. This means that the deepening depression is moving in the direction of the positive X-axis. The velocity is about 30 m per second, a value much higher than in reality where the depression moved with a velocity of 10 m per second during the twelve hours of strongest deepening. We must realise, however, that as the depression is moving on, it comes quickly into a region with a less strong upper current, which reduces the above value. If we had taken the value $c_1' = -20$ cb we would have found a deepening of 33 mb within 24 hours, and a velocity of displacement of the depression of 20 m/sec during the first hours, both values being in better agreement with reality.

If we put $c_1' = 0$ we find mere steering without deepening of the depression parallel to the high level isobars.

As for an anticyclone c_2 has the opposite sign as for a cyclone, it is clear from (9.7) that an anticyclone will get stronger when moving along a divergent upper current whereas both a cyclone and an anticyclone will decrease when moving along a convergent upper current. It may be embarrassing at first sight that an anticyclone intensifies in a divergent current. In practice an anticlonic disturbance of any importance will seldom enter a frontal zone which is essentially a place of low pressure at the ground. If it does, it will develop into an important anticyclone for instance such a one which is closing a depression family.

Perhaps the development of a strong winter anticyclone over European Russia can be explained by an analogous mechanism. Owing to the strong radiation cooling of the air over Siberia we find a closed high-level depression there, whereas the high-level isobars in north-western Europe and over the Northern Atlantic may show a North-South course under some circumstances. We find a weak indication of the possibility of such a pressure field in the free atmosphere over Northern Russia in the normal pressure chart at 4 km of the Northern hemisphere as given in Brunt's "Physical and Dynamical Meteorology". If this is true, the strong increase of arctic anticyclones moving slowly southwards over Northern Russia could be understood. Meanwhile a thorough aerological investigation of a winter situation over Russia is lacking up to now, so that we cannot prove our view.

Nevertheless, in concluding we may remark that the divergence or convergence of the upper air current, that is of the high-level isobars or isobaric contour lines, must be considered of great importance and worth observing with attention.

PART IV

Vertical motions and pressure variations at the ground

10. *Vertical motions and the deepening of tropical cyclones.*

The influence of vertical motions on pressure variations at the ground shows physically some analogy with the effect on these pressure variations due to the horizontal or quasi-horizontal motion of air in a convergent or divergent isobaric field. For in the case of vertical motions the pressure variation is also a consequence of the fact that air arrives in an area with a pressure gradient which is too small or too large compared with the horizontal velocity the air possesses. The pressure variation is given by $\frac{\partial p_0}{\partial t} = -g \int_0^\infty I_3 \, dz$, the integrand being equal to:

$$I_3 = -\frac{1}{l^2}\left[\frac{\partial}{\partial x}\left\{\varrho v_z \frac{\partial}{\partial z}\left(\frac{1}{\varrho}\frac{\partial p}{\partial x}\right)\right\} + \frac{\partial}{\partial y}\left\{\varrho v_z \frac{\partial}{\partial z}\left(\frac{1}{\varrho}\frac{\partial p}{\partial y}\right)\right\}\right] \tag{10.1}$$

As in previous developments we can see here too that the terms containing horizontal derivatives of ϱ can be neglected compared with the other terms. Taking account of this the expression for I may be written:

$$I_3 = -\frac{1}{l^2}\left[\varrho \frac{\partial v_z}{\partial x}\frac{\partial \alpha}{\partial z}\frac{\partial p}{\partial x} + \frac{\partial v_z}{\partial x}\frac{\partial^2 p}{\partial x \partial z} + \varrho v_z \frac{\partial \alpha}{\partial z}\frac{\partial^2 p}{\partial x^2} + v_z \frac{\partial^3 p}{\partial x^2 \partial z} + \varrho \frac{\partial v_z}{\partial y}\frac{\partial \alpha}{\partial z}\frac{\partial p}{\partial y} + \frac{\partial v_z}{\partial y}\frac{\partial^2 p}{\partial y \partial z} + \varrho v_z \frac{\partial \alpha}{\partial z}\frac{\partial^2 p}{\partial y^2} + v_z \frac{\partial^3 p}{\partial y^2 \partial z}\right] \tag{10.2}$$

This is a very complicated expression from which it is difficult to draw any conclusion. Nevertheless vertical motions may be of great importance. This was first emphasized by Durst and Sutcliffe (9) who tried to explain the deepening of tropical cyclones by computing the effect on surface pressure of the ascent of air occurring in these disturbances. They supposed that a particle of air reaching a higher level will find a weaker field of pressure gradient there, and consequently will be driven out of the cyclone, causing fall of pressure in its centre.

If we want to verify their view we do best to transform (10.2) into cylindrical coordinates:

$$I_3 = -\frac{1}{l^2}\left[\varrho \frac{\partial v_z}{\partial r}\frac{\partial p}{\partial r}\frac{\partial \alpha}{\partial z} + \frac{\partial v_z}{\partial r}\frac{\partial^2 p}{\partial r \partial z} + \varrho v_z \frac{\partial \alpha}{\partial z}\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right) + v_z\left(\frac{\partial^3 p}{\partial r^2 \partial z} + \frac{1}{r}\frac{\partial^2 p}{\partial r \partial z}\right)\right] \tag{10.3}$$

From this expression we can see immediately that it is not only the sign of the several quantities $\left(v_z, \frac{\delta p}{\delta r}\right)$ Durst and Sutcliffe considered, which determines the value of $\frac{\delta p_0}{\delta t}$, but that it is the exact distribution of these quantities appearing in the differential quotients as well. To all appearance Durst and Sutcliffe did not stress this point sufficiently.

Comparing the various terms of (10.3) it appears that this expression can still be simplified as the terms containing $\varrho \frac{\delta \alpha}{\delta z}$ turn out to be of an order of magnitude 10 times smaller than the rest. If we neglect these terms we get:

$$I_3 = -\frac{1}{l^2}\left[\frac{\partial v_z}{\partial r}\frac{\partial^2 p}{\partial r \partial z} + v_z\left\{\frac{\partial^3 p}{\partial r^2 \partial z} + \frac{1}{r}\frac{\partial^2 p}{\partial r \partial z}\right\}\right] \tag{10.4}$$

The sign of $\frac{\partial p_0}{\partial t}$, therefore, depends on the ratio of the terms in (10.4). It is easy to see, that in the centre of a cyclone in which the pressure gradients decrease with increasing height $\frac{\partial p_o}{\partial t}$ will be negative if v_z is positive. As there is a descending motion in the centre of a tropical cyclone however, the eye of the cyclone, surface pressure should rise there according to (10.4).

The ratio between the terms containing $\frac{\partial v_z}{\partial r}$ and those containing v_z will depend among other things on friction. So it looks as if the problem of the deepening of tropical cyclones by vertical motions cannot be solved unless we know exactly how friction works with these disturbances.

For the time being it seems reasonable to ascribe the deepening of tropical cyclones for the greater part to the Rodewald effect.

PART V

Pressure variations due to discontinuities

11. *General remarks.*

We have seen that according to (7.5) the influence of advection of air with different density is rather small as long as the air can be considered as to belong to one air-mass, that means as long as no surfaces of discontinuity are present.

Such surfaces being present the various derivatives occurring in (7.5) become much larger so that in this case advection may lead to rather important pressure variations. The derivatives only keep their significance when the frontal surface can be considered as a thin transition layer which is always the case. In principle nothing changes in our former reasoning.

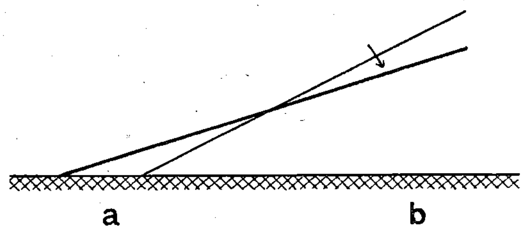

Figure 10.

About the connection between the divergence-theory and surfaces of discontinuity something has already been said. In fact the origin of the strong upper current occurring above frontal surfaces must be seen in the penetration towards the warm air of the cold air in the way which is indicated schematically in fig. 10. Above *a* the air will rise, leading to an increase of pressure in the free atmosphere whereas above *b* the pressure will fall as the air moves downwards there. So in the free atmosphere there will originate a pressure gradient directed from *a* towards *b*, which will be added to the pressure gradient which is present already. As long as this new gradient is growing in the free atmosphere the air will move from *a* to *b* (isallobaric wind) during which motion it is accelerated until it moves nearly geostrophically along the isobars, that is perpendicular to the cross-section given in fig. 10 and with great velocity. At places where the cold air did penetrate less far towards the warmer air the pressure gradients in the free atmosphere are less, that means, the upper isobars diverge (fig. 11). We saw from section 9 how the entrance of a depression into such a "delta" leads to pressure-fall whereas the pressure rises in case of an anticylcone enterring the divergent zone.

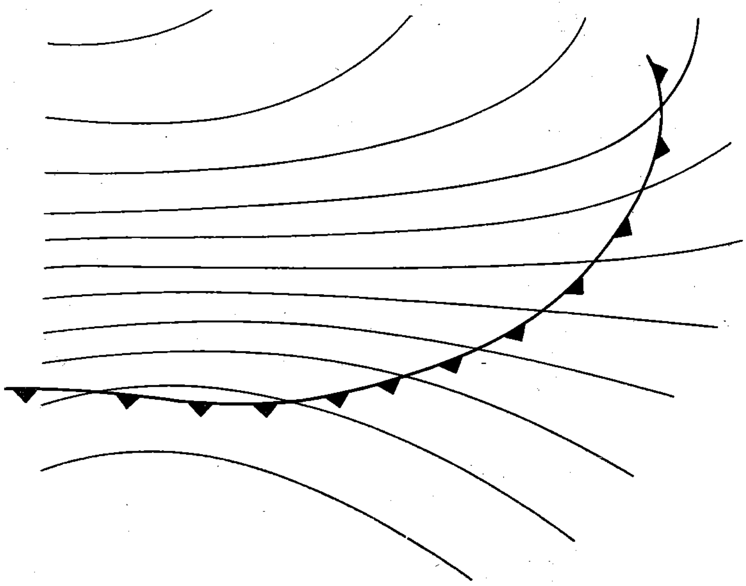
Figure 11.

We can now better understand why it is nearly always depressions which suffer great intensifying in a divergent system of upper isobars whereas a strong increase of pressure in an anticyclone seldom occurs here. We saw already how it is more probable for a depression to enter the frontal zone than it is for an anticyclone. Moreover we see from the present qualitative argument that a depression having the greatest cyclonic rotation in the lowest levels which reaches the "delta" of a frontal zone will try to intensify the rate of divergence near the centre of the depression as the cold air will penetrate deeper towards the warm air in the rear of the depression whereas it will be hampered in its motion in front of the depression. An anticyclone, on the contrary, is inclined to destroy the "delta" of the frontal zone when it enters the latter as the cold air in front of the anticyclone is forced to move towards the warm air whereas at the rear the cold air has the tendency to retire.

We can say, therefore, that although the formal calculations of 9 did not indicate an important difference in the behaviour of a cyclone or an anticyclone reaching a "delta" it follows from a qualitative examination of the flow pattern, that in reality the strong deepening of cyclones will occur much more often than the strong intensifying of an anticyclone.

The foregoing reasoning on the formation of a strong upper current in a frontal zone and the influence of the "delta" on depressions or anticyclones is independent of the fact whether the transition from warm to cold air is a gradual one or whether there exists a real surface of discontinuity between the two.

If there exists a real surface of discontinuity, we have to integrate from 0 to ∞ through this surface in order to compute $\frac{\partial p_o}{\partial t}$. For instance divergence of isobars may occur in both air-masses. We can distinguish two ways in which this integration may be made according to the idea we have about the surface of discontinuity.

In the first place we can understand the surface of discontinuity to be the limit of a transition layer between the two adjacent air-masses. It is clear that in this case the integration from the ground to the upper boundary of the atmosphere of $div\ (\varrho \vec{v})$ can be subdivided into three parts, namely $\int_o^{H-\delta}$ $\int_{H-\delta}^{H+\delta}$ and $\int_{H+\delta}^{\infty}$ where H indicates the height of the surface of discontinuity whereas δ denotes an infinitely small vertical distance approaching zero. If the integrand in the three integrals is represented by the same analytical function $f(z)$, which is always possible, we can combine them into one integral $\int_o^{\infty} f(z)\,dz$. Under these circumstances the existence of a frontal surface does not lead to any analytical deviation of our foregoing reasonings. Of course the existence of a surface of discontinuity will influence the flow and so dynamically it will be of importance.

Secondly we can suppose that the two adjacent air-masses are separated by an imaginary impermeable surface. In that case the integration from the ground to the boundary of the atmosphere is represented by the sum of two integrals $\int_o^{H-\delta}$ and $\int_{H+\delta}^{\infty}$. In this case therefore the values of the integral at $H - \delta$ and $H + \delta$ cannot be omitted. It will be clear that this conception about the surface of discontinuity does not agree with atmospheric reality where there is never a sharp frontal surface but always a more or less thick transition layer. Nevertheless it will appear that even with this idea about an atmospheric surface of discontinuity we can prove that the existence of such a surface does not influence the analytical expressions we formulated in the foregoing sections. Besides in doing so we shall be able to disprove a theory on pressure changes that was founded by Ertel and that got some adherents, especially in Germany.

12. *Singular advection.*

In a great number of articles Ertel (11, 12, 13, 14, 15, 16, 17) introduced the so-called singular advection. Others took over his ideas, extended them to some respect and compared them with practical weather situations.

According to the theory of singular advection, pressure variations at the ground do not of necessity originate in divergent motions of the air above the place of observation but may be due to the mere existence of surfaces of discontinuity of the order zero or one. According to this theory these surfaces need not move themselves, which is already an amazing supposition. As was said already Ertel's considerations were based on the above mentioned second conception of a surface of discontinuity. According to this conception the values of the integral at $H - \delta$ and at $H + \delta$ appear in the final result. If these values are different ones and H is a function of the coordinates, the integration from 0 to ∞ gives a result different from zero even if the values of the integral between 0 and $H - \delta$ and between $H + \delta$ and ∞ are zero themselves. For in that case we have:

$$\int_{o}^{H-\delta} f'(x)\,dx + \int_{H+\delta}^{\infty} f'(x)\,dx = \int_{o}^{\infty} f'(x)\,dx + \{ f(H-\delta) - f(H+\delta) \} \frac{dH}{dx} \tag{12.1}$$

so that $\int_{o}^{\infty} f'(x)\,dx$ does not determine the right hand side of the equation. As a matter of fact results are analogous if there are several surfaces of discontinuity, $H_1, H_2, \ldots H_n$, H_i being a function of the several coordinates.

We shall now determine the influence of surfaces of discontinuity on pressure variations at the ground according to (12.1). We do best to follow the reasoning of van Mieghem (29). According to him starting from the equation of continuity, the pressure variation at the ground is given by

$$\frac{\partial p_o}{\partial t} = g \left[- \int_{o}^{\infty} \operatorname{div} (\varrho \vec{v})\, dz + \sum_{i=1}^{m} \left\{ [\varrho v_N] \sqrt{H_x^2 + H_y^2 + 1} \right\}_i \right] \tag{12.2}$$

an expression that can be easily deduced. In it the newly introduced symbols have the following meaning:

i numbers the surfaces of dicsontinuity, m of which are present. Each such surface is given by a relation:

$$z = H_i (x, y, t) \tag{12.3}$$

$$H_x = \frac{\partial H}{\partial x} \text{ and } H_y = \frac{\partial H}{\partial y}.$$

$[\varrho v_N]$ is equal to $(\varrho v_N)_{H-\delta} - (\varrho v_N)_{H+\delta}$ and indicates the difference of momentum on both sides of a surface of discontinuity, pointing in the positive direction of the normal to that surface.

Now we have

$$[\varrho v_N] = [\varrho v_x] \cos (x, N) + [\varrho v_y] \cos (y, N) + [\varrho v_z] \cos (z, N) \text{ and as } \cos (x, N) \equiv$$

$$\equiv \frac{-H_x}{\sqrt{H_x^2 + H_y^2 + 1}}; \cos (y, N) \equiv \frac{-H_y}{\sqrt{H_x^2 + H_y^2 + 1}}; \cos (z, N) \equiv \frac{1}{\sqrt{H_x^2 + H_y^2 + 1}}$$

we can write consequently

$$[\varrho v_N] \sqrt{H_x^2 + H_y^2 + 1} = - [\varrho v_x] H_x - [\varrho v_y] H_y + [\varrho v_z].$$

From this relation it follows that

$$\frac{\partial p_o}{\partial t} = - g \int_{o}^{\infty} \operatorname{div} \varrho \vec{v}\, dz - g \sum_{i=1}^{m} \left\{ [\varrho v_x] H_x + [\varrho v_y] H_y - [\varrho v_z] \right\}_i \tag{12.4}$$

This expression for $\frac{\partial p_o}{\partial t}$ differs from the one Ertel deduced by the term $-[\varrho v_z]$ which is missing in Ertel's expression. This is a consequence of the fact that Ertel started falsely from $\frac{\partial(\varrho v_x)}{\partial x}+\frac{\partial(\varrho v_y)}{\partial y}$ instead of from the three dimensional expression $div(\varrho\vec{v})$. It means that he considers the discontinuity of ϱv_x and ϱv_y whereas he deals with ϱv_z as if being a continuous quantity. It is clear that this is necessarily in defiance of the kinematical condition of discontinuity surfaces.

Van Mieghem, apparently in imitation of Ertel omitted $-(\varrho v)$ further on in his article by introducing it separately for a second time with opposite sign.

Maintaining the term we can see immediately that

$$-g\{[\varrho v_x]H_x+[\varrho v_y]H_y-[\varrho v_z]\}\equiv g[\varrho]H_t \tag{12.5}$$

in which relation $[\varrho]$ denotes $\varrho_{H-\delta}-\varrho_{H+\delta}$, whereas $H_t\equiv\frac{\partial H}{\partial t}$. By this result (12.4) gets a very simple meaning. It is the mathematical expression for the absolutely plausible fact that as a consequence of divergence in both adjacent air-masses, surface pressure falls where the surface of discontinuity descends locally and vice versa. If the air moves parallel to the frontal surface, $H_t=0$ and no variation of surface pressure occurs at all, a quite intelligible result which shows once more clearly that the conception of "singular advection of the order zero" is untrue.

This is even clearer with the so called "singular advection of the first order" which should be due to surfaces of discontinuity of the first order. We discuss it briefly according to the developments of Lucht (27).

He starts from the equation

$$\frac{\partial}{\partial t}\int_0^\infty \varrho\,dz+\frac{\partial}{\partial x}\int_0^\infty \varrho\,v_x\,dz+\frac{\partial}{\partial y}\int_0^\infty \varrho\,v_y\,dz=0 \tag{12.6}$$

and supposes, that ϱv_x and ϱv_y are continuous everywhere, but that

$$\left[\frac{\partial\varrho v_x}{\partial z}\right]=\left(\frac{\partial\varrho v_x}{\partial z}\right)_{H-\delta}-\left(\frac{\partial\varrho v_x}{\partial z}\right)_{H+\delta}\neq 0 \tag{12.7}$$

and

$$\left[\frac{\partial\varrho v_y}{\delta z}\right]=\left(\frac{\partial\varrho v_y}{\delta z}\right)_{H-\delta}-\left(\frac{\partial\varrho v_y}{\partial z}\right)_{H+\delta}\neq 0 \tag{12.8}$$

in a surface of discontinuity of the first order, the tropopause for instance which is given by the equation $z=H(x,y,t)$ again.

In order to take account of these relations Lucht transforms (12.6) into:

$$\frac{\partial}{\partial t}\int_0^\infty \varrho\,dz+\frac{\partial}{\partial x}\int_0^{H-\delta} \varrho v_x\,dz+\frac{\partial}{\partial x}\int_{H+\delta}^\infty \varrho\,v_x\,dz+\frac{\partial}{\partial y}\int_0^{H-\delta} \varrho\,v_y\,dz+\frac{\partial}{\partial y}\int_{H+\delta}^\infty \varrho v_y\,dz=0 \tag{12.9}$$

Substituting (12.7) and (12.8) into (12.9) he finds the following expression for the pressure variation at the ground:

$$\frac{\partial p_0}{\partial t}=g\,H\left\{\left(\frac{\partial\varrho v_x}{\partial z}\right)_{H-\delta}-\left(\frac{\partial\varrho v_x}{\partial z}\right)_{H+\delta}\right\}\frac{\partial H}{\delta x}+g\,H\left\{\left(\frac{\partial\varrho v_y}{\partial z}\right)_{H-\delta}-\left(\frac{\partial\varrho v_y}{\partial z}\right)_{H+\delta}\right\}\frac{\partial H}{\partial y}+$$

$$+g\int_0^\infty z\frac{\partial}{\partial z}\left\{\left(\frac{\partial\varrho v_x}{\partial x}\right)+\left(\frac{\partial\varrho v_y}{\partial y}\right)\right\}dz \tag{12.10}$$

Now the first term of (12.10) is equal to $g.\ H$ times the difference of ϱv_x on both sides of the surface of discontinuity measured in the x-direction along the surface of discontinuity. And this quantity is zero by definition, so that the first term of (12.10) disappears. The same reasoning holds for the second term.

The last term of (12.10) can be transformed by partial integration into

$$\left[gz\left\{\frac{\partial\varrho v_x}{\partial x}+\frac{\partial\varrho v_y}{\partial y}\right\}\right]_o^\infty - g\int_o^\infty\left\{\frac{\partial\varrho v_x}{\partial x}+\frac{\partial\varrho v_y}{\partial y}\right\}dz$$

and as the first term is equal to zero we find again the ordinary relation between the pressure variation at some point of the surface of the earth and the divergence of momentum in the whole vertical column above that point. So we can say, that "singular advection of the first order" is a fiction too.

The fact that the theory as developed by Ertel and Lucht apparently leads to quantitatively good results does not form a proof for its correctness. For in applying the theory Lucht for instance uses the slope of the tropopause which is very difficult to determine. It is never known exactly in the place near the centre of a depression and an extrapolation is very dangerous as the slope varies considerably from point to point. Besides he transforms his formulae in such a way that the difference between the vertical temperature lapse-rates below and above the tropopause enters in them. Now on the average this difference amounts to about 0,5° C per 100 meter, the lapse-rates themselves being of the same order of magnitude. A mistake of some tenth's of degrees centigrade is very well possible, especially as it is difficult to determine the exact height of the tropopause. As both the slope of the tropopause and the difference between the two lapse-rates enter in Lucht's formulae as multiplicative factors, his good results must be considered as having no practical value at all.

13. *Pressure variations connected with frontal surfaces.*

So we have seen that even according to the second conception about a frontal surface pressure variations at the ground cannot be a consequence of the mere existence of such a surface.

Let us try to make some qualitative remarks on the influence of a moving surface of discontinuity of the order zero on pressure variations at the ground.

If we base ourselves on the equations (12.4) and (12.5) and if we suppose for the moment that $div\ \varrho\vec{v}$ can be zero, H_t being $\neq 0$, we can distinguish two extreme cases which in reality will always occur in combination with each other:

a. local variations of the height H of a frontal surface may be due to a simple horizontal motion of both adjacent air-masses and their common boundary surface. We call this frontal advection. To estimate the effect this process has on the surface pressure we put the velocity perpendicular to the front to 10 m per second, the slope to $^1/_{100}$ and $[\varrho]$ equal to 10^{-5} which are all values of the right order of magnitude. Then $g\,[\varrho]\,H_t$ equals $10^{-1}\,\frac{\text{dyne}}{\text{cm}^2\ \text{sec}} = 1$ mb/3 hours, an order of magnitude that occurs in reality;

b. local variations of the height H of a frontal surface may be due to slope variations of the frontal surface, originating in the motions and deformations the both adjacent air-masses undergo. When its slope diminishes we call the frontal surface kataklinic, when its slope does not change we call it isoklinic and when the slope increases we call it anaklinic according to Bleeker (4). A good deal of the observed pressure variations are due to these slope variations. Finally according to Margules (28) the kinetic energy of depressions must be transformed from the potential energy converted when frontal slopes diminish, causing a descent of the centre of gravity.

In 11. we saw that we do best to consider a frontal surface as a thin transition layer so that we can use the old relation:

$$\frac{\partial p_o}{\partial t} = -g \int_o^\infty \operatorname{div} (\varrho \vec{v})\, dz \tag{13.1}$$

In this case frontal advection will mean an advection of air of different density and, therefore, it must be described with the aid of (7.5).

Slope variations will inevitably be connected with departures form geostrophic winddistribution in both air-masses and these departures will contribute to *div* $(\varrho \vec{v})$.

Let us imagine for example a depression with a kataklinic cold front as often occurs. The cold air in the rear of the depression is subsiding in this case. This subsidence leads to strong pressure gradients in the free atmosphere above the cold frontal surface. As these strong gradients do not exist in front of the depression, the depressioncentre is situated in a "delta", that is to say the depression will deepen. We saw already how an anticyclone has the tendency to destroy the "delta".

As cold air always has the tendency to flatten, the warm frontal surface of a depression will usually also be kataklinic. It seems, however, as if the behaviour of the cold air in the rear of a depression is the most important for the development of a depression. This was already supposed by other authors and may be confirmed by the fact that strongly deepening depressions occur in which only a cold front can be recognized (Canada-depressions) whereas depressions only showing a warm front are never observed.

We see by this how Margules' statement about the conversion of potential energy into kinetical energy by means of slope variations of a surface of discontinuity may be connected with Scherhag's and Rodewald's divergence rules.

It will remain an important task to establish the exact physical connection between Margules' energy rules, Bjerknes' depression scheme and Rodewald's empirical divergence theory. It is the writer's conviction that the exact knowledge of this connection is the indispensable basis for a better understanding of the general circulation of middle latitudes.

The author wishes to express his thanks to Dr. W. Bleeker, K. R. Postma and W. Schalkwijk who assisted him in finishing this paper by suggestions and discussions.

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