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SOME SPECULATIONS ON THE RESISTANCE TO THE MOTION OF CUMULIFORM CLOUDS

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CONTENTS

CHAPTER I. GENERAL INTRODUCTION

1. Form drag and skin-friction drag

A body moving through a liquid undergoes a drag which is composed of two parts:

1°. a form drag,

2°. a skin-friction drag.

The *form drag* is that part of the total drag which originates from fluid being pushed aside at the front of the body and from fluid streaming towards the back of the body. As a result a certain amount of fluid will move from the front to the back of the body. The kinetic energy of this movement is supplied by the body.

The *skin-friction drag* is that part of the total drag which is caused by a film of liquid adhering to the body. The velocity of the moving body is then partly transferred to the surrounding fluid by viscosity. As a result, the kinetic energy of the body is diminished.

The moving body may be solid, liquid or gaseous. The movement of solid bodies in a liquid or gaseous medium is studied in hydrodynamics. The value of the form drag depends among other things on the profile of the moving body. So called stream-line bodies show a minimum form drag for a given cross-section and a given velocity. The skin-friction drag on a solid body appears to depend among other things on the roughness of its surface. Moreover both drags depend on the viscosity of the fluid. In a turbulent flow the order of magnitude of the phenomena is quite different from that in a laminar flow.

A special case of skin-friction drag presents itself, when a jet discharges itself into fluid at rest. Kinetic energy is then transferred from the jet to the originally resting environment.

In practice, however, it is usually impossible to separate the two drags.

2. Drag on bodies in the atmosphere

In meteorology the drag on bodies moving in the atmosphere has been examined for two cases:

1°. *Cloud elements or particles of precipitation falling through the atmosphere.*

These are small solid or liquid bodies moving in a gaseous medium. After some time a state of equilibrium is established in which the acceleration by gravity is counterbalanced by the retardation by the atmospheric resistance. For cloud elements and small rain drops this resistance is mainly a matter of skin-friction.

In the case of small spherical bodies and laminar flow, the value of the total drag is given by STOKES' expression:

$$R = 6\pi\mu r W \tag{2.1}$$

where R is the drag, μ the viscosity, r the radius of the sphere and W its vertical velocity relatively to the air. If the cross-sections of the falling bodies are larger, while these bodies, however, can still be considered as being solid, STOKES' expression must be transformed in the way indicated by OSEEN.

Another difficulty arises when a raindrop falls through the atmosphere; small drops behave like solid bodies, whereas large drops behave like liquid ones. This was pointed out for the first time by LENARD (1904). When the velocity of a large drop surpasses a certain critical value, the drop splits up into a number of smaller drops.

2°. *Pilot balloons*

The drag on larger bodies such as pilot balloons is given by NEWTON's expression:

$$R = cF\varrho W^2 \tag{2.2}$$

where c is a numerical constant, F the cross-section of the body perpendicular to the direction of motion, ϱ the density of the fluid in which the body moves, and W the velocity of the body. Here R is for the greater part due to form drag.

The value of c can be determined as soon as the balloon has obtained a constant rate of ascent, which in practice occurs in a rather short time. It appears that the values of c obtained from various measurements differ widely. This is explained by the fact that the turbulence elements in the atmosphere are as a rule, much larger than the pilot balloon itself, so that this is carried along by the turbulence elements. This means that, strictly speaking, NEWTON's expression may not be used. (See for instance LETTAU, 1939).

It will be clear from these examples that the introduction of drag in meteorology leads to great difficulties, and cannot be treated in a general and simple way.

3. Form drag and skin-friction drag on clouds

A gaseous body moving in the atmosphere suffers also a drag. A cumuliform cloud is an example of such a body. The cloud loses kinetic energy by the drag in two ways:

1°. At the top of the cloud an increase of pressure arises as the cloud tries to push aside the above lying air. Owing to this increase of pressure the air above the cloud will flow away. If large-scale horizontal convergent or divergent flows are absent and if the density of the air above the cloud remains constant, this air will move downwards in such a way, that on an average through any arbitrary horizontal section equal masses of air flow upwards and downwards in unit time.

The condition that local changes in density shall not occur above the top of the cloud means, strictly speaking, that the atmosphere must be considered as an incompressible fluid. This supposition does not agree with reality as the density of the air decreases with increasing height. However, it is shown in hydrodynamics, that with respect to the motion of a body the atmosphere may be considered as an incompressible fluid, when the velocity of that body is small compared with the velocity of sound in the atmosphere. As indicated by PRANDTL (1929) the error made by using the equation of continuity as if the atmosphere were incompressible amounts to 1 % when the velocity of the body moving in the atmosphere is 48 m per sec. As the velocities in cumuliform clouds lie below this value, it follows that for questions of form drag henceforth the atmosphere may be treated as an imcompressible though inhomogeneous fluid.

The downward moving air forms the so called *counter-current*. The energy of this counter-current is drawn from the same source as the kinetic energy of the cloud; this means, that the kinetic energy available for the cloud is diminished by the counter-current. The counter-current can, therefore, be considered as the cause of *the form drag on the cloud.*

2°. Particles inside the cloud change places with particles outside the cloud as a result of turbulence. By this process a certain amount of momentum is transferred from the cloud to its environment. This part of the total drag may be called the *skin-friction drag on the cloud.*

It is clear that there is some analogy with the jet mentioned above. Expressions have been deduced in hydrodynamics which represent excellently the velocity-distribution in such jets.

We will suppose for the present, that these expressions are also valid for vertical currents in the atmosphere, in particular for the flow in cumuliform clouds. From this supposition conclusions will be drawn concerning the energy dissipation in these clouds.

4. The calculations of Christians

Up to now, only few computations have been made concerning the effect of form drag and skin-friction drag on the motion of cumuliform clouds.

The form drag was mentioned by RAETHJEN (1929), while the counter-current was theoretically investigated by J. BJERKNES (1938) and S. PETTERSSEN (1939). We shall refer to these theories in the next section.

The friction due to turbulence appearing in vertical air-currents (for example cumuliform clouds) was mentioned only incidentally by SHAW (1920), and REFSDAL (1930; 1932).

Only one estimate exists of the influence of both drags on the motion of cumuliform clouds, namely the estimate made by CHRISTIANS (1935).

In computing the *form drag* he started from the supposition, that the vertical velocity of the air above the cloud decreases with height, according to the formula

$$W(z) = W(o)e^{-cz} \tag{4.1}$$

where $c = \frac{\gamma_d - \gamma}{T}$ is a measure for the stability of the atmosphere, $W(z)$ the vertical velocity at the height z, $W(o)$ at the height o, γ_d the dry-adiabatic lapse-rate, γ the geometric one and T the absolute temperature. CHRISTIANS did not take into account the change in the lapse-rate, that must result from the vertical motion above the cloud. As, according to the above formula, the velocity decreases upwards, air must diverge horizontally over the cloud. The velocity of this flow of air can be expressed as a function of $W(z)$.

It follows from (4.1) that a decreasing vertical motion exists up to an infinite height above the top of the cloud, and hence a horizontal divergence too. To avoid this, CHRISTIANS introduced the rather unsatisfactory supposition, that these motions are no longer perceptible when 90 % of the air that originally filled the space occupied by the cloud, has flown off laterally. Nevertheless his conclusion is, that the vertical motion persists to an altitude of more than 50 km above the cloud when the vertical lapse-rate is 0,5 centigrade per 100 m, a quite improbable result. Gliding pilots for instance have never observed important vertical currents above cumuliform clouds. It is true that this altitude can be decreased by introducing a constant factor in c, but this is not very satisfactory, the less so as in a dry-adiabatic atmosphere the motions remain perceptible up to an infinite height, in spite of all restrictions.

Examining the *skin-friction drag*, CHRISTIANS started from the following expression for that drag in the direction of the Z-axis:

$$R_z = \frac{\mu}{\varrho}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right) \tag{4.2}$$

and the analogous expressions for R_x and R_y. It was supposed that μ/ϱ is a constant equal to 10 m^2 sec^{-1}. From this the conclusion was drawn, that the skin-friction drag dissipated about 1 % of the total energy of the cloud. These considerations are inadequate, for it follows from the theories of turbulence that μ/ϱ is proportional to the velocity-gradient in the flow and, therefore, generally speaking a function of the coordinates. Moreover, the value 10 m^2 sec^{-1} is too small. μ/ϱ is of this order of magnitude for horizontal flows, it is true, but here the vertical velocity-gradients are smaller than the velocity-gradients that are apt to appear at the border of a vertically rising cumuliform cloud. This was shown, for instance, by WALKER (1939), who pointed out, that the velocity-gradients, occurring at the border of cumuliform clouds may be about 7 m/sec per 100 m horizontal distance, a value larger than those usually occurring in horizontal flows. By putting μ/ϱ 100 m^2 sec^{-1}, the energy-dissipation would increase to 10 % of the total available energy. Moreover, there is also a dissipation due to turbulence in the interior of the cloud. In any case CHRISTIANS' method give no more than a very rough estimate of the skin-friction drag.

In this treatise an attempt will be made to give another deduction of the form drag and the skin-friction drag experienced by cumuliform clouds.

In order to discuss the form drag we will start from some considerations on the counter-current. Our investigation on the skin-friction drag will be based on the hydrodynamical theories of jets.

5. The slice-method

As mentioned already, CHRISTIANS' attempt to consider both drags is the only one existing up to now. We can even say that by far the most numerical examinations on the growth of cumuliform clouds ignore all interaction between the cloud and its environment (apart of course from the fact that the energy is obtained by the changing places of the cloudparcel with another one).

This method of investigation, very important in spite of its theoretical shortcomings, is often called the *parcel-method*.

One part of the drag-problem, namely the influence of the counter-current i. e. the form drag, was thoroughly investigated by J. BJERKNES (1938) and S. PETTERSSEN (1939). They did not mention the form drag however and their results do not depend on the vertical dimensions of the cloud. BJERKNES already pointed out the desirability of considering the entire cloud.

Developing the so called *slice-method*, PETTERSSEN distinguished the following three cases:

1. An ascending saturated current, surrounded by dry-adiabatically descending air.
2. Ascending as well as descending currents are and remain saturated. This means, that liquid water must be present in the descending air.
3. Ascending as well as descending currents are and remain non-saturated.

Besides PETTERSSEN introduced three assumptions, limiting the problem:

a. The horizontal motion does not at any level maintain any net inflow to or outflow from the region under consideration.

b. The conditions are barotropic at the initial moment, that means, the isobaric surfaces coincide with the isosteric surfaces.

c. The temperature-changes are adiabatic.

It follows from *a* that an equal transport of mass upwards and downwards exists through any arbitrary isobaric surface, or through any isobaric slice of unit thickness, supposing no local variations in density occur. We may write in this case:

$$M'W' = -MW \tag{5.1}$$

where M' and W' denote the mass and the vertical velocity of the ascending air and M and W denote the mass and the vertical velocity of the descending environment.

Assumption *c* involves, that radiation and turbulent interchange of particles are neglected and that all temperature changes may be expressed by the following equations:

$$\frac{\partial T'}{\partial t} = W'(\gamma - \gamma_d) \text{ in ascending dry air} \tag{5.2}$$

$$\frac{\partial T'}{\partial t} = W'(\gamma - \gamma_m) \text{ in ascending wet air} \tag{5.3}$$

$$\frac{\partial T}{\partial t} = W(\gamma - \gamma_d) \text{ in descending dry air} \tag{5.4}$$

$$\frac{\partial T}{\partial t} = W(\gamma - \gamma_m) \text{ in descending wet air} \tag{5.5}$$

Here $\frac{\partial T'}{\partial t}$ and $\frac{\partial T}{\partial t}$ are the temperature changes per unit time in the ascending and the descending currents. These four equations will be criticized later on; it will then be shown that they form the weak point in the theory of BJERKNES and PETTERSEN.

As a rule the three assumptions *a, b, c* will not be realised in the atmosphere. Convergent and divergent currents will always occur to some extent, especially near the cloudbase where a field of convergence exists, and at the cloud top where there is divergence. The temperature of the cloud in its base may differ from the temperature in its environment, this being in contradiction with assumption *b*. Finally radiation and turbulence will assert their influence, so that the third assumption too is usually not satisfied.

If all three assumptions are satisfied however, and an impulse is applied, raising a mass of air M', while a mass M in the environment of M' descends, in the isobaric slice under consideration an amount of heat is released per unit time, given by the equation:

$$\frac{\partial Q}{\partial t} = c_p \left(M \frac{\partial T}{\partial t} + M' \frac{\partial T'}{\partial t} \right) = c_p (M + M') \frac{\partial T}{\partial t} + c_p M' \left(\frac{\partial T'}{\partial t} - \frac{\partial T}{\partial t} \right) \qquad (5.6)$$

The meaning of the symbols used here is as follows: $\partial Q/\partial t$ is the heat gained (or lost) per unit time in the slice with total mass $(M + M')$; c_p is the specific heat of moist air at constant pressure; the remaining symbols have already been defined above.

In equation (5.6) $c_p (M + M') \frac{\partial T}{\partial t}$ means the uniform heating (or cooling) of the whole slice, while the last term $c_p M' \left(\frac{\partial T'}{\partial t} - \frac{\partial T}{\partial t} \right)$ represents the excess heating (or cooling) of the ascending air relative to the descending environment. Only the last term contributes to the production or consumption of kinetic energy.

It is possible to apply formula (5.6) to the three cases 1, 2, 3, mentioned above. For practical applications the most important case is 1: An ascending saturated current surrounded by dry-adiabatically descending air. In this case formulae (5.1), (5.3) and (5.4) hold. Substituting them in (5.6) we obtain:

$$\frac{\partial Q}{\partial t} = c_p W'M' \left(1 + \frac{M'}{M} \right) (\gamma_d - \gamma) + c_p W'M' \left\{ (\gamma - \gamma_m) - (\gamma_d - \gamma) \frac{M'}{M} \right\} \qquad (5.7)$$

Putting $\gamma_d - \gamma = \alpha$ and $\gamma - \gamma_m = \beta$ (fig. 1) we obtain the following equation:

$$\frac{\partial Q}{\partial t} = c_p W'M' \left(1 + \frac{M'}{M} \right) \alpha + c_p W'M' \left(\beta - \alpha \frac{M'}{M} \right) \qquad (5.8)$$

The first term on the right hand side represents the uniform heating of the slice, the second the heating of the ascending current of air relative to the surrounding descending air. The cloud is found to keep the same temperature as its environment when $\beta = \alpha M'/M$, whereas energy is converted when $\beta > \alpha M'/M$. Equation (5.8) leads, therefore, to another stability limit than the particle-method. A striking example is, for instance, the case $\beta = o$ and $\alpha > o$, meaning that the geometric lapse-rate is equal to the saturated adiabatic one. In that case β is smaller than $\alpha M'/M$, and it follows from (5.8) that then the ascending cloud becomes colder than its environment. The atmosphere, therefore, is stable; the particle-method, however, leads to indifferent equilibrium.

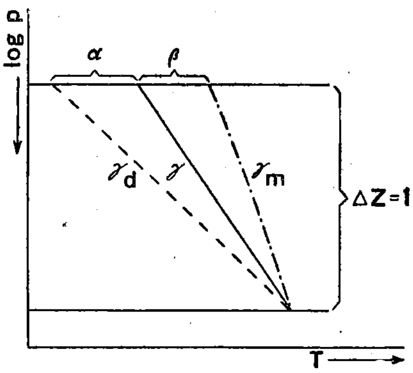

Fig. 1

The most important case to which the slice-method can be applied is that of conditional instability, that means the case in which both α and β are positive numbers. The sign of the *solenoid producing* term $c_p W'M' (\beta - \alpha M'/M)$ depends not only on α and β, but also on the ratio M'/M, that is the ratio of the ascending to the descending mass. If $c_p W'M' (\beta - \alpha M'/M)$ should be positive for given α and β, M'/M must have a value below β/α. When, therefore, β is smaller than α, M' must be smaller than M, in case any energy is produced by convection. When $M'/M > \beta/\alpha$ the mass M' is retarded in the direction of the initial impulse. When β is smaller than α, the space occupied by ascending air (clouds) must be smaller than the space occupied by descending air (compensating counter-current). From this PETTERSSEN concluded, that when the rate of conditional instability is small ($\beta << \alpha$) the cloudiness must consist of small cumuli, separated by a relatively large space containing no clouds. This appears to be confirmed by observation. As it follows from this that only such impulses are selected for which $M'/M < \beta/\alpha$, PETTERSSEN called this kind of instability *selective instability*.

Meanwhile, the conclusion, that a large number of small cumuli occurs in space when β is much smaller than α, is not necessarily the only one to be drawn from the above considerations, as the condition $M'/M < \beta/\alpha$ may also be satisfied by *one* huge cumulus in a large space. But for reasons of probability PETTERSSEN's conclusion may be considered to be correct.

We are not allowed to suppose, that the whole atmosphere outside the cumuliform clouds is descending. The counter-currents concentrate around the clouds and at some distance from them no vertical motion will be found at all. Moreover the supposition that the velocity has a constant value over the cross-section of the cloud as well as over the cross-section of the counter-current is only a rough approximation. Some of these imperfections were indicated by J. BJERKNES (1938).

However, without further considering these imperfections which it will be difficult to avoid as we do not know the exact motions in the atmosphere, we are in a position to criticize the speculations of BJERKNES and PETTERSSEN. This criticism is based on the fundamental equations (5.2), (5.3), (5.4) and (5.5), and we will apply it to the case of a wet upward current in a dry adiabatically descending environment.

According to PETTERSSEN the following equations, mentioned already, are valid for the temperature-changes in an arbitrary level:

$$\frac{\partial T'}{dt} = W'(\gamma - \gamma_m) \text{ for the ascending cloud} \tag{5.3}$$

$$\frac{\partial T}{\partial t} = W(\gamma - \gamma_d) \text{ for the descending counter-current} \tag{5.4}$$

From these equations it follows that both the temperature in the cloud and the environment of the cloud will increase continually when the atmosphere is conditionally unstable. It is clear, that this conclusion cannot be maintained for the cloud (fig. 2). As γ means the vertical lapse-rate in the atmosphere before the convection starts, the equation $\partial T'/\partial t = W'(\gamma - \gamma_m)$ only holds for the slice lying W' meters above the convective condensation level, during the unit of time following the initial moment of ascent of wet air in this level. Once the cloud has reached the slice in question $\partial T'/\partial t = o$, if turbulent mixing is not considered. This follows from the fact, that in the cloud $\gamma = \gamma_m$, so that we may not use equation (5.3) when the top of the cloud has passed the slice in question.

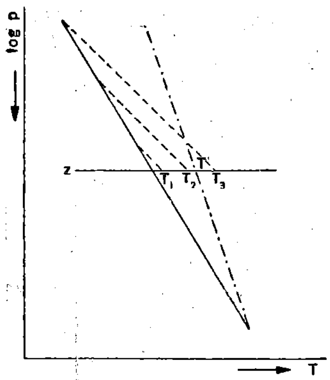

Fig. 2

In the counter-current, on the contrary, the temperature increases continually with time as is shown in figure 8. However, equation (5.4) applies only to the levels which have been surpassed by the top of the cloud and only from the moment that this occurred. Before this moment there is no counter-current at all at this level if at least, we suppose that there is no important vertical velocity of the air above the cloudtop. For that reason it follows that (5.3) and (5.4) can never be satisfied at the same time, as (5.3) applies when the top of the cloud lies under the level in question, whereas (5.4) applies when the top of the cloud has passed it. Therefore, the conclusions deduced from (5.8), an equation which was obtained by combining (5.3) and (5.4) cannot be correct.

Starting from these views a new calculation will be developed in chapter II. The counter-current will then appear to be connected with the form drag which the cloud suffers in the atmosphere. Afterwards we will try to find a solution for the problem of the skin-friction drag.

CHAPTER II. THE FORM DRAG ON CUMULIFORM CLOUDS (COLUMN-METHOD)

6. Form drag and displaced fluid

As was pointed out in the preceding chapter, the counter-current is connected with the form drag. A theory of the energy dissipation due to the counter-current must, therefore, be based on considerations concerning this form drag.

The form drag originates from the fact, that the moving body transfers part of its momentum to the fluid pushed aside so that the form drag is chiefly a matter of inertia. As was already mentioned for the case of the pilot-balloon, the form drag can be given by NEWTON's expression:

$$R = cF\varrho W^2 \tag{6.1}$$

c being a constant, varying for each case. NEWTON supposed c to be function of the form of the body but independent of the dimensions of the body, and of its velocity with respect to the fluid. Later investigations showed, that this is not quite true and that c must be considered as a function $c\,(Re)$ of REYNOLD's number $Re = \frac{Wl}{\mu/\varrho}$, l being a measure for the dimensions of the body and μ the dynamical viscosity.

The energy lost by the moving body as a result of the form drag is transferred to the replaced fluid. This energy is equal to $\int_{s_1}^{s_2} Rds$, s being the length of the way covered by the drag. Hence it follows, that the energy lost by the body, can be represented by:

$$\int_{s_1}^{s_2} Rds = \int_{s_1}^{s_2} c\varrho F W^2 ds = E_{k2} - E_{k1} \tag{6.2}$$

E_k being the kinetic energy of the moving body. Equivalent to this expression is the following one:

$$\int_{s_1}^{s_2} Rds = E'_2 - E'_1 \tag{6.3}$$

E' being the average total energy of the displaced fluid consisting of its kinetic energy E'_k and its internal and potential energy E'_p. If we know, therefore, the total energy of the displaced fluid at any moment, the value of the form drag is likewise known.

In hydrodynamics this is not generally the case. There R must be determined with the aid of the function $c\,(Re)$.

7. Form drag and large-scale convergence

In determining the form drag on cumuliform clouds we will start, however, from a known kinetic, internal and potential energy of the displaced air. This amounts, in fact, to the prescription of $\int_{s_1}^{s_2} Rds = E'_2 - E'_1$. It will appear that this method, though unsatisfactory at first sight, allows us to draw some conclusions as to the behaviour of cumuliform clouds.

As a consequence of the rising of a cumuliform cloud a shortage of air arises under the base of the cloud, whereas, on the other hand, a surplus of air will be found above the top of the cloud.

We might suppose a priori that the original conditions will be restored by large-scale horizontal *convergence* beneath the cloud and by large-scale horizontal *divergence* above the top of the cloud. A simple overall computation shows, that this is rather improbable.

Suppose that in an area of 100 km length and 100 km width cumuliform clouds ascend with an average velocity of 1 meter per second, and cover $^1/_5$ of the sky, these being rather moderate values.

By the ascent of the clouds $^1/_5 \times 1 \times 10^{10}$ m^3 of air are withdrawn per second from the space under the base of the clouds. If we now suppose that the wind U is directed along the X-axis, a new supply by horizontal convergence of the air withdrawn from the space requires that $\int_0^\tau \frac{\partial U}{\partial x} d\tau$ shall be $^1/_5 \cdot 10^{10}$ m^3/sec, τ being the volume under the cloud base. When the height of the cloud base is 1000 m, this means, that $\partial U/\partial x$ must amount to 0,2 m/sec per km, a very large value, which, however, might occur in some exceptional cases.

As no important vertical motion has ever been observed above the top of cumuliform clouds, an equally strong *divergence* should prevail in the top level. This is rather difficult to imagine. Moreover, it is not clear why a large-scale convergence should not give rise to a uniform slow ascent of the whole mass of air. Such ascents of 10^{-2} — 10^{-1} m/sec are often observed. In these cases we find a veil of stratified clouds in the area with convergence. Finally a descending motion is observed between cumuliform clouds. So we suppose henceforth that the same volume of air which ascends in the clouds, descends in the space between the clouds. This descending current is the *counter-current*, mentioned already.

Nevertheless convergence and divergence are of importance for the arising of cumuliform clouds. Convergence is favourable in this respect that it makes the vertical temperature lapse-rate larger, and moreover may lead to instability when the atmosphere is potentially unstable. Divergence on the contrary is unfavourable for the occurrence of convection as it stabilizes the atmosphere. In an anticyclone, where there is generally divergence, cumuliform clouds do not occur as a rule, excepting cumulus humilis.

Concluding we may say that a counter-current will always be present, but that it is possible that part of the air defect in the lower layers is compensated by horizontal convergence and part of the air surplus at the top is carried away by horizontal divergence.

8. The variation of density with height

In chapter I a short summary was given of BJERKNES' and PETTERSSEN's results concerning the formation of cumuliform clouds in a conditionally unstable atmosphere. It appeared, that the simultaneous application of the equations:

$$\frac{\partial T'}{\partial t} = W'(\gamma - \gamma_m) \tag{8.1}$$

$$\frac{\partial T}{\partial t} = W(\gamma - \gamma_d) \tag{8.2}$$

was not quite correct.

Now in his article BJERKNES pointed out the desirability of computing the energy converted by the motion of the cumuliform cloud as a whole, that means, integrating in some way or other equation (5.8) from the base of the cloud to its top. This means however, that we must know W and W' as functions of the height z.

We can obtain these functions in different ways. First we can suppose that the clouds are cylindrical and that no horizontal convergent or divergent currents occur between base and top. In this case the vertical velocities can be written as

$$W' = W_o' \frac{\varrho_o'}{\varrho'} \tag{8.3}$$

and

$$W = W_o \frac{\varrho_o}{\varrho} \tag{8.4}$$

ϱ' and ϱ representing the densities in- and outside the cloud. Though this method is rather simple in principle, it leads to calculation-difficulties, so that it has not been developed further.

In the second place we can introduce such horizontal convergent and divergent currents, that the vertical velocities W and W' remain constant. This means that an additional kinetic energy due to the horizontal currents must be taken into account. This too leads to difficulties.

Henceforth we will apply the second method but abstract from the energy of the horizontal currents. This means that actually we consider a homogeneous atmosphere. This is, of course, a rather rough approximation especially when very high cumuliform clouds are considered in which case we have to introduce some mean density in our formulae. Many inconsistencies arise which must be neglected, such as the fact that the vertical temperature lapse-rate in a homogeneous atmosphere amounts invariably to 3,42° C per 100 meters.

Nevertheless, we may expect that our approximate solution of the problem is somewhat better than the one BJERKNES and PETTERSSEN deduced, especially because we start from suppositions as to the temperature in and outside the cloud, which approach reality more closely than the simultaneous application of (8.1) and (8.2).

9. Method of calculation

If we wish to compute the energy converted by some proces in an amount of air, we have to investigate how the internal and potential energies of the air change, the quantity of heat liberated and the quantity of energy dissipated by friction. MARGULES (1903) followed this method, calculating the energy of storms. The calculation, by the way, of all these forms of energy is rather laborious.

REFSDAL (1932) indicated a method for arriving at the determination of the kinetic energy in a simpler manner, at least for cumuliform clouds. He pointed out that when friction is neglected, the lability energy, liberated when unit mass moves through a medium, and which is nothing else than the energy-balance of gravity when two masses change places, is converted into kinetic energy of the moving unit mass and its environment. This lability energy equals the sum of the internal energy, the potential energy and the heat supplied.

The lability energy so defined of unit mass moving from a level z_1 to a level z_2 is equal to:

$$\int_{z_1}^{z_2} g \frac{\varrho_d - \varrho_m}{\varrho_m} dz \tag{9.1}$$

ϱ_m being the density of the moving unit mass and ϱ_d the density of its environment. When this relation is applied to cumuliform clouds, ϱ_m represents the density in the cloud and ϱ_d the density in the environment of the cloud. Computing these densities or plotting them graphically is usually much simpler than calculating the different energy-forms separately. In considering the form drag we shall start from REFSDAL's expression for the lability-energy. Applying the usual supposition that the vertical motion takes place quasi-statically, that is to say, that everywhere $p_d(z) = p_m(z)_1$ expression (9.1) becomes:

$$\int_{z_1}^{z_2} g \frac{T_m - T_d}{T_d} dz \tag{9.2}$$

T denoting absolute temperature like before.

In nature the velocity profile through cloud and counter-current will always show a smooth shape as shown in figure 3 as a result of the continually acting turbulent friction.

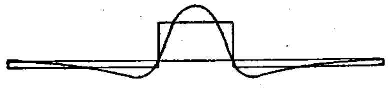

Fig. 3

As we have observed already, in this chapter friction will be considered as negligibly small. We can then abstract from the smooth course of the velocity profile and in particular we can start from the supposition, that the velocities in cloud and counter-current are constant over a cross-section (see fig. 3). This means, that in the cloud as well as in the counter-current we shall only consider mean velocities. This same velocity distribution was used by BJERKNES and PETTERSSEN. We shall consider a large area and suppose that all the air in this area outside the clouds has a descending motion, so that the counter-current is spread out over the whole cloud-free space.

We have to make another remark with respect to

$$M'W' = -MW \tag{9.2}$$

This relation involves, that in an arbitrary horizontal layer we do not take into account any horizontal convergent or divergent motions. Strictly speaking this relation does not hold for the top of the cloud where air is pushed aside, nor for the base of the cloud, where air converges. As we shall only compute the energy which is converted between the top and the base of the cloud, we ignore this complication. The part of the counter-current situated under the base of the cloud will also be excluded. Of course all this means more approximations.

Here equation (9.2) will be used in a somewhat other form:

$$M_m W_m = M_d W_d \tag{9.3}$$

the suffix m indicating quantities referring to the cloud and the suffix d indicating those referring to the counter-current. W_m and W_d represent the absolute values of the vertical velocities in cloud and counter-current respectively.

A cumuliform cloud, ascending in a descending environment gives rise to the consideration of three quantities of energy:

E, the lability-energy, released by the simultaneous motion of the cloud and its environment;

E_m, the kinetic energy the cloud obtains during the motion;

E_d, the kinetic energy of the descending counter-current.

We shall determine these three quantities, supposing the atmosphere to be homogeneous. Although this is only a rough approximation it will appear that this method gives results which can be used in practice.

10. Selective instability according to the column-method

Let us consider a cumuliform cloud, ascending as a coherent body (a *column*) in a descending environment. In consequence of the supposed homogeneity of the atmosphere it follows from the equation of continuity and the absence of convergent or divergent horizontal currents, that the vertical velocities at a certain moment are the same in every level and likewise in the counter-current. In order to determine E, it is necessary, first of all, to consider the vertical temperature distribution in the counter-current. In fig. 4 z_{t_0} represents the level in which the top of the cumuliform cloud is at the time t_0. At that moment the environment in z_{t_0} begins to descend, and after the time $dt = \frac{z_{t_0} - z}{W_d}$ this part of the counter-current reaches the level z.

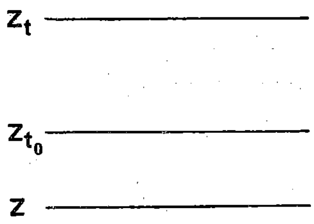

Fig. 4

The top of the cloud has ascended to the level z_t in the same time, so that dt is equal to $\frac{z_t - z_{t_0}}{W_m}$ also.

W_d and W_m must be understood to be mean values during the time dt. From the above z_{t_0} can be eliminated so that dt becomes a function of z and z_t only:

$$dt = \frac{z_{t_0} - z}{W_d} = \frac{z_t - z_{t_0}}{W_m} = \frac{z_t - z}{W_d + W_m} \tag{10.1}$$

In the level z the temperature [1]) of the descending environment has, therefore, increased between the moment the cloud top passed this level and the moment it reaches the level z by an amount:

$$T_d - T = (\gamma_d - \gamma)(z_{t_0} - z) = (\gamma_d - \gamma) W_d dt = (\gamma_d - \gamma) W_d \frac{z_t - z}{W_d + W_m} \qquad (10.2)$$

As $\frac{Wd}{W_d + W_m}$ is constant with time, because it is equal to $\frac{1/M_d}{1/M_d + 1/M_m} = \frac{M_m}{M_m + M_d}$, expression (10.2) is only a function of $(z_t - z)$ and $(\gamma_d - \gamma)$, so that the increase of temperature occurring in the descending environment in the level z, after the moment the top of the cloud has passed this level, is proportional to the height of the top above it.

The temperature difference between the cloud and the undisturbed environment is given by:

$$T_m - T = \tau_b + \int_0^z (\gamma - \gamma_m)\, dz \qquad (10.3)$$

τ_b denoting the initial temperature difference between the cloud and the undisturbed environment in the base level of the cloud, that is in $z = 0$.

By taking an average for γ, $(\gamma_d - \gamma)$ can be considered to be a constant. Approximately this holds for $(\gamma - \gamma_m)$ too. For clouds with very large vertical dimensions an average value for γ_m can be introduced.

Therefore we may write for the temperature difference between the cloud and its dry-adiabatically descending environment:

$$T_m - T_d = (T_m - T) - (T_d - T) = \tau_b + (\gamma - \gamma_m) z - (\gamma_d - \gamma) W_d \frac{z_t - z}{W_d + W_m} \qquad (10.4)$$

From this relation it follows, that $T_m - T_d$ in any level is a function of the height of this level above the base of the cloud as well as of the height z_t of the top of the cloud, a result which does not appear in the slice-method of BJERKNES and PETTERSSEN.

In order to compute the energy converted by the motion of the cloud we start from REFSDAL's expression for the energy of lability. For a unit mass of the cloud the energy converted when this unit mass moves from $z = 0$ to $z = z$ is:

$$E_1 = g \int_0^z \frac{\varrho_d - \varrho_m}{\varrho_m} dz = g \int_0^z \frac{T_m - T_d}{T_d} dz \qquad (10.5)$$

Applying WEIERSTRASZ' theorem we may put:

$$E_1 = g \left[\overline{\frac{1}{T_d}}\right] \int_0^z \left\{ \tau_b + (\gamma - \gamma_m) z - (\gamma_d - \gamma) \frac{W_d}{W_d + W_m} (z_t - z) \right\} dz \qquad (10.6)$$

As (10.6) holds for unit mass of the cloud, ascending with the same velocity as the top of the cloud $(z_t - z)$ is constant. For that reason integration of (10.6) gives:

$$E_1 = g \left[\overline{\frac{1}{T_d}}\right] \left\{ \tau_b z + {}^1/_2 (\gamma - \gamma_m) z^2 - (\gamma_d - \gamma) \frac{W_d}{W_d + W_m} (z_t - z) z \right\} \qquad (10.7)$$

The energy of lability, converted by the motion of the cloud as a whole is obtained by the integration $\int_0^\mu E_1 d\mu$ where μ represents the total mass of the cloud. For μ we can write $M_m z_t$, M_m being the mass of the unit layer under consideration, and z_t the height

[1]) Strictly speaking one ought to consider the virtual temperature, SCHNAIDT (1942) pointed out, however, that the use of the ordinary temperature in clouds is at least as accurate.

of the cloud. As M_m is independent of z, owing to the homogeneity of the atmosphere, the total energy is equal to $E = M_m \int_0^{z_t} E_1 \, dz$. This applied to (10.7) gives:

$$E = M_m g \int_0^{z_t} \overline{\left[\frac{1}{T_d}\right]} \left\{ \tau_b z + {}^1/_2 (\gamma - \gamma_m) z^2 - (\gamma_d - \gamma) \frac{W_d}{W_d + W_m} (z_t - z) z \right\} dz \tag{10.8}$$

E is, therefore, the lability-energy converted into kinetic energy (neglecting friction) as a result of the ascending of the cloud and the descending of its environment.

To this integral WEIERSTRASZ' theorem is applied again, so that we get finally the following expression for E:

$$E = M_m g \overline{\overline{\left[\frac{1}{T_d}\right]}} \left[{}^1/_2 \tau_b z_t^2 + {}^1/_6 (\gamma - \gamma_m) z_t^3 - {}^1/_6 (\gamma_d - \gamma) \frac{W_d}{W_d + W_m} z_t^3 \right] \tag{10.9}$$

The kinetic energy of the cloud is equal to:

$$E_m = \tfrac{1}{2} M_m W_m^2 z_t \tag{10.10}$$

and the kinetic energy of the descending counter-current to:

$$E_d = \tfrac{1}{2} M_d W_d^2 z_t \tag{10.11}$$

In order to simplify the comparison of the equations to some extent we now introduce the quantity:

$$\sigma = \frac{M_m}{M_d} = \frac{W_d}{W_m} \tag{10.12}$$

Then the equations for the three amounts of energy can be written as:

$$E = M_m g \overline{\overline{\left[\frac{1}{T_d}\right]}} \left[{}^1/_2 \tau_b z_t^2 + {}^1/_6 (\gamma - \gamma_m) z_t^3 - {}^1/_6 (\gamma_d - \gamma) \frac{\sigma}{1 + \sigma} z_t^3 \right] \tag{10.13}$$

$$E_m = {}^1/_2 M_m W_m^2 z_t \tag{10.14}$$

$$E_d = {}^1/_2 \sigma M_m W_m^2 z_t \tag{10.15}$$

Now $\frac{dE}{dt} = \frac{dE_m}{dt} + \frac{dE_d}{dt}$ when we neglect the friction.

If we neglect the starting impulse which, however small it may be, is always present, integration of this last relation gives: $E = E_m + E_d$ or

$$W_m^2 (1 + \sigma) = g \overline{\overline{\left[\frac{1}{T_d}\right]}} \left[\tau_b z_t + {}^1/_3 (\gamma - \gamma_m) z_t^2 - {}^1/_3 (\gamma_d - \gamma) \frac{\sigma}{1 + \sigma} z_t^2 \right] \tag{10.16}$$

From this relation it follows, that the sign of W_m^2 still depends on σ when z_t, $(\gamma - \gamma_m)$ and $(\gamma_d - \gamma)$ are given. In particular there is a value $\sigma = \sigma_l$ making the right hand side zero. The value σ_l represents a limiting value. This means, that if $\sigma > \sigma_l$, the form

$$\left[\tau_b z_t + {}^1/_3 (\gamma - \gamma_m) z_t^2 - {}^1/_3 (\gamma_d - \gamma) \frac{\sigma}{1 + \sigma} z_t^2 \right]$$

becomes negative; an ascent of moist air could only take place under the influence of external forces. This kind of ascent, however, does not lead to the origin of big cumuliform clouds so that relation (10.16) marks the existence of a selective instability. When clouds have different vertical dimensions $\overline{\overline{\left[\frac{1}{T_d}\right]}}$ has different values too. As this fact influences the value of W_m^2 for a few percents only, it will be ignored henceforth.

When we take $\tau_b = 0$ (this being initially the case in the convective condensation level) the relation between the upward velocity in the cloud and σ can be written as:

$$3\,\frac{W_m^2}{z_t^2} = \frac{(\gamma-\gamma_m)(1+\sigma)-(\gamma_d-\gamma)\,\sigma}{(1+\sigma)^2} = (\gamma_d-\gamma_m)\,\frac{\frac{\gamma-\gamma_m}{\gamma_d-\gamma_m}(1+\sigma)-\frac{\gamma_d-\gamma}{\gamma_d-\gamma_m}\,\sigma}{(1+\sigma)^2}$$

$$= (\gamma_d-\gamma_m)\,\frac{\Gamma(1+\sigma)-(1-\Gamma)\,\sigma}{(1+\sigma)^2} \backsim \frac{\Gamma(1+2\sigma)-\sigma}{(1+\sigma)^2} = \varepsilon \qquad (10.17)$$

Here $\Gamma = \frac{\gamma-\gamma_m}{\gamma_d-\gamma_m}$ so that $0 < \Gamma < 1$ means, that the atmosphere is conditionally unstable according to the particle-method.

When the atmosphere is absolutely stable Γ is < 0, while for an absolutely unstable atmosphere Γ is > 1.

From (10.17) it follows that the limiting value of σ is equal to:

$$\sigma_l = \frac{\Gamma}{1-2\Gamma} \qquad (10.18)$$

Now relation (10.17) will be subjected to a more detailed examination.

From fig. 5 it appears, that in an absolutely stable atmosphere ($\Gamma < 0$) the formation of cumuliform clouds is always accompanied by loss of energy; this loss counter-acts the formation of the clouds. In an absolutely unstable atmosphere the formation of cumuliform clouds is always accompanied by the conversion of energy of lability. This is also the case for a conditionally unstable atmosphere and $\Gamma \geq 0{,}5$.

The most interesting case is that of conditional instability and $0 < \Gamma < 0{,}5$. In this case positive as well as negative energy may appear, that is to say the formation of cumuliform clouds can be intensified or suppressed after a quantity of air has for some reason passed beyond the condensation level.

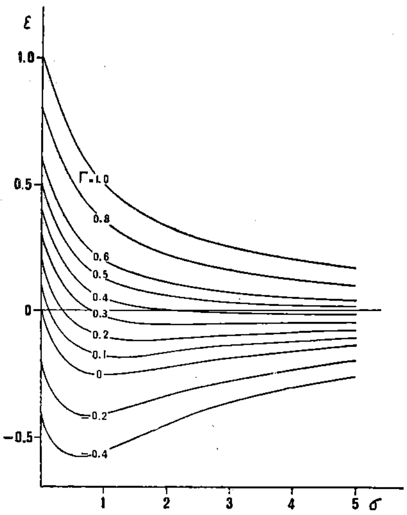

Fig. 5

We can say that approximately $\sigma = \frac{M_m}{M_d} = \frac{F_m}{F_d}$, F_m and F_d being the cross-sections of the cloud and the counter-current respectively, so that for a fixed value of conditional instability the sign of the converted energy depends only on the ratio of the cross-sections of cloud and counter-current. It appears for example, that, when $\Gamma = 0{,}3$, cumuliform clouds with a cross-section larger than 0,74 times that of the counter-current are suppressed, so that we may conclude that there exists a certain selectivity for the external causes of the origin of cumuliform clouds when the atmosphere is conditionally unstable. This means that only those external disturbances in the condensation level give rise to lasting cumuliform clouds for which $\varepsilon(\sigma,\Gamma) > 0$. When for an external cause $\varepsilon(\sigma,\Gamma) > 0$, no lasting cumuliform clouds can be formed, but only a cumulus humilis, disappearing very soon again. This last case may occur in a conditionally unstable as well as in an absolutely stable atmosphere.

To every value, therefore, of Γ belongs a limiting value σ_l indicating that for $\sigma > \sigma_l$ no lasting cumuliform clouds are possible. The theory of Bjerknes and Petterssen as given in chapter I also gives a value σ_l of this kind. In figure 6 the limiting values of σ according to Petterssen and to the theory developed here are plotted against Γ. It appears, that our limiting value is larger than the one found by Petterssen, i. e. the possibility of the formation of big cumuliform clouds according to our theory is greater than according to Petterssen's.

11. The amount of cumuliform clouds

Relation (10.17): $\frac{W_m^2}{z_t^2} \propto \frac{\Gamma(1+\sigma)-\sigma}{(1+\sigma)^2}$ leads to still another interesting conclusion.

In an isolated mechanical system, all conversions of energy will take place in such a way that the potential energy of the system will become a minimum; when this potential energy is totally converted into kinetic energy, this means that the latter becomes a maximum. For the isolated system of a large number of cumuliform clouds in a descending environment, this means that $F_m W_m^2 + F_d W_d^2$ will be a maximum, or $\sigma F W_m^2$ will be a maximum, F being $F_m + F_d$.

We can say, therefore, that in an atmosphere in unstable equilibrium those disturbances will be selected that make σW_m^2 a maximum.

In an atmosphere with a certain Γ, W_m^2 is a function of z_t and σ. We will now suppose, that at a given instant t all cumuliform clouds in the area considered have approximately the same vertical dimension z_t. Then our condition reads:

$$\sigma\varepsilon = \frac{\sigma\Gamma(1+2\sigma)-\sigma^2}{(1+\sigma)^2} = \max. \tag{11.1}$$

Differentiation with respect to σ shows that $\sigma\varepsilon$ has a maximum value for

$$\sigma = \sigma_m = -\frac{\Gamma}{3\Gamma - 2} \tag{11.2}$$

This relation shows, that σ_m is positive for $0 < \Gamma < \frac{2}{3}$. When $\Gamma > \frac{2}{3}$, σ_m becomes negative. This has the following meaning:

When $0 < \Gamma < \frac{2}{3}$ we can find a most probable ratio between the cross-section of the clouds and that of the counter-current, that is to say, a most probable degree of cloudiness by applying (11.2). In figure 6 σ_m too is plotted against Γ. σ_m appears always to be smaller than σ_l, which might be expected a priori. For $\Gamma = 0{,}3$ the most probable degree of cloudiness becomes $\frac{0{,}27}{1+0{,}27} = 0{,}21$.

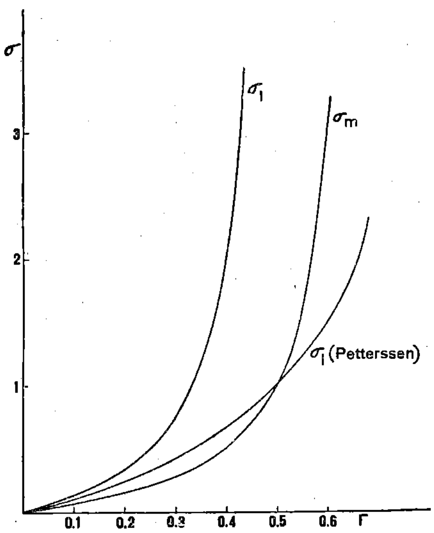

Fig. 6

When $\Gamma > \frac{2}{3}$, we can say nothing as to the most probable cloudiness, as a negative σ_m has no sense. Then all values of σ have the same probability and the cross-sections of the various cumuliform clouds are in that case determined by the external disturbances. *As a consequence we may expect the appearance of the sky to be less regular when Γ is large then when $\Gamma < \frac{2}{3}$.*

Petterssen too concluded in a slightly different way, that the appearance of the sky will be more chaotic for large than for small values of Γ. This conclusion turns out to be confirmed by experience.

It must be remarked here that a first attempt to forecast the degree of cloudiness was made by Poulter (1938). He supposed F_m/F_d to be determined exclusively by the relative humidity of the outer air in the convective condensation level. Hewson (1938) in the discussions following Mr. Poulter's article pointed out already that this supposition was too simple and that it leads to inconsistencies.

It is not necessary to add much to Hewson's arguments. In chapter VI it will be shown, that the distribution of relative humidity plays a part in the growth of cumuliform clouds, but in quite another way than Poulter stated.

12. The column-method and Kopp's measurements

In this chapter a theory of the counter-current is developed in which not only the energy converted in a single layer is considered, but in which the energy converted in the whole cloud and its counter-current from base to top is determined. In order to be able to do so, the cloud is considered to be a rigid system, which, of course, is only an approximation.

Meanwhile this method throws new light on an old question to which no definite answer has been given up to now: Are KOPP's measurements of temperatures inside clouds correct and if they are, how is it possible that they agree with the theoretical results of REFSDAL and others, results which are confirmed in practice?

The theory of the counter-current developed here is able to give a more satisfactory answer to this question than the simple statement that KOPP's measurements are not reliable at all or that they have no bearing on cumuliform clouds at their first stage.

It must be admitted that some of the clouds KOPP (1930, 1933) investigated are to be considered as cumuli humiles, clouds which are indeed generally colder than their environment.

Other clouds he crossed at their very top, where the temperature may be lower than that of the environment as a consequence of the fact, that the cloud by inertia passed beyond its equilibrium position.

In other cases, however, there must be another reason for the phenomenon. At first sight it would appear possible to explain the temperature difference as a consequence of adiabatic compression of air by the thermometer. Experiments by VAN DER MAAS and WYNIA (1938) showed, however, that the dry adiabatic compression at a height of 3000 m and a velocity of 200 km/h amounts to 2° C. As can easily be shown from an adiabatic diagram, this means that the temperature difference due to the difference between dry adiabatic compression in the environment and saturated adiabatic compression in the cloud is at most 0,8° C, a value far too small to explain KOPP's results.

Such an explanation is possible by applying our theory of the counter-current.

If a cumuliform cloud can be considered as a relatively rigid entity, this includes as a consequence of the preceding developments, that in its lower part the cloud may be colder than its environment.

For the counter-current starts at the base level of the cloud so that the temperature will have increased mostly there. Near its top the cloud is warmer than its environment as the counter-current has only lasted a short time in this region so that the temperature of the environment has only slightly increased.

In the upper layers of the cloud, therefore, positive energy of lability is converted into kinetic energy whereas near the base of the cloud the negative energy of lability reduces the kinetic energy of the cloud. If, however, the positive energy near the top of the cloud over-compensates the negative energy near its base, the cloud as a whole will ascend in the atmosphere. In this way the lower layers of the cloud are dragged upwards by the rest of the cloud (see figure 7).

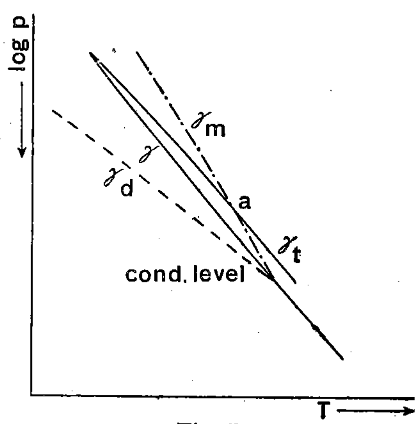

Fig. 7

When the temperature is measured below level *a*, flying almost horizontally from the counter-current into the cloud or vice versa, the temperature measured in the cloud will be lower than that measured in the descending environment the temperature of which is given by the temperature curve γ_t. The deviation of γ_t from γ_d depends on the velocity W_d. The larger W_d, the more γ_t will approach to γ_d. According to this, KOPP's measurements may have been correct and nevertheless we are not obliged to introduce an over-saturation of several hundreds percents; KOPP's results, therefore, do not contradict the principles formulated by REFSDAL and others. As WENZEL (1933) and RENNER (1939) crossed the clouds which they investigated in other directions, it can be understood that they were not able to confirm KOPP's results. For WENZEL only crossed the higher parts of cumuliform clouds, when according to our theory the temperature difference between the cloud and its environment will be positive. RENNER crossed the cumulonimbus he investigated in a vertical direction from base to top and then returned at some distance from the cloud. As remarked already, the supposition of a rectangular velocity profile only forms an approximation. In reality it will show a fluent course with the largest downward velocities in the immediate neighbourhood of the cloud (figure 3). At some distance from the cloud the descent of air and as a consequence the rise of temperature are so small, that the temperature here remains lower than in the cloud at the same level. In this way RENNER's results can be explained.

13. Concluding remarks

The column-method does not furnish the maximum height the cloud will reach, $(\gamma_d - \gamma)$ and $(\gamma - \gamma_m)$ being constant and τ_b being zero.

Only when $\tau_b > 0$ and $\varepsilon < 0$ does the column-method give an upper limit for the height of the cloud, this limit being lower than the limit according to the particle-method.

In reality, the conditions $(\gamma_d - \gamma) =$ constant and $(\gamma - \gamma_m) =$ constant are never satisfied. The introduction of a mean value of γ leads to a constant value of $(\gamma - \gamma_d)$, but $(\gamma - \gamma_m)$ remains a variable quantity, as γ_m is a function of the pressure.

In order to indicate how the calculations would develop if $(\gamma_d - \gamma)$ and $(\gamma - \gamma_m)$ were variable quantities, we give the foregoing calculations in a more general form:

Formula (10.2) then changes into:

$$T_d - T = \int_z^{z_t} (\gamma_d - \gamma) \frac{W_d}{W_d + W_m} dz = \frac{W_d}{W_d + W_m} \int_z^{z_t} (\gamma_d - \gamma)\, dz \qquad (13.1)$$

while the temperature difference between the saturated adiabatic referring to the cloud and the original temperature curve is given by:

$$T_m - T = \tau_b + \int_o^z (\gamma - \gamma_m)\, dz \qquad (13.2)$$

Then (10.4) changes into:

$$T_m - T_d = (T_m - T) - \frac{W_d}{W_d + W_m} \int_z^{z_t} (\gamma_d - \gamma)\, dz \qquad (13.3)$$

For a unit mass of the cloud the following expression for the energy holds:

$$E_1 = g \left[\overline{\frac{1}{T_d}} \right] \int_o^z \left[(T_m - T) - \frac{W_d}{W_d + W_m} \int_z^{z_t} (\gamma_d - \gamma)\, dz \right] dz \qquad (13.4)$$

and for the cloud as a whole:

$$E = M_m g \int_o^{z_t} \left[\overline{\frac{1}{T_d}} \right] \int_o^z \left[(T_m - T) - \frac{W_d}{W_d + W_m} \int_z^{z_t} (\gamma_d - \gamma)\, dz \right] dz \,.\, dz \qquad (13.5)$$

When $(\gamma_d - \gamma)$ is an arbitrary function of z we cannot find a convenient mathematical form for E. A graphical integration is possible though rather complicated. At first we should have to determine the function:

$$\varphi(z) = \int_{0}^{z} \left[(T_m - T) - \frac{W_d}{W_d + W_m} \int_{z}^{z_t} (\gamma_d - \gamma)\, dz \right] dz \tag{13.6}$$

for a large number of values of z, z_t being given. This can conveniently be done on energetic paper possessing a height-scale. The function $\varphi\,(z)$ should be plotted against z on the same paper, eventually multiplied with the associated values of $\left[\frac{1}{T_d}\right]$ and integrated graphically from $z = 0$ to $z = z_t$.

Another method for generalizing the considerations is to find other than constant functions for $\gamma_d - \gamma$. We shall always find the two areas of figure 7, as is easy to understand from this figure. This means that a selective instability also exists in these cases, which was of course to be expected.

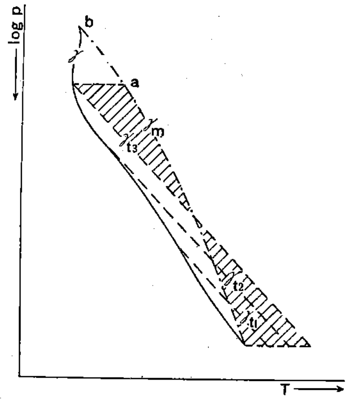

Fig. 8

From these qualitative considerations follows an important conclusion with respect to the vertical dimensions of a cumuliform cloud. In figure 8 γ represents an arbitrary temperature curve and γ_m the saturated adiabatic referring to the cloud; σ being constant the descent of air in the counter-current will give rise to the temperature curves γ_{t1}, γ_{t2}, γ_{t3}. Now the cloud will only ascend as long as the total energy released is positive. That means, that the cloud will only reach level a as above this level the total energy released is negative. For that reason the level b in which the cloud at its top becomes colder than its environment is usually not reached. This means that a spreading of the cumuliform clouds at their tops will occur rather seldom, by which the usual dome-shape is explained.

We will return to this question in chapter VI in connection with both form drag and skin-friction drag.

CHAPTER III. THE AMOUNT OF CUMULIFORM CLOUD

14. The method of investigation

What we have investigated is the extent to which the relation $\sigma_m = -\frac{\Gamma}{3\Gamma - 2}$ or $N_\sigma' = \frac{10\sigma_m}{1+\sigma_m} = \frac{5\Gamma}{1-\Gamma}$, N_σ' being the computed cloudiness in tenths of the sky, agrees with the amount of cumuliform cloud observed from the ground.

In order to do so we have used the aerological data and the amount of cloud observed in Holland.

The aerological material was obtained by means of airplane ascents at Soesterberg from January 1, 1931 to September 1939 and at Schiphol from September 1939 to May 1940.

For the amount of cloud we used the synoptic observations of De Bilt (situated at a distance of 7 km from Soesterberg) at 07.00, 10.00, 13.00, 16.00 and 18.00 gmt, and some abbreviated reports, especially of the morning hours. As regards the aerodrome Schiphol the halfhourly observations during the period favourable for the formation of cumuliform clouds could be used.

The material was carefully selected in order to make certain the correctness of the value Γ. In the first place only those days were examined on which $C_L = 2$ (cumulus congestus) was codified and on which at least two ascents had been made. Some hundred days satisfied this double condition.

The results of the airplane soundings were then further examined. As the registration of the humidity is as a rule not sufficiently reliable, no corrections for virtual temperatures were introduced. As a test of the reliability of the pressure- and temperature-data, the registrations obtained during the ascent were compared with those obtained during the descent. As a rule these two do not coincide in the diagram, partly because the plane never passes the same air during the descent as during the ascent, but mainly owing to the lag of the registering instruments.

Those soundings were selected for which the temperature lapse-rate in a layer of about 100 mb thickness above the convective condensation level was the same during the ascent and during the descent.

Rossi (1940) observed, that the height of the convective condensation level was about 1,44 times that computed from the surface data of the diagram, a fact which must be ascribed either to the turbulent mixing between the ascending air volume and its usually dryer environment or to the existence of a pre-cloud stage. We ignore, however, this effect as we do not possess sufficiently reliable data concerning it. It must be borne in mind however, that this may lead to quantitatively incorrect results.

The equality of the lapse-rates for ascent and descent was considered as an indication that the temperature lapse-rate was well represented, in spite of the lag of the meteorograph. Only those days were examined in detail on which at least one of the soundings satisfied this condition well, while the others do not show too large a deviation. Of the registrations collected in this way, those during the ascents were considered to be correct. The aerological registrations satisfied this condition only on 44 days.

However these 44 days cannot be used directly for a comparison of N_σ' with N_h' (the observed amount of cloud in tenths).

In the first place, conditions are favourable when the time at which clouds $C_L = 2$ are observed falls between two soundings or a short time after the last sounding. Days on which the cumuliform clouds occur a long time before the first or after the last sounding are as unfavourable for our investigation as days on which only one sounding was made. If the time at which the cumuliform clouds were observed falls between two soundings, it is favourable when the lapse-rates according to the two soundings do not differ much.

Attention must be paid to the fact that it is our intention here to check a theoretical formula. Once having stated the agreement between this formula and the observations the theory can be applied to days on which only one sounding is made, if of course this sounding is a reliable one and the atmospheric conditions do not vary rapidly during the day.

In the second place the relation $N_\sigma' = \frac{5\Gamma}{1-\Gamma}$ has been deduced on the supposition that $\gamma - \gamma_m$ and $\gamma_d - \gamma$ are constant quantities, that means that γ is constant above the condensation level. For that reason registrations showing large variations in the lapse rate, especially inversions, in a layer of 100 mb thickness above the condensation level have not been used. Hence the number of days providing data for our investigation is once more reduced.

A further difficulty in comparing N_σ' with N_h' is that N_h' usually is not known, but must be deduced from the code-figure for the amount of cloud. For this purpose, we applied the following schema:

N_h	=	0	1	2	3	4	5	6	7	8
N_h'	=	0	0,5	1,0	2,5	5,0	7,5	9,0	9,5	10

When different values of N_h are reported the arithmetic mean of the corresponding N_h' is computed. It is also possible to express the computed amount of cloud in the synoptic code N_h. We call this number N_σ.

It might be possible to utilize the climatological data in which the degree of cloudiness is given in tehnths. It appears, however, that only on a very small number of days this application would be possible, as generally medium and high clouds also cover part of the sky and only the total cloudiness is available.

When N_σ' has a very large value, the real degree of cloudiness N_h' must be expected to be smaller than N_σ'. For in deriving the formula for N_σ' we neglected the fact that the velocities in the counter-current become very large if N_σ' is large. Owing to these large velocities, the greater part of the impulses which would give rise to saturated adiabatic ascending air currents in the cloud-free space will be suppressed, so that the theoretical value N_σ' is not reached. This fact, which was pointed out already by PETTERSSEN (1939), will appear in the result of our investigation.

15. Survey of the data used

In the annexed table the following data are given of the 44 soundings investigated:

- *N°.* The number of the day, from 1 to 44.
- *y.m.d.* The date, indicated by year, month and day.
- t_c The time at which $C_L = 2$ was observed, expressed in gmt.
- t_A The moment at which the aerological sounding was started, also expressed in gmt.
- h The estimated height of the cloud base according to the Copenhagen code.
- N_h The estimated amount of cloud according to the Copenhagen code.
- q_o The specific humidity at ground level, i. e. at a height of 2 meters, from which the convective condensation level is determined.
  We generally took for q_o the specific humidity determined at the aerodrome before the starting of the plane. This has the advantage, that quantities are compared which have been obtained under similar circumstances. As a rule the difference between q_o obtained in this way and that determined at the moment the cloud was observed is small, so that the height of the condensation level and the value of Γ are hardly influenced by the choice of q_o. In some cases, however, q_o as determined at the aerodrome appeared to lead to improbable results. In these cases the specific humidity, determined at De Bilt during the cloud observation was used. At Schiphol such difficulties did not occur. The value of q_o, which have been determined from the humidity-registrations at De Bilt, reduced to the synoptic psychrometer observations, have been marked by an asterisk.

$P_1P_1P_1$ The height of the computed convective condensation level, given by the pressure in millibars. This pressure $P_1P_1P_1$ is obtained in the classical manner by intersecting the q_o-line on a REFSDAL aerogram by the geometric temperature curve.

T_{max} The maximum temperature observed on the day in question at De Bilt, respectively Schiphol.

T_{cond} The temperature that must be reached at ground level in order that the air shall be able to reach its convective condensation level, rising dry adiabatically.

γ A measure for the geometric temperature lapse-rate above the condensation level.

γ_m A measure for the saturated adiabatic temperature lapse-rate above the condensation level.

γ_d A measure for the dry adiabatic temperature lapse-rate above the condensation level. The three quantities γ, γ_m and γ_d are determined from the temperature differences between the pressures $P_1P_1P_1$ and $P_1P_1P_1$-50 measured along the geometric temperature curve, the saturated adiabatic and the dry adiabatic respectively. These temperature differences are determined graphically by means of REFSDAL's aerogram.

Γ In the last column, the value of Γ has been entered.

N°.	y	m	d	t_C	t_A	h	N_h	q_o	$P_1P_1P_1$	T_{max}	T_{cond}	γ	γ_m	γ_d	Γ
1	31	6	20	12.30	06.05	4	4	8.5	965	18.8	10.1	3.3	2.4	4.3	0.47
				13.00	12.20	5	4	8.4	930		17.8	3.3	2.5	4.6	0.38
				18.00		6	3								
2	32	3	9	09.00	08.10	4	4	3.7	950	4.1	3.4	3.4	2.8	4.3	0.40
					17.10			3.2	950		2.0	3.4	2.8	4.3	0.40
3	32	4	27	10.00	08.10	4	5	5.0	950	13.6	8.0	2.6	2.6	4.2	0.00
				11.30	17.55	5	5	4.8*	890		12.0	2.7	2.8	4.5	—0.06
				13.00		5	5								
				13.30		5	4								
				14.00		5	4								
				14.30		5	4								
4	32	6	8	09.30	07.25	4	5	7.1	930	16.5	15.8	3.8	2.6	4.5	0.63
					19.00			6.7	940		14.1	3.4	2.6	4.5	0.42
5	32	6	23	08.30	07.10	5	4	7.8	950	19.7	15.5	3.3	2.4	4.4	0.45
				09.00	13.00	5	5	8.5*	960		16.8	3.6	2.6	4.5	0.53
				09.30		5	5								
				10.00		5	5								
				10.30		4	5								
				12.00		5	4								
				12.30		5	5								
				13.00		5	5								
				13.30		5	5								
				14.00		5	4								
6	32	7	13	12.00	07.35	5	2	13.7	980	26.9	20.8	2.6	2.0	4.2	0.27
				12.30	18.55	5	2	13.1	905		20.6	3.6	2.3	4.7	0.54
				13.00		5	4								
				13.30		5	4								
				14.00		5	4								
				14.30		5	4								
7	32	7	27	09.00	07.40	4	6	8.2	900	19.7	18.8	3.1	2.5	4.6	0.29
				10.00	18.50	4	5	9.3	950		16.9	3.4	2.3	4.4	0.52
8	32	8	25	10.30	07.10	4	3	7.9	900	23.9	20.0	1.7	2.5	4.7	—0.36
				12.30	13.40	5	5	9.3	905		21.8	4.6	2.5	4.7	0.95
9	32	9	12	08.30	08.40	4	4	7.2	930	17.6	15.2	4.1	2.7	4.4	0.83
					20.55			7.3	900		17.3	4.3	2.5	4.5	0.90
10	32	10	27	13.00	07.05	5	2	6.0	910	12.2	12.5	3.6	2.7	4.4	0.53
				13.30	13.10	5	3	6.0	910		12.5	3.5	2.7	4.4	0.47
				14.00		5	4								
				14.30		5	3								
11	32	11	24	13.00	07.40	4	4	4.3	895	8.4	10.0	2.6	2.8	4.5	—0.12
					12.55			4.2	890		10.0	3.6	2.3	4.5	0.59
12	33	2	22	08.00	10.00	—	1	3.4	930	2.6	4.0	2.8	2.6	4.3	0.12
				08.30	14.40	—	1	3.2	890		6.0	3.3	2.8	4.6	0.28
				11.00		4	3								
				11.30		4	3								
13	33	3	8	11.30	10.25	3	4	6.0*	950	11.9	13.8	3.2	2.6	4.4	0.33
				13.00	14.55	5	3	6.4	970		11.8	3.4	2.5	4.3	0.50
14	33	5	11	11.00	07.00	4	5	6.5	945	16.6	12.3	4.2	2.5	4.8	0.74
				11.30	14.40	5	4	8.4*	970		14.7	4.0	2.4	4.3	0.84

N°.	y	m	d	t_C	t_A	h	N_h	q_o	$P_1P_1P_1$	$T_{\max}$	T_{cond}	γ	γ_m	γ_d	Γ
15	33	7	12	13.00	07.10	4	3	9.6	970	20.9	16.4	2.8	2.3	4.4	0.24
					18.30			8.7	900		19.8	3.8	2.5	4.7	0.59
16	33	8	24	14.00	07.10	5	5	8.0	930	21.1	16.6	3.5	2.5	4.5	0.50
					14.00			8.6*	910		19.5	3.7	2.5	4.6	0.57
17	33	8	30	10.30	07.10	3	3	14.6*	910	24.8	28.0	2.8	1.9	4.8	0.31
				12.00	14.50	4	4	13.5*	975		22.0	2.8	2.0	4.6	0.31
				12.30		4	4								
18	34	2	26	11.30	07.05	4	4	5.0	995	8.2	5.0	2.6	2.4	4.1	0.12
				13.00	13.45	5	3	3.1	920		4.0	3.8	2.9	4.4	0.60
				13.30		5	4								
				14.00		5	4								
19	34	5	18	14.00	07.10	6	4	4.8*	890	17.1	12.2	3.3	2.7	4.6	0.32
					14.35			4.8*	840		16.0	3.5	3.0	4.9	0.26
20	34	7	26	07.30	07.00	4	5	8.9	900	21.3	20.8	0.0	2.3	4.8	— 0.92
				08.00	14.00	4	4	11.2	960		20.0	2.3	2.2	4.5	0.04
				08.30		4	4								
21	34	8	2	18.00	07.05	5	2	9.6	885	21.2	22.3	4.2	2.4	4.7	0.78
					09.25			9.9	880		23.0	3.6	2.5	4.8	0.48
22	34	8	23	12.30	07.02	5	3	9.6	920	20.1	20.0	3.5	2.5	4.6	0.48
				13.00	15.15	5	2	8.4*	900		19.9	3.2	2.4	4.5	0.38
				13.30		5	3								
				14.00		5	3								
23	34	9	10	12.00	08.05	5	4	8.0	890	22.2	20.2	3.3	2.4	4.6	0.41
				12.30	13.45	5	3	8.8	930		19.0	3.3	2.4	4.6	0.41
24	34	10	4	12.30	07.00	5	3	7.0*	840	17.7	19.6	2.9	2.6	4.9	0.31
				13.00	13.30	5	3	7.7	865		19.3	3.9	2.6	4.8	0.59
				13.30		5	3								
				14.00		5	3								
25	35	2	26	09.30	07.05	6	1	4.0	920	7.8	6.0	3.8	2.8	4.4	0.62
				11.00	13.55	5	3	3.6	900		6.4	3.6	2.9	4.6	0.41
				13.00		6	4								
				13.30		6	4								
				14.00		6	4								
26	35	5	8	13.30	07.05	5	4	6.8	970	15.8	12.2	3.5	2.5	4.3	0.56
					13.45			5.0	860		16.0	0.3	2.8	4.9	— 1.19
27	36	10	6	11.30	06.50	5	4	4.1*	940	9.9	6.5	3.7	2.7	4.4	0.59
					13.45			4.1*	910		9.0	3.7	2.7	4.4	0.59
28	37	3	18	11.00	06.55	5	4	5.2	870	12.8	14.3	3.6	2.9	4.8	0.37
				13.00	09.40	4	5	6.4*	930		12.4	3.7	2.7	4.5	0.55
				13.30	12.40	4	4	5.8	890		14.5	3.6	2.7	4.5	0.50
				14.00	15.35	4	4	5.8*	890		13.8	3.5	2.7	4.5	0.44
29	37	6	21	13.00	07.00	5	5	7.7	980	17.2	12.6	2.9	2.3	4.2	0.32
				13.30	13.50	5	5	7.0*	900		17.0	4.2	2.5	4.7	0.77
30	37	7	5	07.30	06.55	4	3	8.0	940	18.5	16.2	0.7	2.5	4.6	— 0.86
					10.10			8.2	910		19.0	0.8	2.5	4.6	— 0.81
31	38	4	28	09.30	07.05	5	3	4.5*	840	16.6	15.7	4.0	3.0	4.9	0.53
				14.00	13.10	6	3	4.5*	840		15.7	3.8	3.0	4.9	0.42
				14.30		6	3								
32	39	3	7	10.00	07.05	4	3	4.9	950	8.8	8.3	3.0	2.5	4.2	0.29
				11.00	15.20	4	5	4.0	930		6.9	3.7	2.7	4.4	0.59
33	39	4	3	08.30	07.45	5	2	5.6	920	13.9	10.8	3.8	2.5	4.4	0.68
				09.00	13.10	5	2	5.0	880		12.8	4.0	2.6	4.8	0.64
				15.00	19.00	5	4	5.8	920		11.3	3.4	2.5	4.4	0.47
34	39	4	19	16.00	17.10	4	4	5.0	920	14.7	15.0	1.7	2.5	4.5	— 0.40
				18.00	17.25	—	1	5.6	940		12.6	1.7	2.6	4.6	— 0.45
35	39	4	27	15.00	07.00	5	4	4.1	940	11.5	7.0	3.3	2.6	4.3	0.41
					16.00			3.6	870		10.4	3.7	2.9	4.6	0.47
36	39	6	22	09.00	06.15	5	4	12.4	880	29.6	27.2	2.4	2.2	4.9	0.07
				13.00	19.50	6	2	12.6	890		27.0	3.0	2.3	4.9	0.27
				14.00		6	2								
				15.00		6	2								
				16.00		6	2								
				18.00		—	2								
37	39	6	23	13.00	09.05	4	5	10.7	810	25.3	30.4	3.0	2.6	5.1	0.16
					19.50			10.9	800		31.5	2.9	2.5	5.1	0.15
38	39	9	5	17.00	06.00	6	1	10.3	950	22.0	19.5	2.4	2.2	4.4	0.09
				17.30	16.50	6	2	9.9	870		25.5	3.1	2.5	4.9	0.25
				18.00		6	2								
				18.30		6	1								
				19.00		6	1								
				19.30		6	1								
39	39	9	15	08.30	05.50	4	3	7.7	950	18.0	14.4	3.2	2.5	4.5	0.35
				09.00	15.30	4	3	8.0	890		19.5	3.7	2.6	4.6	0.55
				09.30		4	4								
				17.30		5	1								

N°.	y	m	d	t_C	t_A	h	N_h	q_o	$P_1P_1P_1$	T_{max}	T_{cond}	γ	γ_m	γ_d	Γ
				18.00		5	1								
				18.30		5	1								
40	39	9	19	05.30	05.55	6	2	8.3	930	19.0	18.2	3.2	2.4	4.5	0.38
				06.00	15.30	6	2	7.7	890		20.0	3.3	2.6	4.7	0.31
				06.30		6	2								
				07.00		6	2								
				07.30		6	3								
41	39	9	21	06.30	05.50	5	3	8.5	925	18.5	18.5	2.9	2.6	4.6	0.15
				07.00	15.30	5	2	7.4	875		20.1	3.2	2.6	4.7	0.29
				07.30		5	3								
				08.00		5	3								
				12.30		5	2								
				13.00		5	2								
				13.30		5	2								
				14.00		5	3								
				14.30		6	2								
				15.00		6	2								
42	39	10	6	12.30	07.40	4	4	7.8	920	16.0	16.9	2.9	2.4	4.4	0.25
				13.00	15.25	5	4	6.5	885		16.7	4.1	2.7	4.7	0.70
				13.30		5	3								
43	39	10	16	15.30	06.05	4	4	6.5	960	13.8	11.0	2.8	2.5	4.3	0.17
				17.00	15.10	4	2	6.5	920		15.0	3.3	2.6	4.5	0.37
44	40	4	16	07.00	06.32	4	4	3.8	910	8.8	6.0	3.2	2.7	4.4	0.29
				07.30	15.20	4	4	3.5	865		8.5	3.4	2.8	4.6	0.33
				08.00		4	4								
				08.30		4	4								
				09.00		5	4								
				09.30		5	4								
				12.00		5	4								
				12.30		5	4								
				13.00		5	4								
				13.30		5	4								
				14.00		5	4								
				14.30		5	3								

16. Discussion of the data

Here follows a short discussion concerning the results which can be deduced from the preceding table.

We shall take for N_σ' the value which corresponds to the expression

$$\Gamma(t) = \Gamma(t_1) + \left\{ \Gamma(t_2) - \Gamma(t_1) \right\} \frac{t - t_1}{t_2 - t_1} \tag{16.1}$$

where t_1 means the time of the first sounding, t_2 the time of the second one and t is the mean time at which the maximum cloudiness was observed.

Here a difficulty arises: It is possible, and in the following examples it actually occurs, that one of the soundings takes place at a time at which the clouds have already partly been formed. That means, that the value of γ registrated by such a sounding has been influenced by the counter-current. As it is difficult to calculate this influence we ignore it henceforth. This is even the more permissible, as the profile through the counter-current is in reality of a shape as shown in figure 3. When therefore the sounding takes place at some distance from the cloud, as will usually be the case, we are allowed to neglect the influence of the already existing counter-current on the vertical temperature lapse-rate.

1. As the cumulus observation of 18.00^h occurs a long time after the last aerological sounding only the observations of 12.30^h and 13.00^h are of importance; we take $t = 12.30$ so that $\Gamma = 0{,}38$, from which we obtain $N_\sigma' = 3{,}1$ or $N_\sigma = 3$. According to the observations $N_h' = 5{,}0$ and $N_h = 4$.

2. Both soundings satisfy the conditions. It follows from the data that $\Gamma\,(09.00) = 0{,}40$; therefore $N_\sigma' = 3{,}3$ and $N_\sigma = 3$. One single cumulus observation shows that $N' = 5{,}0$, $N_h = 4$.

3. The first sounding shows indifferent equilibrium. The second does not satisfy the condition that γ shall not depend on the height, so that this day cannot be used.

4. Both soundings are applicable, Γ (09.30) $= 0{,}63 - {}^2/_{12} \times 0{,}21 = 0{,}60$. $N_\sigma' = 7{,}5$, $N_\sigma = 5$; $N_h' = 7{,}5$ and $N_h = 5$.

5. Both soundings are tolerably suitable. The mean time is 10.30. Γ (10.30) $= 0{,}45 + {}^1/_2 \times 0{,}08 = 0{,}49$. $N_\sigma' = 4{,}8$, $N_\sigma = 4$; $N_h' = {}^1/_{10}\,(3 \times 5 + 5 \times 7{,}5) = 6{,}7$ or $N_h = 5$.

6. Both soundings can be used. For the maximum cloudiness $t = 13.45$. Γ (13.45) $= 0{,}27 + {}^6/_{11} \times 0{,}27 = 0{,}44$. $N_\sigma' = 3{,}9$, $N_\sigma = 4$; $N_h' = 5{,}0$ and $N_h = 4$.

7. The first sounding has no constant temperature lapse-rate. The second took place about seven hours after the cumulus observation. We cannot use this day.

8. A strong inversion 50 mbar above the condensation level makes both soundings useless.

9. The soundings took place after the cumulus observation. Moreover, the first sounding shows a temperature lapse-rate which has no constant value. Of no use.

10. Both soundings satisfy the conditions. Γ (14.00) $= 0{,}47$. $N_\sigma' = 4{,}4$; $N_\sigma = 4$. $N_h' = 5{,}0$; $N_h = 4$.

11. The first sounding shows a variation of lapse-rate just above the condensation-level and takes place long before the cumulus observation. The second satisfies the conditions very well and coincides with the cumulus observation. Γ (13.00) $= 0{,}59$, $N_\sigma' = 7{,}2$; $N_\sigma = 5$; $N_h' = 5{,}0$ and $N_h = 4$.

12. Although $T_{cond} > T_{max}$, both soundings seem very suitable. The difference can be ascribed to the fact, that T_{max} has not been determined at the same place as the data leading to T_{cond}. Γ (11.00) $= 0{,}12 + {}^1/_5 \times 0{,}16 = 0{,}15$. $N_\sigma' = 0{,}9$; $N_\sigma = 2$; $N_h' = 2{,}5$ and $N_h = 3$.

13. Both soundings are suitable. For Γ we take a mean value: $\Gamma = 0{,}42$. $N_\sigma' = 3{,}6$; $N_\sigma = 4$; $N_h' = {}^1/_2\,(5 + 2{,}5) = 3{,}8$ and $N_h = 4$.

14. Both soundings satisfy the conditions. Γ (11.00) $= 0{,}74 + 0{,}05 = 0{,}79$. $N_\sigma' = 19$ so that it is considerably higher than 10. $N_h' = 7{,}5$ and $N_h = 5$.

15. Both soundings are suitable. The lapse-rates, however, differ rather much. Γ (13.00) $= 0{,}24 + \frac{5{,}5}{11{,}5} \times 0{,}35 = 0{,}41$. $N_\sigma' = 3{,}5$; $N_\sigma = 4$; $N_h' = 2{,}5$ and $N_h = 3$.

16. Both soundings are suitable. Γ (14.00) $= 0{,}57$. $N_\sigma' = 6{,}6$, $N_\sigma = 5$; $N_h' = 7{,}5$ and $N_h = 5$.

17. In both soundings the lapse-rate varies with height.

18. The Γ-difference is very large. The morning-sounding is only known up to 800 mbar, while the lapse-rate of the second is not constant.

19. Not suitable owing to strong variations in the lapse-rate of the afternoon-sounding.

20. Owing to strong variations in the lapse-rate of both soundings not suitable.

21. As the only cumulus observation took place eight hours after the last sounding this day cannot be used.

22. Both soundings satisfy the conditions. Γ (13.00) $= 0{,}48 - {}^6/_8 \times 0{,}10 = 0{,}40$. $N_\sigma' = 3{,}3$; $N_\sigma = 3$; $N_h' = {}^1/_4\,(1 + 3 \times 2{,}5) = 2{,}1$ and $N_h = 3$.

23. The first sounding shows an inversion at 90 mbars above the condensation level. The second one, which is near to the moment of cumulus observation is suitable however. $\Gamma = 0{,}41$. $N_\sigma' = 3{,}5$, $N_\sigma = 4$; $N_h' = {}^1/_2\,(5 + 2{,}5) = 3{,}8$ and $N_h = 4$.

24. The second sounding, which has been achieved during the cumulus observation has no constant γ. So we must cancel this day.

25. Both soundings satisfy the conditions. Γ (13.30) = 0,41. $N_\sigma' = 3{,}5$; $N_\sigma = 4$; $N_h' = 5{,}0$ and $N_h = 4$.

26. The afternoon sounding shows a strong inversion just above the convective condensation-level. The difference between q_o and q_o^* is very large. On the whole these soundings look rather unreliable.

27. When q_o^* is used both soundings satisfy the conditions. Γ (11.30) = 0,59. $N_\sigma' = 7{,}2$; $N_\sigma = 5$; $N_h' = 5{,}0$ and $N_h = 4$.

28. The first sounding is of no importance for our purpose. The three remaining ones give Γ (13.00) = 0,50. $N_\sigma' = 5{,}0$; $N_\sigma = 4$; $N_h' = {}^1/_4 (3 \times 5 + 7{,}5) = 5{,}6$ and $N_h = 4$.

29. The afternoon sounding, the most important one for our investigation as the cumulus congestus was observed at 13.00^h and 13.30^h, shows a lapse-rate variation at 50 mbars above the condensation-level.

30. Owing to the occurrence of very strong inversions in both soundings, not suitable.

31. Both soundings satisfy the conditions. We can neglect the morning cumulus observation. Γ (14.00) = $0{,}42 - {}^1/_6 \times 0{,}11 = 0{,}40$. $N_\sigma' = 3{,}3$; $N_\sigma = 3$; $N_h' = 2{,}5$ and $N_h = 3$.

32. Both soundings satisfy the conditions. Γ (10.30) = $0{,}29 + {}^7/_{16} \times 0{,}30 = 0{,}42$. $N_\sigma' = 3{,}6$, $N_\sigma = 4$; $N_h' = 5{,}0$ and $N_h = 4$.

33. The 08.30^h sounding is of no value owing to small inversions. The most important cumulus observation took place at 15.00^h, that is between the second and the third sounding, so that Γ (15.00) = $0{,}64 - {}^2/_6 \times 0{,}17 = 0{,}58$. $N_\sigma' = 6{,}9$; $N_\sigma = 5$; $N_h' = 5{,}0$ and $N_h = 4$.

34. The vertical temperature lapse-rate is not constant for both soundings.

35. Both soundings satisfy the conditions. Γ (15.00) = $0{,}41 + {}^8/_9 \times 0{,}06 = 0{,}46$. $N_\sigma' = 4{,}3$; $N_\sigma = 4$; $N_h' = 5{,}0$ and $N_h = 4$.

36. Owing to the occurrence of a number of small inversions and to the fact that the second sounding only the registrations during the downward flight are known, this date must be cancelled.

37. Using q_o, the convective condensation level appears to lie much too high. Using q_o^* the height of the condensation-level agrees with h, but in that case the lapse-rates of both soundings are not constant.

38. As the observation times of the cumuliform clouds fall after the last sounding and the values of Γ following from both soundings differ greatly, we cannot use this case for a comparison of N_σ' with N_h'.

39. Only the cumulus observations of 08.30^h, 09.00^h and 09.30^h can be compared with the results of the soundings. Γ (09.00) = $0{,}35 + {}^3/_{10} \times 0{,}20 = 0{,}41$. $N_\sigma' = 3{,}5$ and $N_\sigma = 4$. If the rate of cloudiness of 08.30^h is ascribed to the initial convection, it follows that $N_h' = {}^1/_2 (2{,}5 + 5) = 3{,}8$ and $N_h = 4$.

40. In the afternoon sounding an inversion occurs at 70 mbars above the condensation-level. As the clouds are formed about eight hours before this sounding and, moreover, the latter appears very suitable, we ignore this complication. Owing to the early hour at which the clouds were observed only the N_h of 7.30^h was considered to be correct. Γ (07.30) = $0{,}38 - {}^2/_{10} \times 0{,}07 = 0{,}37$. $N_\sigma' = 2{,}9$, $N_\sigma = 3$; $N_h' = 3{,}5$ and $N_h = 3$.

41. Both soundings satisfy the conditions. We take the mean value ${}^1/_2 (0{,}15 + 0{,}29) = 0{,}22$ for Γ. $N_\sigma' = 1{,}4$, $N_\sigma = 2$; $N_h' = {}^1/_{10} (6 \times 1 + 4 \times 2{,}5) = 1{,}6$ and $N_h = 3$.

42. Unsuitable owing to strong lapse-rate variations in the second sounding.

43. Both soundings satisfy the conditions. $N_h = 2$ at 17.00^h must be ascribed to the vanishing of the clouds. Γ (15.30) = 0,37. $N_\sigma' = 3{,}0$, $N_\sigma = 3$; $N_h' = 5{,}0$ and $N_h = 4$.

44. Both soundings satisfy the conditions. For Γ we do well to take the mean value 0,31. $N_\sigma' = 2{,}2$; $N_\sigma = 3$; $N_h' = 5{,}0$ and $N_h = 4$.

17. Application to the forecasting of cumuliform clouds

From the above survey it follows, that among the 44 days investigated 26 can be used for a comparison of N_σ with N_h, respectively N_σ' with N_h'. These 26 cases are the numbers: 1, 2, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 22, 23, 25, 27, 28, 31, 32, 33, 35, 39, 40, 41, 43, 44.

The following table gives the values of the computed and the observed cloudiness for these 26 days.

N°.	N_σ'	N_h'	N_σ	N_h	N°.	N_σ'	N_h'	N_σ	N_h
1	3,1	5,0	3	4	23	3,5	3,8	4	4
2	3,3	5,0	3	4	25	3,5	5,0	4	4
4	7,5	7,5	5	5	27	7,2	5,0	5	4
5	4,8	6,7	4	5	28	5,0	5,6	4	4
6	3,9	5,0	4	4	31	3,3	2,5	3	3
10	4,4	5,0	4	4	32	3,6	5,0	4	4
11	7,2	5,0	5	4	33	6,9	5,0	5	4
12	0,9	2,5	2	3	35	4,3	5,0	4	4
13	3,6	3,8	4	4	39	3,5	3,8	4	4
14	19	7,5	>8	5	40	2,9	2,5	3	3
15	3,5	2,5	4	3	41	1,4	1,6	2	3
16	6,6	7,5	5	5	43	3,0	5,0	3	4
22	3,3	2,1	3	3	44	2,2	5,0	3	4

From this table we can compute the mean value of N_σ' belonging to various values of N_h'. The result is plotted in figure 9 which shows a rather good agreement between the theoretical and the observed amounts of cloud. Only for large values of N_h' does the theoretical value seem to be somewhat larger than the observed one. This can be explained

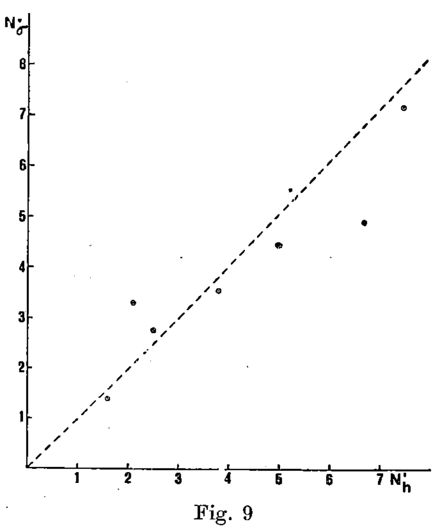

Fig. 9

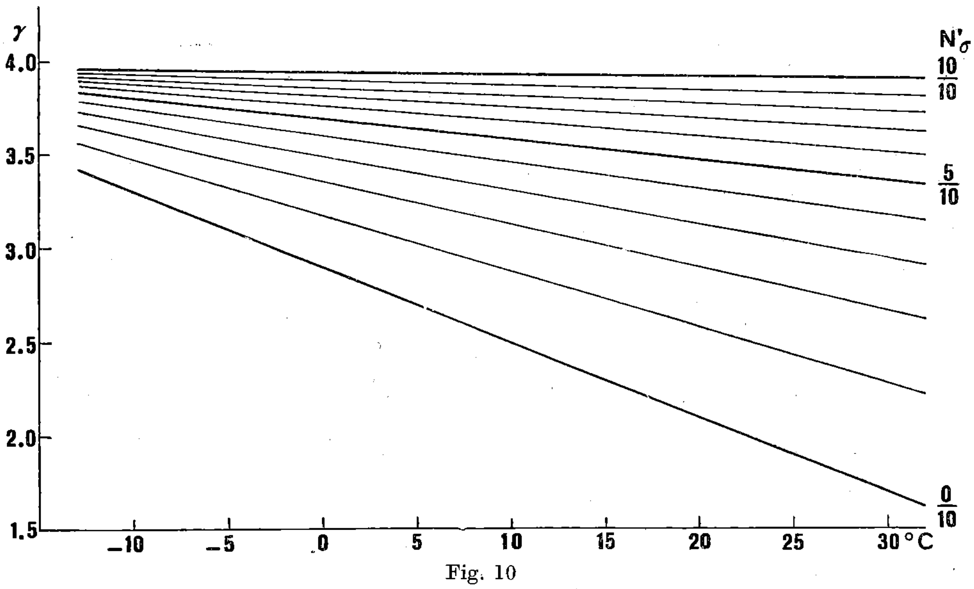

Fig. 10

by the fact that the counter-current suppresses part of the initial impulses for large values of σ, which is not taken into consideration by theory.

Concluding we may say that a succesful forecast of the amount of cumuliform cloud must depend on many factors which are sometimes difficult to ascertain. Especially the variations of the vertical lapse-rate which occur continually can lead to unpleasent surprises.

Moreover, it is evident, that in the first place the aerological data should be sufficiently reliable.

Nevertheless, it may be of importance for practical purposes to be able to derive N_σ' quickly by graphical means. It is possible to construct diagrams, which make this graphical derivation possible. They can be used for one pressure only and allow to find the amount of cloud by cumuliform clouds as a function of temperature in the condensation level and the temperature difference between this level and that laying 50 mbars higher, measured along the geometric temperature curve. The latter yields a value for γ. Figure 10 shows such a diagram; it can be used for a pressure of 900 mb in the condensation level. It is quite easy to construct similar diagrams for other pressures.

If we want to apply the diagrams, the conditions introduced in the foregoing pages must, of course, be satisfied.

CHAPTER IV. THE PROPAGATION OF JETS

18. Introduction

In considering the propagation of a jet of fluid in a space filled with the same fluid we can distinguish between several problems.

In the first place we can distinguish between jets according to their form. Two forms have been investigated up to now, two-dimensional jets and axially symmetrical ones. As in general, a better approximation is obtained by considering a cumulus congestus to be axially symmetrical than two-dimensional we shall only consider axially symmetrical jets.

We can divide the latter into three kinds:

a. Jets with the same temperature as their environment;

b. Jets that show a temperature difference with their environment, this temperature difference not affecting the velocities in the jet.

c. Jets that owing to a temperature difference, show a density difference with the environment affecting the velocities in the vertically placed jet.

In computing the motion of these jets we can use two conceptions concerning turbulent mixing, the one of PRANDTL (1925) who suggested that impulse is a transferable quantity and the one of TAYLOR (1915, 1932) who supposed vorticity to be transferable.

In PRANDTL's opinion the velocity- and temperature distributions are identical, according to TAYLOR there is a slight difference between the two. Though neither of the two theories is absolutely satisfactory in every respect (see e. g. GOLDSTEIN II, 1938), we can say, that generally TAYLOR's conception accounts the best for the experimental data. We shall nevertheless take PRANDTL's standpoint as this leads to simpler considerations. We are the more allowed to do so as our calculations only lead to a rough approximation of the third problem mentioned above.

PRANDTL introduced the so-called mixing length l, this quantity playing a part more or less analogous to the free path in kinetic theories.

A summary of the theoretical investigations on turbulence can be found in GOLDSTEIN I (1938). Later investigations by REICHARDT (1942) do not influence importantly our considerations.

19. a. The propagation of jets having the same density as their environment

The problem of the propagation of a stationary symmetrical jet in a space filled with fluid of the same density, was solved by TOLLMIEN (1926). Let the mouth of the nozzle through which the jet discharges itself be situated at $z = o$ and the jet be propagated in the direction of the positive z-axis with a velocity W. It appears that such a jet is spread out conically in the direction of propagation. This spreading is caused by the turbulent mixing of fluid from the jet with fluid from the space in which the jet is discharged. It follows from experiment that the linear dimensions of the area where this mixing takes place are proportional to z. On the other hand theory teaches that the intensity of mixing must be proportional to the mixing length l. TOLLMIEN therefore supposed l to be proportional to z, $l = cz$. As always in turbulence theory c must be determined by experiment.

In order to make the mathematical solution possible TOLLMIEN supposed the nozzle to be infinitely small. To such a nozzle corresponds an infinite initial velocity. Moreover, he introduced the limitation, that the pressure gradient in the direction of the flow is negligeably small, so that friction is the only force acting in the direction of the z-axis. Testing the theory by experiment provides an a posteriori justification of this neglect.

From these suppositions it follows, that the total momentum in the z-direction is constant:

$$M = \int_0^\infty 2\pi\varrho W r\, dr = \text{constant} \tag{19.1}$$

As we stated above, the boundary of the jet is defined by

$$r_l = kz \tag{19.2}$$

The combination of (19.1) and (19.2) leads to the expression: $W = \frac{1}{z} f(\eta)$, η denoting r/z. For on substitution (19.1) becomes:

$$M = \int_0^\infty 2\pi\varrho f(\eta)^2 \eta d\eta = \int_0^k 2\pi\varrho f(\eta)^2 \eta d\eta = \text{constant} \tag{19.3}$$

The form of $f(\eta)$ is obtained by solving the equation of motion for the stationary case:

$$W\frac{\partial W}{\partial z} + V\frac{\partial W}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}\left\{ l^2 r\left(\frac{\partial W}{\partial r}\right)^2 \right\}, \tag{19.4}$$

V being the radial velocity. The solution is given in the form of an infinite series. It appears that W has a maximum value W_{max} in the axis of the jet and decreases gradually

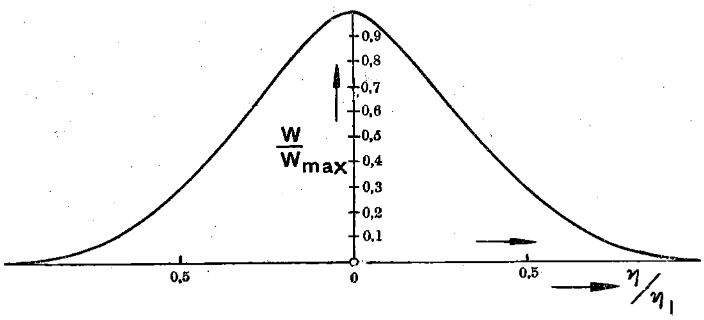

Fig. 11

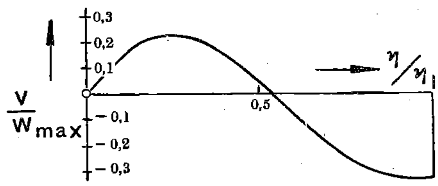

Fig. 12

to zero at the boundary of the jet. In figure 11 W/W_{max} is plotted against η/η_l. It turns out from experiment that $r_l = 0{,}214z$, *i. e.* $\eta_l = k = 0{,}214$. $W = 1/zf(\eta)$ so that in the axis of the jet W_{max} is proportional to $1/z$. In what follows this velocity distribution along the axis will be considered in detail.

Tollmien also calculated the lapse of the radial velocity V. It follows from continuity, that near the axis of the jet a divergent motion must exist, whereas near the boundary of the jet the fluid converges radially. In figure 12 V/W_{max} is plotted against η/η_l.

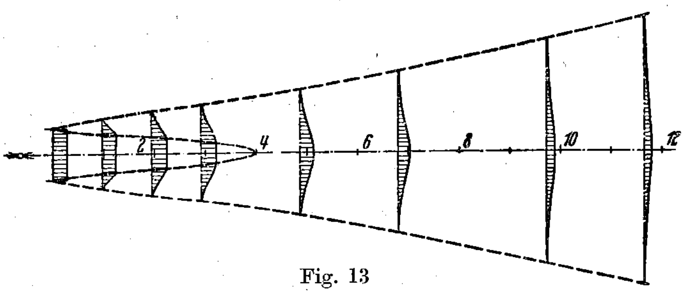

Fig. 13

A consequence of the fact that W is inversely proportional and r_l is directly proportional to z, is that the amount of fluid, flowing through a cross-section of the jet in a unit time is directly proportional to z. In equal cross-section of two jets, that is for equal values of z, the amounts of fluid passing in unit time will be in the proportion of W_{max} in both cross-sections.

Practice differs, however, from theory in some respects.

First, in reality the cross-section of the nozzle is not infinitely small. As a result, the flow pattern deduced theoretically by Tollmien is not found immediately, but only at a distance $z = 5D$ from the mouth of the nozzle, D being the diameter of the nozzle. For $z < 5D$ W_{max} is constant, whilst originally the distribution of W is different from the one indicated in figure 11 (fig. 13). Moreover, it appears that the origin of co-ordinates must be chosen $0,66D$ below the mouth of the nozzle if we wish to apply TOLLMIEN's results. When the various quantities are fixed in this way, we obtain the following relation as was pointed out by RUDEN (1933):

$$W_{\max} = \frac{6{,}45\ D}{z_0}\ W_0 \tag{19.5}$$

where W_0 denotes the velocity in the mouth of the nozzle. With the aid of figure 11 the distribution of the velocity over a cross-section can then be determined from $W_{\max}$, defined by (19.5). In investigating the influence of friction on the velocity distribution in vertical currents of air, we shall start from this relation.

A first difficulty met with on applying the theoretical results of TOLLMIEN and RUDEN to real cases in the atmosphere arises from the existence of the counter-currents, appearing round cumuli congesti. They are the logical consequence of the equation of continuity when the density remains locally constant. The counter-current should be considered as a whole together with the upward motion in the cloud. Together these two form the circulation around the isobaric-isosteric solenoids correlated to a cumulus cloud, which ascends in a conditonally unstable atmosphere. Hence the ascending current, which shows itself as a cumuliform cloud, forms only part of a circulation.

In the case of the jet, no counter-current occurs. The feeding of the jet takes place through the mouth of the nozzle, that means from another space than that into which the jet emerges. The streaming off of air at the end of the jet takes place outside the wind tunnel, or at least far away from the place where the jet is measured. There is no question of isobaric-isosteric solenoids here. The jet is an independent entity.

In considering the surface-friction drag of clouds, we shall not take into account the counter-current. In the last chapter the combined effect of surface-friction drag and form drag (counter-current) will be considered qualitatively.

Even if we neglect the counter-current, the analogy between a cloud ascending in the atmosphere and a jet which is discharged in a space filled with fluid, is not complete.

In the first place a fixed outlet is missing in the case of the cloud. Henceforth it will be supposed, that the velocity distribution of fig. 11 exists already at the base of the cloud. This supposition is not at variance with the forms, observed at the tops of cumuli. The flattening that ought to exist if the velocity distribution were as in fig. 13 for $z > 5D$ is never observed at the top of growing cumuli. Moreover, the current forming the cloud above the condensation level usually exists already at some distance under the condensation level. For that reason the current below the condensation level has usually the opportunity to acquire TOLLMIEN's velocity distribution. Besides, as the fixed nozzle leaks, air from the environment of the cloud will immediately be dragged along by turbulence. The supposition, that at the base of the cloud TOLLMIEN's velocity distribution exists seems, therefore, to be admissible.

In the second place the cloud particles often undergo an upward directed acceleration, due to the temperature difference between the cloud and its environment, in other words: the cloud particles move in the upward branch of the circulation around the isobaric-isosteric solenoids. Moreover, this temperature difference is altered by the turbulent mixing. It will appear, however, that TOLLMIEN's solution for the jet can be generalised in such a way, that this temperature difference can be taken into account.

20. b. The turbulent diffusion of heat in a jet

In order to compute the temperature distribution in a jet we must consider the heat content per unit volume $\varrho c T$ as a transferable quantity. Here ϱ denotes the density of the air, c the specific heat and T the absolute temperature. When the heat content is interchanged, only the difference of $\varrho c T$ between the two layers between which the interchange takes place plays a role. In a homogeneous liquid we can confine, therefore, ourselves to $\varrho c\tau$ where τ denotes the temperature difference between the layers. According to PRANDTL the same laws hold in a homogeneous liquid for this quantity $\varrho c\tau$ as for momentum.

According to this the equation for the heat transport becomes:

$$W \frac{\partial(c\tau)}{\partial z} + V \frac{\partial(c\tau)}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}\left\{ l^2 r \frac{\partial(c\tau)}{\partial r}\frac{\partial W}{\partial r} \right\} \tag{20.1}$$

In this equation $\partial\varrho/\partial r$ has been neglected against $\partial\tau/\partial r$. In the atmosphere c is a constant $= c_p$.

In considering atmospherical processes we ought to use potential temperature ϑ instead of temperature. As, however, the turbulent interaction between cloud and environment takes place horizontally, we can omit the introduction of potential temperature.

In analogy to (19.5) we write now:

$$\tau_{\max} = \frac{6{,}45}{z} D\tau_0 \tag{20.2}$$

Applied to the jet, τ_o denotes the constant temperature difference between the fluid in the mouth of the nozzle and the fluid in the undisturbed space outside the jet. In applying formula (20.2) to a cloud in the atmosphere, the introduction of a constant τ_o means, that according to the particle-method a constant temperature difference exists between the cloud and its environment, independent of height.

Later experimental investigations, for example by HOWARTH (1938) have shown, that the heat transport, occurring in jets cannot be explained completely by PRANDTL's theory, that is to say, heat transport and impulse transport do not obey the same laws.

A better insight into the mechanism of transport in jets is probably obtained by starting from TAYLOR's vorticity transfer theory. This results in the fact that the constants of (20.2) should have slightly different values then.

This difference between the velocity and the temperature distribution in jets, was pointed out by RUDEN (1933). The differences between the results of the two theories are rather small however, and we neglect them henceforth.

21. c. A jet that shows a density difference with the environment

The third problem mentioned above was considered by W. SCHMIDT (1941). He supposed, that variations in the total momentum integrated over the whole cross-section of the two-dimensional or axially symmetrical jet are due to the density-difference between the jet and its environment. This density-difference is a consequence of the temperature-difference τ between jet and environment which is supposed to be an exponential function of the distance from the nozzle: $\tau = Ae^{-mz}$, A being independent of z, but of course varying over the cross-section, and m denoting a positive constant.

Now the following equation must hold:

$$\frac{dJ}{dz} dz = k dz \tag{21.1}$$

J being the flux of momentum in the z-direction and k the lifting power $g\,\frac{\varrho_o - \varrho}{\varrho}$.

Now J is proportional to ϱ and k to $1/\varrho$, ϱ being the density in the jet and ϱ_o the constant density of the environment.

As ϱ is a function of τ, relation (21.1) becomes very complicated. In order to avoid this the author substitutes ϱ_o for ϱ in both sides of the equation.

It is obvious that this may lead to mistakes, this becoming especially clear when we compare the theory with SCHMIDT's experimental results for the axially symmetrical case.

In the experiment, done with a heated jet of air, τ is of an order of 100 degrees. It is evident that in that case we may not replace ϱ by ϱ_o.

This can also be seen from the theoretical velocity distribution SCHMIDT finds on the basis of the above assumptions. He finds in the case of a plane jet a constant velocity in the z-direction, whereas in the axially symmetrical jet the velocity decreases exponentially with z. This is in contradiction with the supposition that the starting velocity in the nozzle is zero.

We must, therefore, admit that a rigorous solution of the problem presents serious difficulties.

As the case of a jet with another density than its environment is very important for our meteorological investigation, we shall give an approximate solution of the problem in the next chapter.

In doing so we shall apply TOLLMIEN's results for the jet without density difference and suppose that the temperature difference is distributed in the same way as the velocity.

22. The inhomogeneity of the atmosphere

Meanwhile another difficulty presents itself when we apply the foregoing results to the atmosphere. Equation (19.4) was originally solved by TOLLMIEN for a homogeneous fluid by the use of the equation of continuity

$$\frac{\partial W}{\partial z}+\frac{1}{r}\frac{\partial V}{\partial r}=0 \tag{22.1}$$

The fact that this equation does not hold for an inhomogeneous fluid complicates the development into series as applied by TOLLMIEN very materially in this case. The general solution is unknown up to now and for that reason the skin-friction drag can likewise be only calculated for cumuliform clouds ascending in a vertically homogeneous atmosphere. As already pointed out, $\partial\varrho/\partial r$ does not influence TOLLMIEN's results for more than 1 %.

Finally we must remember, that a cumuliform cloud never represents a perfectly symmetrical vertical current. This follows already from direct observation. Moreover, gliding pilots have made it clear, that a cumuliform cloud generally consists of a number of „cores" in which a strong ascent of air exists. Between these cores are areas with relatively weak ascent or even descent of air. See for instance U. PIELSTICKER (1940).

When henceforth cumuliform clouds are treated as symmetrical upward currents, this also means a simplification, compared with the forms usually occurring in nature.

CHAPTER V. THE SKIN-FRICTION DRAG OF CUMULIFORM CLOUDS ASCENDING IN A HOMOGENEOUS ATMOSPHERE

23. Introduction

A cumulus, ascending in a homogeneous atmosphere, shows some resemblance to a jet being discharged in a space filled with fluid. The relations derived for the velocity- and temperature distribution in such a jet can be generalized and applied to cumuliform clouds. We shall apply TOLMIEN's and RUDEN's results for axially symmetrical jets, as most cumuliform clouds are approximately axially symmetrical.

When cumuliform clouds ascend in the atmosphere the flow is turbulent owing to the velocity differences between the air inside and outside the cloud. By this turbulent motion particles of a cloud can change places with particles from its environment. Just as in the case of the jet, the consequence of this turbulent interchange is that particles of the outer air are caused to take part in the vertical motion, whilst the particles of the cloud are retarded by the transfer of momentum. For the vertical motion of a particle of the cloud with unit mass the following equation holds when we neglect the vertical component of CORIOLIS' force:

$$\frac{dW_m}{dt}+\frac{\partial\Phi}{\partial z}+\frac{1}{\varrho_m}\frac{\partial p}{\partial z}-\frac{1}{\varrho_m}R=0 \tag{23.1}$$

In this equation W_m denotes the vertical velocity of the particle, t the time, Φ the geopotential, z the height, ϱ_m the density of the cloud particle, p the pressure and R the friction per unit volume. In this chapter the index m will be applied to the quantities referring to the inside of the cloud and the index d to the quantities referring to the outside of the cloud.

Now it is possible to modify (23.1) in such a manner, that the new equation is analogous to equation (19.4).

In the first place $\Phi = gdz$ so that $\partial\Phi/\partial z = g$. If the outer air is at rest we may put $dp_d = -\varrho_d g dz$, where ϱ_d is the density of the outer air. We suppose once more that the vertical motions take place quasistatically, that is to say: the pressure is considered to be the same inside and outside the clouds at each level.

Accordingly (23.1) changes into:

$$\frac{dW_m}{dt}+g-\frac{g\varrho_d}{\varrho_m}-\frac{1}{\varrho_m}R=0 \tag{23.2}$$

As the vertical velocity in the environment is assumed to be zero (no counter-current) we omit in this chapter the index m in W_m.

If we now suppose, that the motion in the cloud is stationary, and if we substitute PRANDL's expression $-1/r\,\partial/\partial r\{\varrho_m l^2 r(\partial W/\partial r)^2\}$ for the friction R, the equation for the motion of a cloud-particle becomes:

$$W\frac{\partial W}{\partial z}+V\frac{\partial W}{\partial r}=\frac{\varrho_d-\varrho_m}{\varrho_m}g-\frac{1}{r}\frac{\partial}{\partial r}\left\{l^2 r\left(\frac{\partial W}{\partial r}\right)^2\right\} \tag{23.3}$$

As we have seen in chapter II, where we considered the form drag, the vertical motion in a cumuliform cloud cannot be stationary owing to the variation of density with height; we shall, however, ignore this complication in the present chapter. This means, that we consider a homogeneous atmosphere again.

Apart from $\frac{\varrho_d-\varrho_m}{\varrho_m}g$, equation (23.3) is identical with equation (19.4), the equation for the motion in a jet. Owing to the presence of the term $\frac{\varrho_d-\varrho_m}{\varrho_m}g$, the force per unit mass responsible for the vertical motion in the cumuliform cloud, TOLLMIEN's results for the jet must undergo a correction.

Apart from momentum, other qualities are also interchanged between the cloud and its environment, for example the water content per unit volume, A, and the heat content [1]) per unit volume. By water content we shall understand the absolute humidity in the outer air. In the cloud the water content is composed of the (maximum) absolute humidity and the liquid or solid water per unit volume in the cloud, so that A_m is usually greater than A_d. For that reason A_m decreases towards the edges of the cloud owing to the turbulent mixing and this leads to the evaporation of water or ice in the cloud and consequently to a decrease of temperature in the cloud. So by this interchange of A the density of the cloud increases and it is retarded relatively. We shall ignore this effect henceforth.

The process of heat-interchange between the cloud and its environment is described by equation (20.1):

$$W\frac{\partial(c\tau)}{\partial z}+V\frac{\partial(c\tau)}{dr}=-\frac{1}{r}\frac{\partial}{\partial r}\left\{l^2 r\frac{\partial(c\tau)}{\partial r}\frac{\partial W}{\partial r}\right\} \tag{23.4}$$

Dealing with the jet, the temperature difference between the jet in the aperture and the undisturbed fluid outside the jet, τ, was supposed to be constant. If no mixing occurred, the temperature difference between the jet and its environment would remain τ everywhere. No mixing (and no counter-current) in the atmosphere, means that the motions can be examined by means of the particle-method. This means that (setting aside the counter-current) the meteorological problem is a complete analogon of the hydrodynamical problem of the jet, if the temperature difference between the cloud and its environment remains constant, as it would be according to the particle-method.

Now usually the temperature difference between the cloud and its environment varies with altitude. As mentioned already in the preceding chapter, TOLLMIEN's and RUDEN's results can be generalized to the case of a variable temperature difference.

Finally we must mention another effect, although it will be neglected in this chapter.

When the vertical temperature lapse-rate in the atmosphere is not the dry adiabatic one, the particles which are carried along and in which no condensation occurs will become colder than their environment owing to their dry adiabatic cooling. They will, for that reason, even retard the upward motion of the cloud still more. That an ascent of air outside the upward flow which is visible as a cloud, occurs in reality was shown by soaring pilots (WALKER (1939)). We shall return to this question in the last chapter.

All effects neglected here (A-transport, dry adiabatic cooling of air carried along by the cloud, counter-current) retard the motion of the cloud. Our result will, therefore, give the minimum retardation of the cloud caused by the turbulent friction and turbulent heat transfer.

24. Scheme of investigation

The investigation of the influence of turbulent friction on the motion of cumuliform clouds can take place in three steps.

a. As the most simple case we can investigate the influence of turbulent friction if the cloud does not show any density difference, i. e. no temperature difference with its environments. This case corresponds directly to the normal case of a jet pouring into a space filled with fluid.

b. The temperature difference between the cloud and its environment is constant according to the parcel-method. This case corresponds with that of a jet showing in the nozzle a constant temperature difference with the undisturbed outer fluid.

c. The temperature difference between the cloud and its environment varies with height. In the case of the jet this would mean a variable temperature difference between the jet in the nozzle and the surrounding fluid.

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[1]) In meteorology $\varrho(c_p T + Lx)$ is often called the heat content per unit volume. It is difficult, however, to consider this quantity here, as x (the mixing-ratio) usually varies very irregularly with height. L is the condensation heat.

25. a. The cloud has an upward velocity and has the same density as its environment

In this case the turbulent friction shows itself exclusively as a retardation owing to the transfer of momentum from the cloud to its environment. An interchange of heat content does not take place, as $\tau = o$. The case may occur in nature when the geometric temperature curve and the saturated adiabatic which is followed by the cloud, coincide [1]). The cumulus will then be a cumulus humulis, in which case the kinetic energy of the cloud may arise from an impulse under the condensation level.

The equation of motion of the cloud is:

$$W \frac{\partial W}{\partial z} + V \frac{\partial W}{\partial r} = -\frac{1}{r} \frac{\partial}{\partial r} \left\{ l^2 r \left(\frac{\partial W}{\partial r} \right)^2 \right\} \tag{25.1}$$

Hence the same solution holds for the cloud as for the jet. This means, according to Ruden, that the cloud is spread out conically in the direction of the positive z-axis, with $dD_z/dz = 0{,}428$, where D_z is the cross-section of the cloud in the level z.

From chapter III it appeared, that we may suppose approximately Tollmien's velocity distribution to exist already at the cloudbase. This means that $z = o$ is situated at a distance $2{,}34\ D_b$ under the cloudbase, where D_b denotes the cross-section of the base. By analogy to the velocity distribution along the axis of the jet, as pointed out by Ruden the following expression holds for the velocity in the axis of the cloud:

$$W_z = \frac{6{,}45\ D_o}{z} W_o \tag{25.2}$$

Here W_z denotes the velocity in the axis of the cloud at the level z and W_o and D_o the velocity of outflow in and the cross-section of a fictitious aperture, situated somewhere under the base of the cloud. This means, that in a jet of air, flowing with the velocity W_o out of this aperture with cross-section D_o, the same velocity distribution would be established as in and above the base of the cloud.

As in the base of the cloud Tollmien's velocity distribution prevails, we can write

$$W_b = \frac{6{,}45\ D_o}{z_b} W_o = \frac{6{,}45\ D_o}{2{,}34\ D_b} W_o \tag{25.3}$$

where W_b and z_b denote the maximum velocity in the base and its height respectively, the latter being $2{,}34\ D_b$. From (25.2) and (25.3) it follows that:

$$W_z = \frac{2{,}34\ D_b}{z} W_b \tag{25.4}$$

If now we put $z = 2{,}34\ D_b + \lambda D_b$, where λ means the height above the cloudbase, expressed in terms of the cross-section of that base, we find for the maximum velocity at an arbitrary height λD_b above the base

$$W_\lambda = \frac{2{,}34}{2.34 + \lambda} W_b \tag{25.5}$$

The velocity profile in the cross-section λ of the cloud is Tollmien's profile of figure 11. From W_λ the velocity in every point of the cross-section λ can be obtained by means of this figure.

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[1]) In reality there is not *one* saturated adiabatic which the clouds follows. Different parts of the cloud originate mostly in different levels, whilst the mixing too leads to the result, that the same saturated adiabatic is not followed everywhere in the cloud. Besides, we cannot speak of one saturated adiabatic, because the various particles of the cloud do not contain the same quantity of water, as part of the water leaves the cloud as precipitation (Fjeldstad 1925). We will neglect this effect here, and will have in mind some average saturated adiabatic when we speak of „the saturated adiabatic".

For the case of an ascending air current of the same density as its environment the particle-method gives a constant velocity and a constant kinetic energy. When the turbulent skin-friction drag is taken into account, it appears, that these quantities decrease gradually. In figure 14 the velocity and the kinetic energy of unit mass in the axis of the cloud are plotted against the height, and compared with the constant velocity and the constant kinetic energy which are the results of the particle-method. It appears, that for $\lambda = 1$, that is at a height D_b above the cloudbase, the kinetic energy is reduced to half its value at the base, so that the motion of the cloud is strongly influenced by surface-friction.

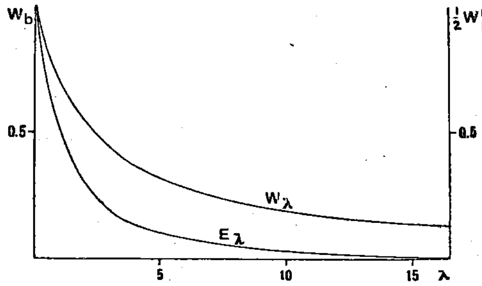

Fig. 14

26. b. The temperature difference between the geometric temperature curve and the saturated adiabatic, which a particle of the cloud would follow according to the parcel-method, is constant

According tot Refsdal's theory the conversion of potential energy is a necessary condition for the formation of cumuli congesti or cumulonimbi. This means, that the cloud must be lighter than its environment. For unit mass the following equation then holds:

$$W\frac{\partial W}{\partial z} + V\frac{\partial W}{\partial r} = \frac{\varrho_d - \varrho_m}{\varrho_m} g - \frac{1}{\varrho_m} R \tag{26.1}$$

When the turbulent friction is taken into account, we must, first, write $\frac{1}{r}\frac{\partial}{\partial r}\left\{ l^2 r \left(\frac{\partial W}{\partial r}\right)^2 \right\}$ for $\frac{1}{\varrho_m} R$, while, in the second place, the influence of the turbulent mixing on the term $\frac{\varrho_d - \varrho_m}{\varrho_m} g$ must be examined. In order to do so, we transform this expression into $\frac{T_m - T_d}{T_d} g = \frac{\tau}{T_d} g$, by applying the gas equation. We here consider the case that τ as furnished by the particle-method is a constant (figure 15).

We must now take into account two effects: 1°. the cloud is retarded by the interchange of momentum, 2°. the force $\frac{\tau}{T_d} g$ is reduced by the interchange of heat content.

Both effects give rise to a diminution of the acceleration due to the density difference. The individual change of velocity which a particle in the axis of the cloud undergoes, can be divided in the following way:

$$\frac{dW_\lambda}{dt} = \left(\frac{dW_\lambda}{dt}\right)_R + \left(\frac{dW_\lambda}{dt}\right)_\tau \tag{26.2}$$

Here $\left(\frac{dW_\lambda}{dt}\right)_R$ means the individual variation of velocity of a particle at the height λD_b above the cloud base arising from the turbulent interchange of momentum and $\left(\frac{dW_\lambda}{dt}\right)_\tau$ the individual variation of velocity of the same particle caused by the temperature-surplus of this particle with respect to the undisturbed outer air.

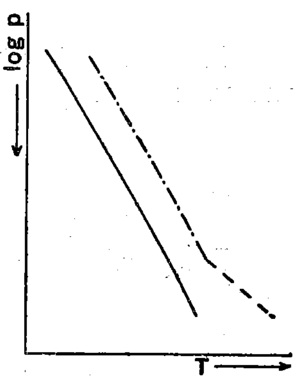

Fig. 15

We can suppose now, that at the base of the cloud, in addition to Tollmien's velocity distribution, a temperature distribution, as treated in chapter IV, exists. For the temperature difference τ_λ between a particle in the axis of the cloud at a height λD_b above the base of the cloud and the outer air the following expression holds, analogous to equation (25.5):

$$\tau_\lambda = \frac{2{,}34}{2{,}34 + \lambda} \tau_b \tag{26.3}$$

Here τ_b denotes the temperature difference between the axis of the cloud and the outer air at the base level of the cloud. We suppose that this temperature difference τ_b is equal to the temperature difference that would exist at the base level of the cloud according to the particle-method.

From this expression for τ_λ follows:

$$\left(\frac{dW_\lambda}{dt}\right)_\tau = \frac{2{,}34\ \tau_b}{(2{,}34 + \lambda)\ T_d}\, g \tag{26.4}$$

Analogous to (25.5), the velocity in the axis of the cloud can be written as:

$$W_\lambda = \frac{2{,}34}{2{,}34 + \lambda} f(\lambda), \tag{26.5}$$

so that W_b is replaced by a function $f(\lambda)$ and this function varies with λ under the influence of $\left(\frac{dW_\lambda}{dt}\right)_\tau$.

For a stationary flow in the cloud, the variation of the velocity with time is described by:

$$\frac{dW_\lambda}{dt} = W_\lambda \frac{\partial W_\lambda}{\partial z} = \frac{2{,}34}{2{,}34 + \lambda} f(\lambda) \left\{ f(\lambda) \frac{\partial}{\partial z}\left(\frac{2{,}34}{2{,}34 + \lambda}\right) + \frac{2{,}34}{2{,}34 + \lambda} \frac{\partial f(\lambda)}{\partial z} \right\} \tag{26.6}$$

The first term of the right-hand side represents the variation of the velocity caused by the interchange of momentum, that is $\left(\frac{dW_\lambda}{dt}\right)_R$, analogous to (25.5).

For the second term on the right hand side of (26.6) the following expression holds:

$$\frac{2{,}34^2}{(2{,}34 + \lambda)^2} f(\lambda) \frac{\partial f(\lambda)}{\partial z} = \left(\frac{dW_\lambda}{dt}\right)_\tau = \frac{2{,}34\ \tau_b\, g}{(2{,}34 + \lambda)\ T_d} \tag{26.7}$$

This equation can be solved using the relation $z = 2{,}34\ D_b + \lambda D_b$. From this relation follows:

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \lambda}\frac{d\lambda}{dz} = \frac{\partial}{\partial \lambda}\frac{d(\,^z/_{D_b} - 2{,}34)}{dz} = \frac{1}{D_b}\frac{\partial}{\partial \lambda}$$

The determination of the velocity distribution along the axis of the cloud depends on the solving of the following differential equation:

$$f(\lambda) \frac{\partial f(\lambda)}{\partial \lambda} = D_b \frac{2{,}34 + \lambda}{2{,}34} \frac{\tau_b}{T_d}\, g \tag{26.8}$$

Integration from o to λ gives:

$$\tfrac{1}{2}\left\{ f(\lambda)^2 - f(o)^2 \right\} = D_b \int_o^\lambda \frac{2{,}34 + \lambda}{2{,}34} \frac{\tau_b}{T_d}\, g\, d\lambda \tag{26.9}$$

Besides $\frac{2{,}34 + \lambda}{\lambda}$, $\frac{1}{T_d}$ too is a function of λ. In order to keep the integral simple, we accept an average value for $\frac{1}{T_d}$ according to WEIERSTRASS' theorem:

$$\tfrac{1}{2} f(\lambda)^2 = D_b \frac{\tau_b}{2{,}34}\, g \left[\overline{\frac{1}{T_d}}\right] \int_o^\lambda (2{,}34 + \lambda)\, d\lambda + \tfrac{1}{2} f(o)^2 = D_b \frac{\tau_b g}{2{,}34} \left[\overline{\frac{1}{T_d}}\right] \left\{ \tfrac{1}{2}(2{,}34 + \lambda)^2 - \tfrac{1}{2} \cdot 2{,}34^2 \right\} + \tfrac{1}{2} f(o)^2. \tag{26.10}$$

The kinetic energy of unit mass in the axis of the cloud at height D_b above its base is $E_\lambda = \frac{1}{2} W_\lambda{}^2 = \frac{1}{2}\left(\frac{2{,}34}{2{,}34+\lambda}\right)^2 f(\lambda)^2$. Applying (26.10) we can write for this expression:

$$E_\lambda = D_b \frac{\tau_b g}{2{,}34}\left[\overline{\frac{1}{T_d}}\right]\left\{\frac{1}{2}\cdot 2{,}34^2 - \frac{1}{2}\frac{2{,}34^4}{(2{,}34+\lambda)^2}\right\} + \frac{1}{2}\left(\frac{2{,}34}{2{,}34+\lambda}\right)^2 f(o)^2 =$$

$$= \frac{1}{2}\left[\overline{\frac{1}{T_d}}\right] D_b \tau_b g \,.\, 2{,}34\left\{1-\left(\frac{2{,}34}{2{,}34+\lambda}\right)^2\right\} + \left(\frac{2{,}34}{2{,}34+\lambda}\right)^2 E_b \; ^{1)} \qquad (26.11)$$

where E_b denotes the kinetic energy of an axis-particle in the base. This expression for E_λ must be compared with the value for the kinetic energy of unit mass furnished by the particle-method, that is, without interchange of momentum and heat content. The particle-method gives for this kinetic energy:

$$E_{\lambda p} = \int_{z_b}^{z} \frac{\varrho_d - \varrho_m}{\varrho_m} g\, dz + E_{bp} = \int_{o}^{\lambda} \frac{\tau_b}{T_d} g\, D_b\, d\lambda + E_{bp} = \left[\overline{\frac{1}{T_d}}\right] \tau_b\, g\, D_b \lambda + E_{bp} \qquad (26.12)$$

Here E_{bp} denotes the kinetic energy at the base and z_b the height of the base above the origin of coordinates. Comparing (26.11) with (26.12) we can put $E_b = E_{bp}$. There will usually be a difference between the average inversed outer temperatures $\left[\overline{\frac{1}{T_d}}\right]$ of (26.11) and (26.12). This difference is comparatively small however and can be neglected. The error we make in doing so is at most a few per cents, as the average temperatures of the outer air, computed according to the two methods will not differ more than 10 degrees.

From (26.12) it follows, that $E_{\lambda p}$ increases linearly with λD_b, that is with the height in meters above the base, so that $E_{\lambda p}$ does not depend on D_b as was to be expected.

Usually E_b will be unequal to zero, because an impulse is needed for starting the motion. If we ignore this and put $E_b = E_{bp} = 0$, we find for the ratio of the kinetic energies according to the two methods here considered:

$$\frac{E_\lambda}{E_{\lambda p}} = \frac{\frac{1}{2} D_b \tau_b g \,.\, 2{,}34\left[\overline{\frac{1}{T_d}}\right]}{D_b \tau_b g \left[\overline{\frac{1}{T_d}}\right]} \cdot \frac{1-\frac{2{,}34^2}{(2{,}34+\lambda)^2}}{\lambda} = 2{,}34\,\frac{\frac{1}{2}\lambda + 2{,}34}{(\lambda+2{,}34)^2} \qquad (26.13)$$

In figure 16 $E_\lambda/E_{\lambda p}$ is plotted against λ. As shown in this figure, $E_\lambda/E_{\lambda p}$ is the smaller the larger λ is, that is the smaller D_b is or in other words:

The influence of turbulent friction is the more important the smaller the cross-section D_b of the cloud-base.

At a constant height z above the base and for the same value τ_b that cloud will show the largest velocity, that has the largest cross-section in the base-level.

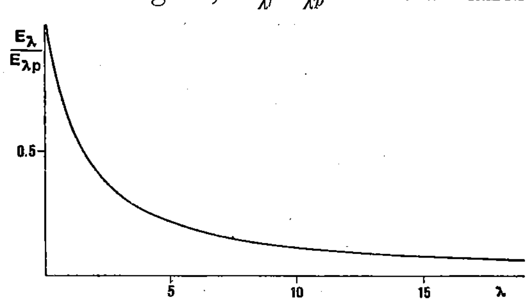

Fig. 16

Figure 17 shows how the kinetic energy of an air particle in the axis of a cumulus congestus or a cumulonimbus changes according to the particle-method and according to the theory developed here. The kinetic energy is plotted against λ and is expressed in terms of $1.17\, D_b\, \tau_b\, g \left[\overline{\frac{1}{T_d}}\right]$. E_b is supposed to be zero. We see that E_λ approaches asymptotically to a finite value.

[1]) In the special case, that $E_b = \frac{1}{2} D_b \tau_b g \,.\, 2{,}34\, [\overline{1/T_d}]$, E_λ appears to be a constant. This agrees with the result of W. Schmidt. (see 21). If $E_b > \frac{1}{2} D_b \tau_b \cdot g \,.\, 2{,}34\, [\overline{1/T_d}]$, E_λ decreases with increasing λ and conversely.

The different behaviour of clouds with unequal cross-section in the base level is shown by figure 18, where the kinetic energies, again expressed in terms of $1.17\, D_b \tau_b g \left[\overline{\frac{1}{T_d}}\right]$, are plotted against $\lambda D_b = z - z_b$ for different cross-sections.

From figures 17 and 18 similar figures can easily be constructed for the velocities.

Finally, we can show briefly, that the velocity-distribution over a cross-section λ of the cloud is again TOLLMIEN's distribution by approximation: It has been assumed, that in

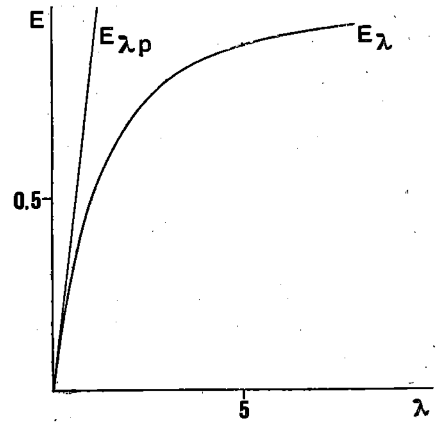

Fig. 17

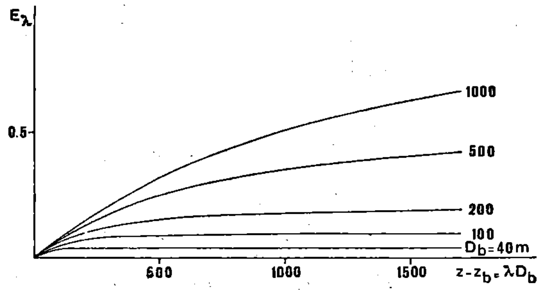

Fig. 18

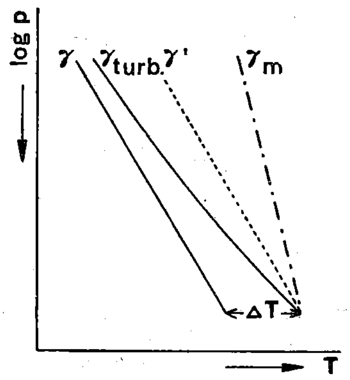

Fig. 19

the base of the cloud the velocity and τ_b are distributed as shown in figure 11. Owing to the temperature differences the velocity of each particle has increased after a short time dt with an amount, proportional to the original velocity of the particle. In this way the type of the velocity-distribution remains unaltered. When the velocity in the axis of the cloud W_λ is known, the velocity in any other point of the cross-section follows from figure 11.

27. c. The temperature difference between the geometric temperature curve and the saturated adiabatic, which the cloud would follow according to the parcel-method, is a function of height

The case treated under b represents only a theoretical approximation of the conditions occurring in the atmosphere in reality. The temperature difference between the cloud and its environment will usually vary with height. This temperature difference will often be zero in the convective condensation level, and then increase with height.

In order to solve the general problem of a temperature difference variable with altitude, we first investigate how the temperature difference between the cloud and the environment of the cloud is changed by turbulent mixing. Using the temperature difference τ_λ, computed in this way, the velocity distribution along the axis of the cloud may be calculated from the equation:

$$W \frac{\partial W}{\partial z} + V \frac{\partial W}{dr} = \frac{\tau_\lambda}{T_d} g - \frac{1}{\varrho_m} R \qquad (27.1)$$

In order to examine the variation of τ_λ with λ, we consider $\tau_{\lambda+d\lambda}$ and examine the relation between $\tau_{\lambda+d\lambda}$ and τ_λ. $\tau_{\lambda+d\lambda}$ is formed from τ_λ by means of two processes (see figure 19):

α. The temperature difference between the cloud and its environment varies owing to the difference in temperature lapse-rate inside and outside the cloud.

β. The temperature difference differs from the value obtained from α by the turbulent interchange of heat content.

Hence it is possible to divide the change of τ_λ, due to a small change of λ, into two parts:

$$\frac{d\tau_\lambda}{d\lambda} = \left(\frac{d\tau_\lambda}{d\lambda}\right)_\gamma + \left(\frac{d\tau_\lambda}{d\lambda}\right)_R \qquad (27.2)$$

where $\frac{d\tau_\lambda}{d\lambda}$ denotes the total variation of τ_λ with λ, $\left(\frac{d\tau_\lambda}{d\lambda}\right)_\gamma$ the part of the variation which must be ascribed to α and $\left(\frac{d\tau_\lambda}{d\lambda}\right)_R$ the part of the variation, which must be ascribed to β.

When we write γ_m for the average saturated adiabatic in the cloud and γ for the temperature lapse-rate in the outer air, we can write:

$$\left(\frac{d\tau_\lambda}{d\lambda}\right)_\gamma = D_b\left(\frac{d\tau_\lambda}{dz}\right)_\gamma = (\gamma_m - \gamma) D_b \qquad (27.3)$$

According to (26.3) $\left(\frac{d\tau_\lambda}{d\lambda}\right)_R$ can be determined in the following way:

$$(\tau_\lambda)_R = \frac{2,34}{2,34 + \lambda}\tau_b$$

and

$$(\tau_{\lambda + d\lambda})_R = \frac{2,34}{2,34 + \lambda + d\lambda}\tau_b,$$

so that

$$(\tau_{\lambda + d\lambda})_R = (\tau_\lambda)_R \frac{2,34 + \lambda}{2,34 + \lambda + d\lambda}$$

and

$$(\tau_{\lambda + d\lambda} - \tau_\lambda)_R = d(\tau_\lambda)_R = \frac{-d\lambda}{2,34 + \lambda}\tau_\lambda \qquad (27.4)$$

or

$$\left(\frac{d\tau_\lambda}{d\lambda}\right)_R = -\frac{\tau_\lambda}{2,34 + \lambda} \qquad (27.5)$$

For the total variation of τ_λ with λ we can write:

$$\frac{d\tau_\lambda}{d\lambda} = (\gamma_m - \gamma) D_b - \frac{\tau_\lambda}{2,34 + \lambda} \qquad (27.6)$$

Equation (27.6) is a linear differential equation of the first order for τ_λ. The general solution of this equation is:

$$\tau_\lambda\, e^{\int \frac{d\lambda}{2,34 + \lambda}} = \int (\gamma_m - \gamma) D_b\, e^{\int \frac{d\lambda}{2,34 + \lambda}} d\lambda + C \qquad (27.7)$$

or after the integration of the exponential function and the introduction of the limits of integration o and λ and a new variable $\varkappa$ in order to avoid confusion with the integration limit:

$$(2,34 + \lambda)\tau_\lambda = \int_o^\lambda (\gamma_m - \gamma) D_b (2,34 + \varkappa)\, d\varkappa \qquad (27.8)$$

The value of the integral can be obtained by integration by parts:

$$(2,34 + \lambda)\tau_\lambda = \int_o^\lambda (\gamma_m - \gamma) D_b (2,34 + \varkappa)\, d\varkappa = \left[(2,34 + \varkappa)\int (\gamma_m - \gamma) D_b\, d\varkappa\right]_o^\lambda - \int_o^\lambda \left[\int (\gamma_m - \gamma) D_b\, d\varkappa\right] d\varkappa =$$

$$= (2,34 + \lambda)(\tau_{\lambda p} - \tau_{bp}) + 2,34\,\tau_{bp} - \int_o^\lambda (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa \qquad (27.9)$$

where $\tau_{\lambda p}$ denotes the temperature difference between the geometric temperature curve and the saturated adiabatic in the level λ, and τ_{bp} the corresponding difference in the base. If the temperature difference in the base of the cloud is zero, (27.9) can be simplified to:

$$\tau_\lambda = \tau_{\lambda p} - \frac{1}{2{,}34 + \lambda} \int_o^\lambda \tau_{\varkappa p}\, d\varkappa \tag{27.10}$$

Under the influence of turbulent mixing the temperature difference between the cloud and its environment in the level becomes, therefore, smaller than the value $\tau_{\lambda p}$ resulting from the particle-method.

For (27.10) we can write:

$$\tau_\lambda = \tau_{\lambda p} - \frac{1}{2{,}34\, D_b + \lambda D_b} \int_{2{,}34\, D_b}^{\zeta} \tau_{zp}\, dz \tag{27.11}$$

For a definite height ζ above the condensation level, the integral does not depend on D_b, just as λD_b. From the factor $\frac{1}{2{,}34\, D_b + \lambda\, D_b}$ it follows then, that $\tau_\lambda - \tau_{\lambda p}$ is the smaller the larger D_b is. This leads to the conclusion, that with a variable temperature difference between the cloud and its environment the influence of mixing is greater for a smaller cross-section of the cloud base.

In order to deduce an expression for the velocity distribution along the axis of the cloud, the general case is here considered:

$$\tau_\lambda = (\tau_{\lambda p} - \tau_{bp}) - \frac{1}{2{,}34 + \lambda} \int_o^\lambda (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa + \frac{2{,}34}{2{,}34 + \lambda} \tau_{bp} \tag{27.12}$$

The force, acting on unit mass in the axis of the cloud due to the density difference between the cloud and the surrounding air in the level λ now becomes:

$$\frac{\tau_\lambda}{T_d} g = \frac{g}{T_d} \left\{ (\tau_{\lambda p} - \tau_{bp}) - \frac{1}{2{,}34 + \lambda} \int_o^\lambda (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa + \frac{2{,}34}{2{,}34 + \lambda} \tau_{bp} \right\} \tag{27.13}$$

As in the case of (26.7) we can put here:

$$\left(\frac{dW_\lambda}{dt}\right)_\tau = \left(W_\lambda \frac{\partial W_\lambda}{\partial z}\right)_\tau = \left(\frac{W_\lambda}{D_b} \frac{\partial W_\lambda}{\partial \lambda}\right)_\tau = \frac{1}{D_b} \frac{2{,}34^2}{(2{,}34 + \lambda)^2} f(\lambda) \frac{\partial f(\lambda)}{\partial \lambda} \tag{27.14}$$

Combination of (27.13) and (27.14) gives:

$$f(\lambda) f'(\lambda) = \frac{D_b g}{T_d} \left\{ \frac{(2{,}34 + \lambda)^2}{2{,}34^2} (\tau_{\lambda p} - \tau_{bp}) - \frac{2{,}34 + \lambda}{2{,}34^2} \int_o^\lambda (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa + \frac{2{,}34 + \lambda}{2{,}34} \tau_{bp} \right\} \tag{27.15}$$

Integration of (27.15) leads in an analogous way as in the case of a constant temperature difference to the kinetic energy, where again according to WEIERSTRASZ' theorem $\left[\overline{\frac{1}{T_d}}\right]$ is written as a factor before the integral. When the integration is carried out between the limits o and Λ, we obtain:

$$\begin{aligned} \tfrac{1}{2} f(\Lambda)^2 - \tfrac{1}{2} f(o)^2 = \left[\overline{\frac{1}{T_d}}\right] D_b\, g \Bigg[& \int_o^\Lambda \frac{(2{,}34 + \lambda)^2}{2{,}34^2} (\tau_{\lambda p} + \tau_{bp})\, d\lambda - \\ & \int_o^\Lambda \frac{2{,}34 + \lambda}{2{,}34^2} \left\{ \int_o^\lambda (\tau_{\varkappa_p} - \tau_{bp})\, d\varkappa \right\} d\lambda + \int_o^\Lambda \frac{2{,}34 + \lambda}{2{,}34} \tau_{bp}\, d\lambda \Bigg] \end{aligned} \tag{27.16}$$

This equation can be written in a somewhat more manageable form by transforming the second integral of the right-hand side by integration by parts.

If we write:

$$-\int_0^{\Lambda} \frac{2,34+\lambda}{2,34^2} \left\{ \int_0^{\lambda} (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa \right\} d\lambda = -\int_0^{\Lambda} u'\, v\, d\lambda \tag{27.17}$$

u' being $\frac{2,34+\lambda}{2.34^2}$ and v being $\int_0^{\lambda} (\tau_{\varkappa p} - \tau_{bp}) d\varkappa$, partial integration gives:

$$-\int_0^{\Lambda} u'\, v\, d\lambda = \left[-uv\right]_0^{\Lambda} + \int_0^{\Lambda} u\, v'\, d\lambda$$

or

$$-\int_0^{\Lambda} \frac{2,34+\lambda}{2,34^2} \left\{ \int_0^{\lambda} (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa \right\} d\lambda = \left[-\int \frac{2,34+\lambda}{2,34^2}\, d\lambda \int_0^{\lambda} (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa \right]_0^{\Lambda} +$$

$$+ \int_0^{\Lambda} (\tau_{\lambda p} - \tau_{bp}) \int \frac{2,34+\lambda}{2,34^2}\, d\lambda \,.\, d\lambda = -\tfrac{1}{2} \left[\frac{(2,34+\lambda)^2}{2,34^2} \int_0^{\lambda} (\tau_{\varkappa p} - \tau_{bp})\, d\varkappa \right]_0^{\Lambda} + \tfrac{1}{2} \int_0^{\Lambda} \frac{(2,34+\lambda)^2}{2,34^2} (\tau_{\lambda p} - \tau_{bp})\, d\lambda =$$

$$= -\tfrac{1}{2} \frac{(2,34+\lambda)^2}{2,34^2} \int_0^{\Lambda} (\tau_{\lambda p} - \tau_{bp})\, d\lambda + \tfrac{1}{2} \int_0^{\Lambda} \frac{(2,34+\lambda)^2}{2,34^2} (\tau_{\lambda p} - \tau_{bp})\, d\lambda \tag{27.18}$$

where in the first integral $\varkappa$ is again replaced by λ.

Finally it follows that:

$$\tfrac{1}{2} f(\Lambda)^2 - \tfrac{1}{2} f(o)^2 = \left[\frac{1}{T_d}\right] D_b\, g \left[\frac{3}{2} \int_0^{\Lambda} \frac{(2,34+\lambda)^2}{2,34^2} (\tau_{\lambda p} - \tau_{bp})\, d\lambda - \tfrac{1}{2} \frac{(2,34+\Lambda)^2}{2,34^2} \int_0^{\Lambda} (\tau_{\lambda p} - \tau_{bp})\, d\lambda + \right.$$

$$\left. + \int_0^{\Lambda} \frac{2,34+\lambda}{2,34} \tau_{bp}\, d\lambda \right] \tag{27.19}$$

If $E_{\Lambda} = \tfrac{1}{2} W_{\Lambda}^2 = \tfrac{1}{2} \left(\frac{2,34}{2,34+\Lambda} \right)^2 f(\Lambda)^2$ represents the kinetic energy per unit mass in the axis of the cloud in the level Λ, it follows from (27.19) that

$$E_{\Lambda} = \left(\frac{2,34}{2,34+\Lambda} \right)^2 \left[\frac{1}{T_d}\right] D_b\, g \left[\frac{3}{2} \int_0^{\Lambda} \frac{(2,34+\lambda)^2}{2,34^2} (\tau_{\lambda p} - \tau_{bp})\, d\lambda - \tfrac{1}{2} \frac{(2,34+\Lambda)^2}{2,34^2} \int_0^{\Lambda} (\tau_{\lambda p} - \tau_{bp})\, d\lambda + \right.$$

$$\left. + \int_0^{\Lambda} \frac{2,34+\lambda}{2,34} \tau_{bp}\, d\lambda \right] + \left(\frac{2,34}{2,34+\lambda} \right)^2 E_b \tag{27.20}$$

where E_b is the kinetic energy of unit mass in the axis of the cloud at its base. For a constant temperature difference $\tau_{bp} = \tau_b$ between the saturated adiabatic, which the cloud follows and the geometric temperature curve, showing the temperature distribution in the undisturbed atmosphere, it follows from (27.20) that:

$$E_{\Lambda} = \left(\frac{2,34}{2,34+\Lambda} \right)^2 \left\{ \left[\frac{1}{T_d}\right] D_b\, g \int_0^{\Lambda} \frac{2,34+\lambda}{2,34} \tau_b\, d\lambda + E_b \right\} \tag{27.21}$$

which form is identical with (26.12) as can be shown by integration.

A second way to simplify (27.20) is by putting τ_b zero. This means that at the base of the cloud the temperature in the cloud is equal to that of the external air. In nature this case occurs in the convective condensation level, that is with a large number of cumuliform clouds. Putting $\tau_b = 0$ leads to the following expression for the kinetic energy of unit mass in the axis of the cloud:

$$E_\Lambda = \overline{\left[\frac{1}{T_d}\right]}\, D_b g \left[\frac{3}{2}\left(\frac{2{,}34}{2{,}34+\lambda}\right)^2 \int_0^\Lambda \left(\frac{2{,}34+\lambda}{2{,}34}\right)^2 \tau_{\lambda p}\, d\lambda - \frac{1}{2}\int_0^\Lambda \tau_{\lambda p}\, d\lambda\right] + \left(\frac{2{,}34}{2{,}34+\lambda}\right)^2 E_b \qquad (27.22)$$

This expression for E_Λ now can be compared with the kinetic energy of unit mass of the cloud obtained when turbulent mixing is neglected, that is the kinetic energy computed by means of the particle-method. This „kinetic energy according to the particle-method" is composed of the kinetic energy, the unit mass possessed at the base of the cloud and the amount of potential energy, converted by the vertical displacement of the particle and the displacement in the opposite direction of another particle:

$$E_{\Lambda p} = E_b + \int_{z_b}^{z} \frac{\varrho_d - \varrho_m}{\varrho_m}\, g\, dz = E_b + \int_0^\Lambda \frac{\tau_{\lambda p}}{T_d}\, g\, D_b\, d\lambda = E_b + \overline{\left[\frac{1}{T_d}\right]}\, D_b g \int_0^\lambda \tau_{\lambda p}\, d\lambda \qquad (27.23)$$

If E_b is put zero (this being of course an approximation, as an initial impulse is always present), we find for the ratio of the kinetic energies, computed according to the two methods:

$$\frac{E_\Lambda}{E_{\Lambda p}} = \frac{\frac{3}{2}\int_0^\Lambda (2{,}34+\lambda)^2\, \tau_{\lambda p}\, d\lambda}{(2{,}34+\Lambda)^2 \int_0^\Lambda \tau_{\lambda p}\, d\lambda} - \frac{1}{2} \qquad (27.24)$$

Obviously here too $\overline{\left[\frac{1}{T_d}\right]}$ from (27.22) is put equal to $\overline{\left[\frac{1}{T_d}\right]}$ from (27.23).

In figure 20 the lapse of $E_\Lambda / E_{\Lambda p}$ is plotted against Λ, γ and γ_m being constant. It can be easily understood, that $E_\Lambda / E_{\Lambda p}$ does not depend on γ and γ_m so long as $\tau_{\lambda p}$ is proportional to λ. It appears, that E_Λ is the smaller compared with $E_{\Lambda p}$ the larger Λ is for a constant value ΛD_b, that is the smaller D_b is. So here too we come to the result:

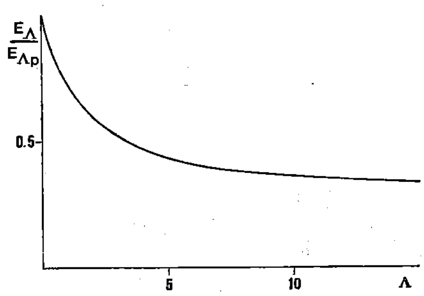

Fig. 20

The influence of the turbulent friction is the more important the smaller the cross-section of the base of the cloud.

From figure 20 or by extraction of roots in (27.22) values of the velocities in the axis of the cloud follow.

Here too the velocity-distribution in a cross-section can be deduced from W_λ by means of Tollmien's profile.

28. Concluding remarks

Summarizing we obtain the following picture of the influence of the skin-friction drag on cumuliform clouds:

We can consider the mechanism of the skin-friction drag arising when a cumuliform cloud ascends in the atmosphere as analogous to the friction occurring when a jet is discharged through an aperture into a space filled with fluid. *This means that non-adiabatic motions are considered.* In this way we can make an estimate of the loss of energy the cloud suffers by this friction. This estimate is rather rough as many restrictions have to be introduced in order to make the problem manageable. The most serious of these restrictions is the one by which the motion in a cumuliform cloud must be considered to be stationary. We found in chapter II, that this is not true and that on the contrary the motion in cumuliform clouds is essentially non-stationary. A further limitation lies in the fact that the relations, deduced here hold only for a homogeneous atmosphere.

A consideration of the skin-friction drag as developed in this chapter permits the finding of a relation between the horizontal dimensions of a cumuliform cloud and the velocities occurring in the cloud. It does not furnish any information concerning the vertical dimensions of the clouds however.

A combination of the results of chapter II and those of this chapter will lead to a qualitative picture of the cumuliform clouds.

CHAPTER VI. SOME CONCLUSIONS CONCERNING THE SHAPE OF CUMULIFORM CLOUDS

29. The combination of form drag and skin-friction drag

In the foregoing chapters II, III and V the influence of the form drag and of the skin-friction drag on the development of cumuliform clouds was examined. The behaviour of cumuliform clouds is governed, however, by the simultaneous action of these two drags. Now form drag and skin-friction drag doubtlessly influence each other, therefore the considerations of the chapters II and V have only a qualitative significance, apart from the simplifying suppositions already introduced.

The importance of this mutual influence follows from the fact that the considerations about the form drag were based on the supposition that a developing cumuliform cloud is necessarily a non stationary current in the atmosphere, whereas in treating the skin-friction drag we found, that the cumuliform cloud must be treated as a stationary current. The discrepancy between the two conceptions is difficult to overcome, so that a synthesis of the results of chapter II and V will not be easy.

It is possible however to derive some qualitative results concerning the form of the clouds, based on the considerations of the preceding chapters.

We will first examine the influence of the skin-friction drag separately.

30. The form of cumuliform clouds

According to the considerations in chapter V each cumulus congestus should have a a conical form. This conical form is, however, never observed with cumuliform clouds; on the contrary, the cross-section of the cloud base is generally larger than a cross-section near the top. We must realize, however, that the conical upward current need not be identical with the visible cloud. For the latter will only occupy that part of the upward current in which condensation occurs.

Now it follows from a simple qualitative consideration, that usually condensation will only occur in and near the central part of the cone.

For, besides momentum and heat content, moisture too is distributed over a cross-section of the cone in the way shown in figure 11 for the vertical velocities, with only this difference, that the moisture-content outside the cone is not zero. Now normal moisture distribution in the atmosphere has as a consequence, that only in the more central parts of the cone the moisture concentration is so large, that condensation occurs and a visible *cloud* appears. As a rule the relative humidity decreases with increasing height on days with convective clouds.

As a result of this fact the visible part of the cone becomes increasingly narrower with height.

A difficulty arises when we apply the computations of the preceding chapter to the moisture distribution in cumuliform clouds. Supposing TOLLMIEN's profile to exist already at the base, it would follow namely, that the bases of cumuliform clouds caused by free convection have a more or less convex form. For, the largest specific humidity will appear in the axis of an ascending air-bubble if the specific humidity in the air-bubble is larger than that of the surroundings, which usually will be the case. Condensation will, therefore, appear first in the centre of the bubble, causing a convex base. DEPPERMANN (1940) pointed out the possibility of the existance of such cloud bases, be it for other reasons.

Now such convex bases are not observed as a rule and we can easily understand this by supposing that the specific humidity of the air under the cloud base is constant with height as a result of usually strong turbulence.

This is partly confirmed by the fact that on clear days one can sometimes observe a hazy layer in the convective condensation level a short time before the occurrence of the convective clouds, which must be ascribed to the large relative humidity in the condensation level.

Owing to this distribution of humidity condensation occurs in the cloudbase as a whole (cross-section D_b) and in one level.

As an example of a numerical computation of the distribution of specific humidity above the condensation level we consider the following case: We suppose, that the condensation level is found at a pressure of 900 mbar and a temperature of 12,5° C. The specific humidity of the cloud-air is then 10 gr/kg. In the environment of the cloud this specific humidity exists also from the ground to 900 mbar. Now let the specific humidity in the external air above the condensation level decrease by 2 gr/kg per 1000 m height. This corresponds to a decrease of relative humidity to about 60 % at 4000 m height above the condensation level. Often this decrease is more rapid, by which the following argument is made still more valuable. The humidity distribution in cloud and environment is given by the following table:

Height above the condensation level	Specific humidity of the atmosphere	Maximum specific humidity in the cloud
0 m	10 gr/kg	10 gr/kg
1000 m	8 gr/kg	8,3 gr/kg
2000 m	6 gr/kg	6,6 gr/kg
3000 m	4 gr/kg	4,9 gr/kg
4000 m	2 gr/kg	3,2 gr/kg

We have obtained the last column by taking the maximum specific humidity along the saturated adiabatic through 900 mbar and 12,5° C. The fact that the cloud has a lower temperature than the one corresponding to the saturated adiabatic (see chapter II) may be neglected here. According to the particle-method an amount of 10 gr/kg is found in any arbitrary level of the cloud as the total moisture-content, liquid water + vapour (abstracted from precipitation).

According to the considerations of chapter V the distribution of the moisture-difference Δq_λ along the axis of the cloud when turbulent mixing is considered is given by the following relation:

$$\Delta q_\lambda = \Delta q_{\lambda p} - \Delta q_{bp} - \frac{1}{2,34 + \lambda} \int_0^\lambda (\Delta q_{\varkappa p} - \Delta q_{bp})\, d\varkappa + C \tag{30.1}$$

In this equation the symbols used have the following meaning:

Δq_λ is the difference of moisture-content between the axis of the cloud and its environment in the level λ;

$\Delta q_{\lambda p}$ is the difference of moisture-content between the cloud and its environment according to the particle-method, that is apart from turbulent mixing;

Δq_{bp} is the difference of moisture-content in the base level;

λ indicates the height of the cloud expressed in terms of the cross-section of the base D_b. This cross-section D_b being 1000 m we obtain for the various levels:

0 m . . . . $\Delta q_0 = C = 0$

1000 m . . . . $\Delta q_1 = 2 - \frac{1}{3,34} \int_0^1 2\varkappa d\varkappa = 2 - \frac{1}{3,34} = 1,7$

2000 m . . . . $\Delta q_2 = 4 - \frac{1}{4,34} \int_0^2 2\varkappa d\varkappa = 4 - \frac{4}{4,34} = 3,1$

3000 m . . . . $\Delta q_3 = 6 - \frac{1}{5,35} \int_0^3 2\varkappa d\varkappa = 6 - \frac{9}{5,34} = 4,3$

4000 m . . . . $\Delta q_4 = 8 - \frac{1}{6,34} \int_0^4 2\varkappa d\varkappa = 8 - \frac{16}{6,34} = 5,5$

The moisture-content in the axis of the cloud is, therefore, at

0 m	1000 m	2000 m	3000 m	4000 m	above the base
10,0 gr/kg	9,7 gr/kg	9,1 gr/kg	8,3 gr/kg	7,5 gr/kg	

At greater heights, therefore, the maximum moisture-content in the cloud is less than the 10 gr/kg required by the particle method. Comparing the last figures with those for the maximum specific humidity in the cloud the water-content in the various levels appears to be 1.4, 2.5, 3.4, 4.3 gr/kg. For that reason it appears, that the cloud is only visible above the „condensation level'' as in this level itself no water is present. If PETTERSSEN's data for watercontent in fog are right, the numerical result, found here confirms the empirical results of ROSSI (1940). We leave this alone henceforth.

Now, the distribution of the moisture-content over a cross-section will be in accordance with TOLLMIEN's velocity profile. The boundary of the cloud will be situated where the value of q in the profile is equal to the maximum specific humidity measured along the saturated adiabatic.

The horizontal cross-section of the visible cloud d_λ can be expressed as a function of D_λ, D_λ being the cross-section of the cone in the relative level λ.

By comparing the numerical results with figure 11 we find:

$\lambda = 0$	$\Delta q_0 = 0$		$d_0 = D_o = D_b$
1	$\Delta q_1 = 1.7$	$q_{1\,max} = 8.3 = q_1 - 1.4$	$d_1 = 0.58\, D_1$
2	$\Delta q_2 = 3.1$	$q_{2max} = 6.6 = q_2 - 2.5$	$d_2 = 0.57\, D_2$
3	$\Delta q_3 = 4.3$	$q_{3max} = 4.9 = q_3 - 3.4$	$d_3 = 0.53\, D_3$
4	$\Delta q_4 = 5.5$	$q_{4max} = 3.2 = q_4 - 4.3$	$d_4 = 0.52\, D_4$

Now the relation $D_\lambda = D_b + 0{,}428\,\lambda D_b$ gives $D_0 = 1000$ m, $D_1 = 1428$ m, $D_2 = 1856$ m, $D_3 = 2284$ m and $D_4 = 2712$ m.

From these values it follows that the cross-sections of the cloud are:

at 0 m above the condensation level	1000 m
„ 1000 „ „ „ „ „	827 „
„ 2000 „ „ „ „ „	1060 „
„ 3000 „ „ „ „ „	1210 „
„ 4000 „ „ „ „ „	1410 „

We find, therefore, in this case also an increase of the cloud diameter with altitude but on a reduced scale. In the lower 1000 m the diameter even decreases in spite of the fact that the total rising column (the cone) increases in diameter.

31. The influence of the counter-current on the form of the clouds

The effect mentioned in 30 becomes more important when the counter-current too is taken into consideration. The profile through cloud and counter-current will have the smooth appearance shown in figure 3. (Compare also chapter II where we ignored this complicated current-pattern.) In the immediate neighbourhood of the cloud a rather strong descending motion will exist, causing a strong decrease of relative humidity. This means that the narrowing of the cloud by the lateral mixing will take place on a larger scale than was the case in the example computed above.

When only the field of motion is taken into account we are allowed to introduce the counter-current qualitatively by imparting to the whole atmosphere a descending motion by letting the cone ascend with respect to this descending environment. The absolutely ascending part of the cone may then be determined by simple addition of the occurring velocities. This absolutely ascending part of the cone is contained in an axially symmetrical body which has everywhere a smaller diameter than the cone. Only in this smaller body (the *convective current*) particles are contained which may condense by adiabatic cooling. If, now, we take, moreover, into account that owing to the lateral mixing the visible cloud again will be smaller than the convective current, we see, that the theoretically obtained conical model can be transformed into a shape nearer the observed cloud form by applying a suitable correction. This is the more so as the motion at the cloud-top is not stationary. This means that the central parts of the cloud will move faster than the more external parts and also that the conical form will only be established at some distance under the top, as the surrounding air will not be dragged away immediately. In this way the typical cupula-shape of the clouds may be understood still better.

32. The air ascending around the cloud

From the considerations concerning the influence of the moisture-distribution it follows, that cumuliform clouds in an atmosphere saturated with water-vapour in all levels will coincide with the convective current and will, therefore, be more or less conical. Comparative observations of moisture-distribution and the form of cumuliform clouds have, however, never been published.

Usually there will be an ascending motion in the immediate neighbourhood of the cloud, that is in the convective current. This has been indeed observed by gliding pilots as was mentioned by WALKER (1939).

As matters stand, further data about ascending motions in the immediate environments of cumuliform clouds are, however, completely lacking. This fact might lead to the conclusion, that these motions are not of a very frequent occurrence or at least that they are suppressed to a great extent. That this must indeed be the case can be easily understood. For in the dry-adiabatically ascending part of the convective current the particles become cooler than their environments, so that they will be retarded and at a given moment the energy necessary for any further upward motion will become so large, that a particle in the neighbourhood of the cloud will cease to ascend. This will occur the sooner, the larger γ_d is with respect to γ, or a cumuliform cloud will preferably have a dry-adiabatically ascending environment when γ differs little from γ_d.

This fact can be discussed in accordance with the division of conditional instability by NORMAND (1938):

1. *conditional instability of the stable type.*
2. *conditional instability of the pseudo-latent type.*
3. *conditional instability of the real latent type.*

In general this distinction is of little importance from the point of view of the particle-method. When, however, the convective current as a whole is considered and not only the cumuliform cloud, it appears that NORMAND's division is of some importance. For the air, ascending dry adiabatically outside the cloud will obtain a gradually increasing relative humidity (leaving the influence of lateral mixing of the water-vapour out of account) and after a slight ascent, condensation may occur in the part of the convective current lying outside the original cloud. If the ascent continues after condensation has occurred, the ascending particle can at a given height become lighter than its environment. Above this level energy is converted by further ascent. It is evident, that the particle will originally hamper the cumulus formation, whereas it will promote it as soon as it is lighter than the environment of the convective current.

For this reason it is possible to distinguish three intensity degrees when a cumuliform cloud is formed in a conditionally unstable atmosphere according to the behaviour of particles ascending outside the cloud:

1. The atmospheric layer from which the particle dragged upward originates is conditonally unstable of the stable type. The air ascending outside the cloud owing to

turbulent friction remains colder than the environment of the convective current and will always retard the cloud.

2. The particle originates from a layer in which the equilibrium is conditionally unstable of the pseudo-latent type. This means, that when unit mass of air is forced to move upwards, the total energy we have to apply is larger than the energy liberated when the particle is raised high enough. The air ascending in the convective current outside the original cloud will become warmer than its environment at a certain given level, so that at greater height it will contribute to the cumulus development. Below this level it retards the cloud. From the definition of the pseudo-latent type it follows, that the retardation by a particle of the convective current is more important than its contribution to the cloud formation at greater altitude. As a whole, therefore, the cloud is retarded in this case also, although less than in case 1.

3. The particle originates from a layer conditionally unstable of the real latent type. We mean by this that the total energy applied is smaller than the energy liberated when the particle of air is forced to ascend.

From the foregoing it follows, that in this case the condition of the air dragged upwards by the cloud will be such as to promote the convection-process.

When in cases 2 and 3 the air dragged upwards and containing condensation products (that means ascending saturated adiabatically) locally passes beyond the level in which negative energy change into positive energy this becomes visible by the appearance of new towers on the cumulus congestus or cumulinimbus. The formation of such towers will therefore occur more in particular in a really latent atmosphere, i. e. in an atmosphere in which the vertical lapse-rate as well as the relative humidity are large.

In order to be correct considerations as to the energy of particles dragged upwards in the convective current ought to be modified to some extent in connection with the turbulent mixing and the counter-current. The qualitative result deduced here may, however, remain true to a certain degree.

33. Vertical oscillations of cumuliform cloud

When we neglect the skin-friction drag as well as the form drag and in doing so start from the point of view of the particle-method, we must expect vertical oscillations to occur in a cumulus congestus (figure 21). For, a particle ascending as a consequence of lability and not suffering any friction will pass level B where it has the same temperature as its surroundings and will continue to ascend to a level C where it has consumed its kinetic energy. As the particle is here heavier than its environment, it will move back downwards to its starting point A (the convective condensation level) whereupon the ascent will set in again. It is easy to see that with the normal density-differences occurring in the atmosphere between a cloud and its environment the period of oscillation must be less than half an hour, even for clouds of large vertical dimensions.

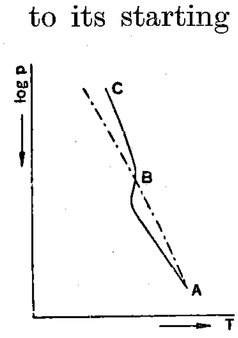

Fig. 21

LETZMANN (1930) observed indeed a pulsation with cumuliform clouds. It is probable, however, that the pulsations observed by LETZMANN are connected with the appearance of new cumulus towers. Moreover, there seemed to be a connection between the pulsation and the intermittant precipitation so that it is quite probable that LETZMANN's observations were concerned with these phenomena rather than with inertia-motions.

Now it appears that the introduction of the skin-friction drag reduces the possibility of such a pulsation considerably. This fact is demonstrated in figure 22 where the alterations to be introduced when the skin-friction drag is taken into account are shown schematically. The figure represents a geometric temperature curve showing an inversion of 4 degrees at 800 m above the convective condensation level. The saturated adiabatic which the cloud would follow according to the particle-method is parallel to the temperature curve lying at a temperature two degrees higher than the latter below the inversion and two degrees lower above the inversion. We suppose that the vertical motion starts in the base level.

When we neglect the skin-friction drag and the temperature variation with height, we can easily compute, that a cloud particle will come to a standstill at 1600 m above the condensation level and that it will afterwards acquire a downward acceleration, bringing it back to the base of the cloud.

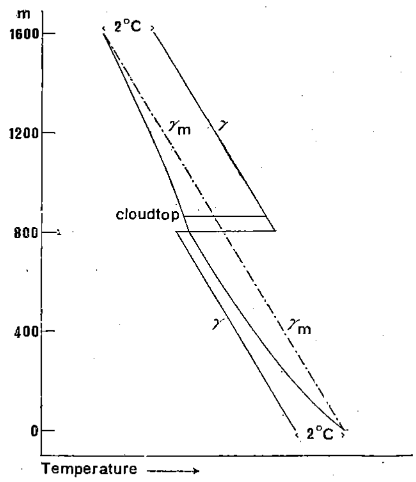

Fig. 22

Applying the relations deduced in chapter V for the case that τ is a constant and starting from the supposition that the cross-section of the cloudbase amounts to 100 m, we can compute that owing to the lateral mixing the cloud will come to a standstill at 860 m above the base. Here too the top of the cloud will try to descend again below the inversion but the pulsation will not acquire the dimensions predicted by the particle-method and it will moreover be damped very soon. As the lateral mixing also influences the temperature in the cumuliform cloud, it will not ascend according to the saturated adiabatic but the cloud-particles will have a lower temperature in each level. The temperature curve which the particles in the axis of the cloud follow is also reproduced in the figure. As the cloud will be retarded by the counter-current too, we cannot expect to find in practice a perceptible pulsation due to inertia.

34. The vertical dimensions of cumuliform clouds

In another respect too, the simultaneous action of skin-friction drag and form drag is of importance. When the top of the cloud comes to a standstill at last in B (figure 21) and the supply of air through the condensation level continues, the cloud will be forced to spread in the level B, forming stratocumulus cumulogenitus or altocumulus cumulogenitus.

According to the particle-method, B is very soon reached: when, for example, the temperature difference between the saturated-adiabatic and the geometric temperature curve amounts in the mean to 1 degree centigrade and the mean temperature of the cloud is 270° while B lies at 5000 m above the condensation level, the top of the cloud reaches B within ten minutes after passing the condensation level. Computing the time necessary for the cloud to reach B and taking into account the skin-friction drag, we shall find a time which is the larger, the smaller the cross-section of the cloud. For a cross-section of the cloud base of 1000 m the time between the moment the top starts from the condensation level and the moment it reaches B will amount to 12 minutes according to chapter V, **26**.

As for most cumuliform clouds the circumstances are such, that B will be reached in a shorter time than in the case mentioned above, we could expect all cumuliform clouds to spread very quickly at their tops. In reality this occurs rather seldom. This means that in most cases B is not reached and this fact must be ascribed to the influence of the counter-current.

We can represent the combined action of the skin-friction drag and the form drag in the following way. The cloud is considered as a body, ascending with respect to a descending environment. By lateral mixing the velocity in the cloud will generally decrease with height but it will always have a finite positive value with respect to the descending environment. The absolute motion of the cloud with respect to the earth will become zero in some definite level, namely where W_λ as defined in chapter V will be equal to W_d of chapter II. When the clouds have a relatively small diameter at their base this level will generally be reached below the level B. As from the moment when $W_\lambda = W_d$ the cloud does not ascend any longer in the atmosphere, no further condensation will occur and the cloud will not spread. When the cloud particles are situated at larger distances from the axis of the cloud, they will show a zero ascent relative to the earth in a lower level and therefore show no longer any condensation. This is another point apt to explain the dome-shape of cumuliform clouds in their final stage (see also § 13, figure 8).

It is evident that generally clouds with the largest cross-section will reach level B first and so be the first to show a flattening, a result which is confirmed by experience. Owing to the combined action of lateral mixing and counter-current most cumuliform clouds do not reach level B at all or if they do, it is only at the final stage of their development: they remain therefore warmer than their environment at their tops.

Finally we may remark that the cloud base gradually ascends after the moment condensation has set in. Two reasons can be pointed out for this fact. In the first place, turbulent mixing will occur in the lowest layers when the temperature gradient is adiabatic or over-adiabatic. This turbulent mixing will lead to a decrease of specific humidity of the air that is in contact with the surface of the earth. In the second place, the air that will form the cloud will mix with the dry descending air of the counter-current. It is obvious that both mixings lead to a higher condensation level. This fact has been shown empirically by PEPPLER (1922).

35. Horizontal velocities around the cloud

In § 19 it has been pointed out, that as a consequence of continuity there exists also a horizontal radial velocity in the neighbourhood of jets which causes the fluid to converge to the jet (fig. 12).

Such a convergent current also exists around cumuliform clouds. This has been observed by balloon crews and during kite ascents.

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LITERATURE

BJERKNES, J., 1938. Quarterly Journal Roy. Met. Soc. **64**, 325.
CHRISTIANS, 1935. Beitr. Ph. fr. Atm. **22**, 149.
FJELDSTAD, 1925. Geof. Publ. **3**, 13.
GOLDSTEIN, 1938. Modern Developments in Fluid Mechanics.
HEWSON, 1938. Quarterly Journal Roy. Met. Soc., **64**, 289.
HOWARTH, 1938. Proc. Cambridge Phil. Soc. **34**, 185.
KOPP, 1930. Beitr. Ph. fr. Atm. **16**, 173.
KOPP, 1933. Beitr. Ph. fr. Atm. **20**, 281.
LENARD, 1904. Met. Zs. **21**, 249.
LETTAU, 1939. Die atmosphärische Turbulenz.
LETZMANN, 1930. Met. Zs. **47**, 236.
MAAS, V. D. and WYNIA, 1938. Rapport V 834 Nat. Luchtvaart Laboratorium.
MARGULES, 1903. Jahrbuch K. K. Zentralanst. Met. u. Geodyn.
NORMAND, 1938. Quarterly Journal Roy. Met. Soc. **64**, 338.
PEPPLER, 1922. Beitr. Ph. fr. Atm. **10**, 130.
PETTERSSEN, 1939. Geof. Publ. **12**, 9.
PIELSTICKER, 1940. Beitr. Ph. fr. Atm. **27**, 1.
POULTER, 1938. Quarterly Journal Roy. Met. Soc. **64**, 277.
PRANDTL and TIETJENS, 1929. Hydro- and Aeromechanik I.
RAETHJEN, 1929. Met. Zs. **46**, 292.
REFSDAL, 1930. Geof. Publ. **5**, 12.
REFSDAL, 1932. Geof. Publ. **9**, 12.
REICHARDT, 1942. V.D.I.-Forschungsheft 414.
RENNER, 1939. Ann. Hydr. **67**, 38.
ROSSI, 1940. Mitt. Met. Inst. Helsinki **44**.
RUDEN, 1933. Naturwiss. **21**, 375.
SCHMIDT, W., 1941. Zs. angew. Math. u. Mech. **21**.
SCHNAIDT, 1942. Met. Zs. **59**, 281.
SHAW, 1920. The air and its ways.
TAYLOR, 1915. Phil. Transactions A **215**, 1.
TAYLOR, 1932. Proc. Roy. Soc. A **135**, 685.
TOLLMIEN, 1926. Zs. angew. Math. u. Mech. **6**, 468.
WENZEL, 1933. Met. Zs. **50**, 65.