ON THE KINEMATIC STRUCTURE OF TRANSITION-LAYERS BETWEEN AIR MASSES

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1. When two air masses are separated by a thermal and kinematic discontinuity surface and the winds are everywhere geostrophic, we have for the slope of this surface the well known formula of Margules.

As a simple form of such a discontinuity surface we may assume a *plane* surface, on the two sides of which we have potential temperatures ϑ_1 and ϑ_2 respectively ($\vartheta_2 > \vartheta_1$), which are constant along it. The latter supposition is in good agreement with actual atmospheric conditions, at least so far as we are concerned with stationary frontal systems (cf. Stüve 1925 [1]), Namias 1938). The formula of Margules can then advantageously be written by means of these potential temperatures instead of densities, as follows:

$$tg\,\alpha = \frac{l}{g}\,\frac{v_2/\vartheta_2 - v_1/\vartheta_1}{1/\vartheta_2 - 1/\vartheta_1}, \qquad (1)$$

where $l = 2\omega \sin\varphi$, ω and φ being the angular velocity of the earth's rotation and the geographic latitude, respectively, while v_1 and v_2 are the velocities of the colder and of the warmer air, respectively, parallel to the frontal surface.

A better approximation to reality might be obtained by replacing the discontinuity surface by a planparallel transition-layer, within which the potential temperature ϑ alters gradually from ϑ_1 on the one side to ϑ_2 on the other; compare fig. 1, where a cross-section of the system is given, in which the fully drawn straight lines represent a set of isentropic surfaces.

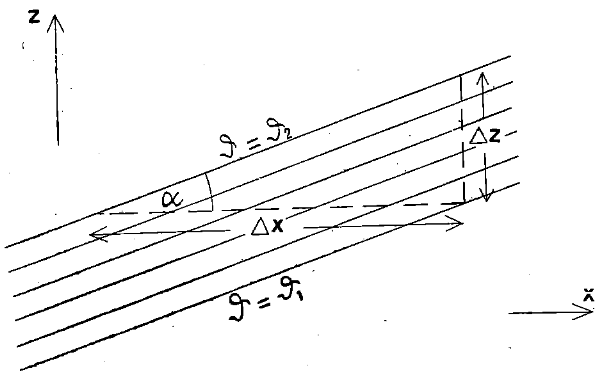


Fig. 1.

We wish to calculate the slope of such a planparallel layer for the case that no accelerations are present and that viscosity is neglected for the present.

To begin with, we might consider the elementary case that we have to deal with *incompressible* fluids of variable density. The calculations would then be somewhat simpler, but the results would turn out to be quite analogous to those we shall arrive at here; in the formulation of our results, derived for *compressible* media, the reciprocal of the potential temperature ($1/\vartheta$) plays the same role as ϱ for incompressible media; the resulting formulas for the latter case may thus directly be written down by replacing $1/\vartheta$ in our results by ϱ; compare formula (1), where $1/\vartheta$ may be replaced by ϱ. *It will appear, that this formula is applicable under much more general suppositions.*

If p_1 be the standard pressure, to which the potential temperature is referred, we have the following relation:

$$p = RT\varrho = \frac{R}{p_1^{\,k}}\, p^k \vartheta \varrho = R^* p^k \vartheta \varrho ,$$

where $k = R/c_p = 0{,}285$.
Therefore

$$\varrho = p^{1-k} / R^* \vartheta .$$

[1]) References will be indicated by mentioning the year of publication; see list of literature at the end of the paper.

Now we have for the geostrophic wind v in the y-direction

$$l v = \frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{R^* \vartheta}{p^{1-k}}\frac{\partial p}{\partial x} = \frac{R^*\vartheta}{k}\frac{\partial}{\partial x}(p^k),$$

or

$$\frac{\partial}{\partial x}(p^k) = \frac{klv}{R^*\vartheta}. \tag{2}$$

Further, the statical equation for pressure yields:

$$\frac{\partial p}{\partial z} = - g \frac{p^{1-k}}{R^*\vartheta}$$

or

$$\frac{\partial}{\partial z}(p^k) = - \frac{kg}{R^*\vartheta}. \tag{3}$$

By differentiating (2) with respect to z and (3) with respect to x, we obtain, in virtue of the continuity of pressure:

$$- g\frac{\partial}{\partial x}\left(\frac{1}{\vartheta}\right) = l\frac{\partial}{\partial z}\left(\frac{v}{\vartheta}\right) \tag{4}$$

Now in a first approximation we may take the quantities $1/\vartheta$ and v/ϑ to be linear functions of x and z within the transition layer under consideration. On multiplying (4) by $\triangle z = tg\,\alpha.\triangle x$(see fig. 1) we obtain therefore:

$$g\,(1/\vartheta_2 - 1/\vartheta_1)\; tg\,\alpha = l\,(v_2/\vartheta_2 - v_1/\vartheta_1),$$

which is equivalent to Margules' formula (1).

But we may still generalize the matter. Let ϑ and v be functions of x and z in such a manner, that both may be written as functions of ax—z; this means that the surfaces ϑ = const. and v/ϑ = const. are parallel planes $ax - z$ = const., with no other restriction. It follows then, that

$$\frac{\partial}{\partial x}\left(\frac{1}{\vartheta}\right) = - a\frac{\partial}{\partial z}\left(\frac{1}{\vartheta}\right) = tg\,\alpha\frac{\partial}{\partial z}\left(\frac{1}{\vartheta}\right),$$

when $tg\,\alpha$ now stands for the slope of the above mentioned planes.

Equation (4) now gives

$$g\; tg\,\alpha\frac{\partial}{\partial z}\left(\frac{1}{\vartheta}\right) = l\frac{\partial}{\partial z}\left(\frac{v}{\vartheta}\right).$$

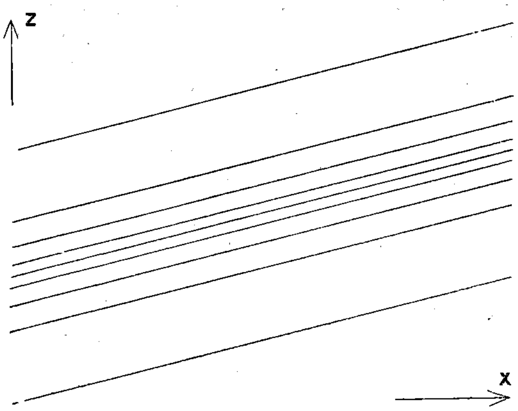


Fig. 2a.

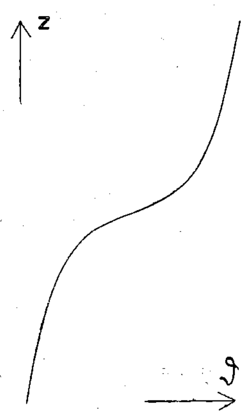


Fig. 2b.

Integrating this along a line x = const. from $z = z_1$ to $z = z_2$ we find

$$g\,(1/\vartheta_2 - 1/\vartheta_1)\; tg\,\alpha = l\,(v_2/\vartheta_2 - v_1/\vartheta_1),$$

as above. *So Margules' formula is also valid for this general case,* where now the indices $_1$ and $_2$ indicate any two planes of the set $ax - z$ = const.; we may, if we like, denote these planes by the values of the coordinate $z^* = \frac{z - ax}{\sqrt{a^2 + 1}}$ perpendicular to all of them. The variation of ϑ in space may, for the rest, be arbitrary. As to the transition-layer between two air masses, we may imagine some such distribution as is shown in fig. 2; fig. 2a represents a cross-section with lines ϑ = const. drawn in it; fig. 2b represents the variation of ϑ with z along a line x = const.

Between the variation of ϑ in space and that of v, however, there exists a close relation. To show this we may first derive as an important result the general differential form of the expression giving the slope of the surfaces ϑ = const., v = const. The above made assumption as to the variation of ϑ and v in space implies first that v/ϑ is a function of ϑ or of $1/\vartheta$ alone. The slope of a surface $1/\vartheta(x,z)$ = const. is now, according to (3)

$$\frac{dz}{dx} = -\frac{\partial(1/\vartheta)\ /\partial x}{\partial(1/\vartheta)\ /\partial z} = -\frac{\partial(1/\vartheta)/\partial x}{\partial(v/\vartheta)/\partial z}\frac{d(v/\vartheta)}{d(1/\vartheta)} = \frac{l}{g}\frac{d\,(v/\vartheta)}{d\,(1/\vartheta)} = \frac{l}{g}\,(v - \vartheta\frac{dv}{d\vartheta}) \qquad (5)$$

This formula is valid independent of any restriction as to the distribution of ϑ in space; in particular the surfaces ϑ = const. (v = const.) need not be plane.

If, however, they are plane and are furthermore parallel to each other, as was assumed above — and for frontal transition layers this assumption is a reasonable one — their slope $dz/dx = tg\,\alpha$ is a constant, so that

$$\frac{d(v/\vartheta)}{d(1/\vartheta)} = \frac{g}{l}tg\,\alpha = \text{const.}$$

It follows, that $v/\vartheta = A/\vartheta + B$, where A and B are constants, $A = g\,tg\,\alpha/l$. Hence:

$$v = A + B\vartheta\,. \qquad (6)$$

This means that the v,z – diagram (see for instance fig. 3b of the following section) should be similar to the ϑ,z – diagram (fig. 2b) but for a multiplicative factor (pos. or neg.) in the v-dimension. Relation (6) is based on the above assumption as regards the thermal field and the velocity-field and on the assumed absence of accelerations and vertical velocity components.

2. *Eddy-viscosity.*

If viscosity terms are taken into account the equations of motion for acceleration-less current-fields may be written:

$$\left.\begin{aligned} -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{1}{\varrho}\frac{\partial}{\partial z}\,(\mu\frac{\partial u}{\partial z}) + lv &= 0 \\ -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \frac{1}{\varrho}\frac{\partial}{\partial z}\,(\mu\frac{\partial v}{\partial z}) - lu &= 0 \end{aligned}\right\}$$

Here, as before, it is assumed that no vertical components of motion are present and that the velocities are independent of y; furthermore the directions perpendicular to the surfaces u = const. and v = const. practically coincide with the z-direction, so that of the viscosity terms only one needs be written down in either equation.

Now we shall try to find out whether these equations can have a solution with still $u(z) \equiv 0$. By supposing this, we are left with the following conditions:

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} + lv = 0,$$

$$-\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\,(\mu\frac{\partial v}{\partial z}) = 0\,. \qquad (7)$$

The first of these two equations is identical with (2). Since (3) also is valid here, the equations (3), (5) and (6) remain valid also.

From (7) it follows, that in general a pressure variation must exist in the y-direction, that is to say in the direction of the current; it serves to compensate the retarding or accelerating effect of internal eddy viscosity friction, if such an effect is present. It would be absent in regions where v is constant with respect to z, or where it varies linearly with z, μ being constant; the latter case might to a certain extent be realized especially in air mass transition layers; see fig. 3a, where such an idealized v, z-diagram is represented. At the two boundary surfaces (P and Q) of such a layer, however, things would be upset and by a drag effect the velocity profile would automatically change from type a (fig. 3) into type b (fig. 3).

Now, in order to make the temperature field stationary and the slope of the frontal layer constant in space and time, we assume ϑ to be independent of y. On differentiating (3) with respect to y, we find then:

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right) = -\frac{g\,(1-k)}{R^*\vartheta}\,p^{-k}\,\frac{\partial p}{\partial y} = (1-k)\,p^{-1}\,\frac{\partial p}{\partial z}\,\frac{\partial p}{\partial y}\,,$$

or:

$$\frac{\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right)}{\frac{\partial p}{\partial y}} = (1-k)\,\frac{\frac{\partial p}{\partial z}}{p},$$

or:

$$\frac{\partial}{\partial z}\left(\log\left|\frac{\partial p}{\partial y}\right|\right) = (1-k)\,\frac{\partial \log p}{\partial z}\,.$$

Hence:

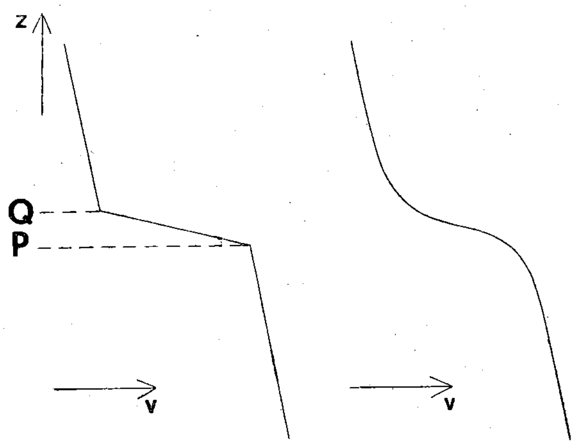
$$\log\left|\frac{\partial p}{\partial y}\right| = (1-k)\,\log p + c\,(x,y)\,,$$

or applying condition (7):

$$\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) = \frac{\partial p}{\partial y} = C\,(x,y)\,p^{1-k} \qquad (8)$$

We have, therefore,

$$\mu\,\frac{\partial v}{\partial z} = C\,(x,y)\int^{z} p^{1-k}(\zeta)\,d\zeta + D(x,y)\,.$$



a Fig. 3. b

Here $C(x,y)$ and $D(x,y)$ are constants with respect to z, which for the present may still depend upon x and y. Now we write

$$T = T_0 - \bar{\gamma}\,z,$$

where $\bar{\gamma}$ is some mean temperature lapse rate which, for the present, we shall treat as a constant. For p we may then easily derive the following expression:

$$p = p_0\left(1 - \frac{\bar{\gamma}}{T_0}\,z\right)^{g/R\bar{\gamma}}$$

Hence:

$$\int^{z} p^{1-k}(\zeta)\, d\zeta = \int^{z} p_o^{\,1-k}\left(1 - \frac{\bar\gamma}{T_o}\zeta\right)^{g(1-k)/R\bar\gamma} d\zeta =$$

$$= -\frac{p_o^{\,1-k} T_o}{\bar\gamma + (1-k)g/R}\left(1 - \frac{\bar\gamma}{T_o} z\right)^{1+g(1-k)/R\bar\gamma} + \text{const.} = -\frac{Tp^{1-k}}{\bar\gamma - \gamma_{ad} + g/R} + \text{const.}$$

We have therefore

$$\mu \frac{\partial v}{\partial z} = -\frac{C\,(x,\,y)}{\bar\gamma - \gamma_{ad} + g/R}\, Tp^{1-k} + D(x,y)\ , \tag{9}$$

or

$$\mu \frac{\partial v}{\partial z} = C_1(x,y)\vartheta p + D(x,y)\ , \tag{10}$$

where

$$C_1(x,y) = -\frac{C\,(x,\,y)}{p_1^{\,k}\,(\bar\gamma - \gamma_{ad} + g/R)}\ . \tag{11}$$

Here γ_{ad} stands for the dry adiabatic lapse rate; $g/R = 3{,}42°C/100$ m is the so called „auto-convection gradient".

A relation (10) may be applied to any height interval, where the temperature lapse rate is sufficiently constant. The connection between adjacent intervals with different $\bar\gamma$ is obtained by giving different values to the constant D, so as to make $\mu\, \frac{\partial v}{\partial z}$ (10) continuous (on account of the fact that it is equal to $\int^{z} \frac{\partial p}{\partial y}\,(z = \zeta)\, d\,\zeta$). By means of a suitable choice of $\bar\gamma$, however, we may also use *one* relation (10) for the whole layer of the atmosphere we are studying without making too large errors.

Applying (6) to equation (10) we obtain:

$$\mu B\, \frac{\partial \vartheta}{\partial z} = C_1 \vartheta p + D,$$

or:

$$\mu \partial \log\vartheta / \partial z = \beta\, p + \gamma/\vartheta, \tag{12}$$

$$\beta = \frac{C_1}{B}\ , \quad \gamma = \frac{D}{B}\ ,$$

where C_1 and D may still be functions of x and y.

Equation (12) should be understood to be a condition of stationariness of a field of motion, and especially of a kinematic transition layer without a wind shift, due to eddy viscosity. It is based upon conditions (6) and (7); the former of these two equations has been derived for a special stratification, such as may be expected to a certain extent in frontal transition zones, as we have seen. It should be borne in mind, that this stationariness is always a rather relative one („quasi-stationariness"), as diverse influencing factors never can be eliminated from actual conditions. The eddy conduction in particular will influence the temperature distribution. This influence, however, will act rather slowly, as we have supposed the transition layer to be parallel to the isentropic surfaces, perpendicular to which the eddy conduction is not large.

In (12) μ will depend on density, or pressure, and on the thermal stratification; these two factors are represented in the above equation by p, ϑ and $\partial\vartheta/\partial z$. Owing to these factors μ also varies with x and y. But a variation of μ with x and y, independent of these factors, and consequently *not* coupled to a variation with z needs not be expected; β and γ will, therefore, have to be constants, independent of x and y, and consequently the same will be true for C_1 (C) and D.

For μ we have:

$$\mu = \frac{\beta p + \gamma/\vartheta}{\partial \log \vartheta/\partial z} \tag{13}$$

Since μ will vanish when p vanishes, we may put $\gamma = 0$ (as ϑ remains finite). Hence:

$$\mu = \frac{\beta p}{\partial \log \vartheta/\partial z} \quad {}^{1)} \tag{14}$$

The question, whether condition (14) can be satisfied in the atmosphere, has to be answered by the theory of turbulence in stratified media, which, as yet, is not very firmly established. At any rate this condition is in qualitative agreement with what is to be expected concerning the influence of *stability*, which is proportional to $\partial \log \vartheta/\partial z$, upon turbulence. Apparently, (14) requires an influence of thermodynamic stability upon μ which is somewhat too large, especially for small values of $\partial\vartheta/\partial z$. A semi-empirical study by Exner, 1927, and two theoretical derivations by Ertel, 1932 and 1937, lead to the conclusion that μ is inversely proportional to $\partial T/\partial z + g/R$, instead of to $\partial \log\vartheta/\partial z = \vartheta^{-1}\partial\vartheta/\partial z$; this would imply that μ would become infinite for $\partial T/\partial z = -g/R$, but it is not very likely that this „auto-convection gradient" $g/R = 3{,}42°C/100$ m plays any real rôle in the atmosphere. Moreover, it may easily be shown that Exner's and Ertel's results are by no means reliable; see, for instance, Rossby and Montgomery 1935; as to Ertel's derivations, the adiabatic expansion and compression of the turbulent elements were not fully taken into account. The calculations by Rossby and Montgomery appear to indicate that the true interdependence of μ and $\partial \log \vartheta/\partial z$ is somewhat more complicated, and that, roughly speaking, we may write down for μ some proportionality to $(\partial \log \vartheta/\partial z + c)^{-1}$, with $c > 0$, so that only for large stabilities we have an eddy conductivity which is inversely proportional to stability, whereas for $\partial\vartheta/\partial z = 0$ it remains finite. On the other hand, we may mention the large values of the eddy conductivity found by Richardson and Proctor, 1925, and by Grimminger, 1938, for lateral mixing in isentropic surfaces, where no stability suppresses it, values which are by a factor 10^4 larger than the normal values of „vertical" eddy conductivity. This again suggests that in a free medium, where no vicinity of a rigid bottom influences the turbulent motions, the eddy conductivity will increase enormously when stability vanishes. — But (even if this be not so), as, in the vicinity of transition layers, we have sufficiently stable stratifications, the presence of a small positive term c in the actual eddy-viscosity-function will, in general, not make a function const. $(\partial \log \vartheta/\partial z)^{-1}$ to be too bad an approximation in those regions, so that, after all, we may conclude that the form of relation (14) will not deviate too far from real distributions of μ in the atmosphere near frontal transition zones; that is to say, that for a given situation a value of β may be found, which makes (14) agree fairly well with reality.

The constant $\beta = C_1/B$ determines the relation between the pressure variation in the y-direction and the vertical velocity distribution. Indeed, $C_1 = \text{const.}\ C = \text{const.}\ p^{k-1}\,\partial p/\partial y$;

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1) If we write $\mu = \varrho\, l^2 \left|\frac{\partial v}{\partial z}\right|$, we may also deduce: $l = \text{const.} \frac{p^{k/2}}{\partial \log \vartheta/\partial z}$, where l = mixing length.

$B = dv/d\vartheta$. In principle, therefore, a pressure- and velocity distribution may be established in such a way, that in the free atmosphere a quasi-stationary situation of a frontal transition zone (in the above mentioned sense) is fairly well maintained *without* an important extra wind shift due to eddy viscosity. Characteristic of this situation is the absence of vertical velocity components. This result differs from a conclusion by Bjerknes et al. 1933, page 499, stating certain upsliding and downsliding motions in such transition zones to follow necessarily from the equations of motion, when viscosity is not neglected [1].

As to the signs of the constants used, we may state the following. In (14) β is positive, so that C_1 has the same sign as B. Let $\partial v/\partial z$ be negative, as in figures 2 and 3 ($tg\ \alpha$ is positive); since $\partial\vartheta/\partial z$ is positive, this means that B is negative, and so is C_1; finally C is positive, and so is $\partial p/\partial y$. The pressure field looks therefore like one of those given in fig. 4. Here we should think of situations in the free atmosphere, as ground friction was not taken into consideration.

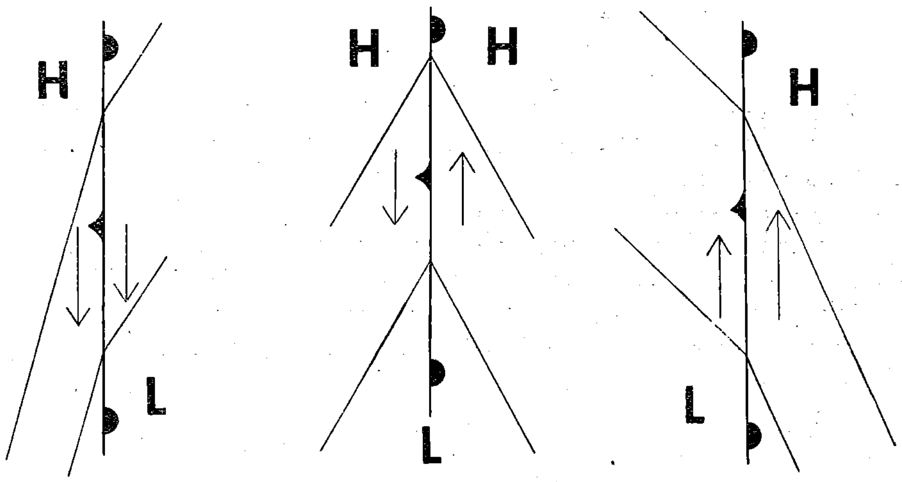

Fig. 4.

If the warmer air were to the right of the front, instead of to the left (so that $tg\ \alpha$ would be negative and $\partial v/\partial z$ positive) the sign of $\partial p/\partial y$ would have to be reversed. (We might, besides, superpose a uniform motion in the x-direction and a corresponding pressure gradient in the y-direction over the whole field).

Finally we shall make an estimate of the order of magnitude of the pressure gradient $\partial p/\partial y$ („anti-viscosity pressure gradient" as we might call it). According to (8) and (11) we have:

$$\partial p/\partial y = Cp^{1-k} = -\ C_1 p_1^{\ k} p^{1-k}\ (\bar{\gamma} - \gamma_{ad} + g/R).$$

Now $C_1 = \beta B = p^{-1}\ \mu\partial \log \vartheta/\partial z \,.\, dv/d\vartheta,$

so that

$$\frac{\partial p}{\partial y} = -\ \mu\ \frac{\partial \vartheta}{\partial z}\ \frac{1}{\vartheta}\ \left(\frac{p_1}{p}\right)^k \frac{dv}{d\vartheta}\ (\bar{\gamma} - \gamma_{ad} + g/R) =$$

$$= \frac{\mu}{T}\ \frac{\partial \vartheta}{\partial z}\ \frac{dv}{d\vartheta}\ (\bar{\gamma} - \gamma_{ad} + g/R).$$

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[1]) This statement is based on the theory of the Ekman-spiral, which assumes constancy of μ. Here lies the cause of the discrepancy.

Here we have to take for μ and $\partial\vartheta/\partial z$ some two corresponding values. Now reliable empirical data concerning the variation of μ with $\partial\vartheta/\partial z$ are rather scanty. We shall, therefore, take some mean values (in the free atmosphere) for both of them: $\mu = 150$ g/cm sec, $\partial\vartheta/\partial z = 0{,}4.10^{-4}$ °C/cm. Further we put: $T = 250°$ C, $\gamma\text{-}\gamma_{ad} + g/R = 3.10^{-4}$ °C/cm. For $dv/d\vartheta$ we may assume a rather large $\triangle v = 20$ m/sec to correspond to a rather small $\triangle\vartheta = 1.5$ °C. With these values we obtain $\partial p/\partial y = 10^{-5}$ g/cm^2 sec^2 = 1 mb/1000 km. On account of the large value of $dv/d\vartheta$ which we have introduced, the „anti-viscosity pressure gradient" $\partial p/\partial y$ may in reality be often still less than this value. Only for very large values of μ (great wind velocities) it might occasionally surpass it, provided $dv/d\vartheta$ be large enough. (A large value of $dv/d\vartheta$ means, according to (5), a steep frontal layer, as the term $\frac{l}{g}v$ in (5) is always very small, representing, as it does, the slope of the isobaric surface).

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