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ON RADIATIONAL COOLING OF THE EARTH'S SURFACE DURING THE NIGHT, ESPECIALLY WITH REGARD TO THE PREDICTION OF GROUNDFROSTS

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Summary

In the present paper the author ventures a new attack on the problem of predicting nocturnal temperature fall in order to make it possible to take account of a number of details in the nocturnal heat exchange processes near the earth's surface without, however, making computations too laborious for practical use. The paper is divided into a theoretical part (I) and a practical part (II). The latter, however, is not much more than a first outline of a practical procedure. Moreover, a lot of experimental work should yet be done in order to make the present results fully paying.

After a critical discussion of the formulas that have formerly been used in theoretically computing nocturnal radiational cooling (sections 1—3) the author's own formula is discussed (section 4). For this new formula the assumption of constancy of the effective radiation (R) of the earth's surface during the night had been dropped, instead of which a linear variation of R with temperature was assumed. The new formula may be a considerable improvement especially for cases when the nights are sufficiently long. When $t \to \infty$ the formula yields a certain lower limit for the temperature.

In sections 5—7 the effects of the initial temperature distribution in the ground, of the eddy conductivity of the air and of condensation of water vapour upon the process of cooling are discussed. It appears that, when we take into account these effects in a reasonable way, a formula of the form described in section 4 may still be used. By transforming this formula into a non-dimensional form it even appears that one main graph is sufficient for computing temperature variations, all details of the heat balance entering as certain constants only into the unit of temperature and into the unit of time that ought to be chosen for the said non-dimensional representation. For the evaluation of some of these constants a few auxiliary graphs or tables might be constructed. It has been assumed that the night is of sufficient uniformity for taking mean characteristics as regards air mass properties and cloudiness.

In section 8 two possible complications are only briefly touched upon.

The rest of the paper is devoted to a short discussion of the possibilities of practically applying the theoretical results arrived at.

Introduction

The problem of predicting groundfrosts has for a long time been realized to be as difficult as it is important. Now, qualitatively speaking, the same is true as to forecasting weather in general. The former problem, however, is much more complicated, as the general synoptic forecast constitutes only part of the data, needed to its solution. Among these data we may distinguish three elements:

a. the general synoptic forecast for a certain district;

b. the given local microclimatological conditions for the piece of land concerned;

c. the local micrometeorological conditions at the beginning of the night and during it, which largely depend upon (1) the general weather in the past and upon (2) the local weather of the preceding day.

As to *a*, we are especially concerned in the following elements, to be expected for the coming night: (1) cloudiness, (2) humidity of the air, especially in the lower parts of the atmosphere, (3) wind velocity.

By *b*, "the given local microclimatological conditions", we understand various microclimatological elements, which are more or less independent of weather (though they may vary with season), as for example: the sort and disposition of the soil, the vegetation on it, the lie of the land (more or less sheltered by neighbouring trees or buildings) as well as the general orographical conditions, etc.

c. For the same land, in the same season, however, the soil constants, which govern the thermal processes in it, are to a high degree dependent on its water content, which in its turn depends upon (1) the general weather in the past, (2) the local weather of the preceding day. The same is true as to the surface temperature at the beginning of the night and the humidity of the air immediately above the ground or above its vegetation. For convenience, we take all these factors together in the term „micrometeorological elements".

For the meteorologist it is impossible to take all these factors into account. With the factors sub *a* and *c*(1) this can still be done, in general, but for the rest the only thing he can do is to give some specialized forecasts for a small number of soil types. The user of the forecasts (the agriculturist) will then have to apply the forecasts intelligently to his special circumstances, of which he has to judge himself.

Apart from this practical aspect of the problem, however, it is possible to study it from a purely physical point of view, in order to find a method for computing the nocturnal minimum, when all meteorological factors, soil constants and other influencing circumstances are known, or rather approximated in the form of a certain number of constants. The principal aim of the present study is to describe such a method (part I). In part II we shall briefly enter into the practical sides of the matter.

As to the existing literature on the subject, the larger part of it differs in character from the present study and results in a rather large number of more or less empirical rules, a survey and discussion of which (up to 1940) one may find in the well known paper by Kessler and Kaempfert, 1940 [1]). We shall not enter upon those rules here. Studies of a similar character as the present one are the papers by Brunt, 1932 and 1939, by Philipps, 1940, and by the author, 1947.

Strictly speaking, the first attack of this kind on the problem of predicting temperature variations at the earth's surface was made by Richardson, 1922, in his great study „Weather Prediction by numerical process" (Ch. 8, 2/15); his physical treatment of the matter was excellent and complete, from a theoretical point of view; it was not, however, worked out into a specialized practical form.

[1]) References are indicated by giving the year of publication, see the list of literature at the end of this paper.

PART I THEORETICAL PART

(This part is a continuation of the author's paper in "Journal of Meteorology" 1947)

1. *Mechanism of cooling.*

In the following we shall suppose the land to be plane.

We write down the energy balance at the surface of the earth in the following form (compare fig. 1):

$$B = R - A - C, \tag{1}$$

where B = heat stream density in the ground; it is reckoned positive, when the heat stream is directed towards the surface; R = density of effective radiation from the earth's surface; A = density of the heat stream in the air, arising from the eddy conduction in it (the molecular conduction will be neglected here); it is reckoned positive, when the stream is directed towards the earth's surface; C = density of the heat stream, which arises from the condensation of water vapour at the surface; by stream density we always understand the stream per unit area. By the "effective radiation" (R) we understand the outgoing radiation of the earth, as given by the Stefan-Boltzmann law, *minus* the incoming radiation from the atmosphere (R′),

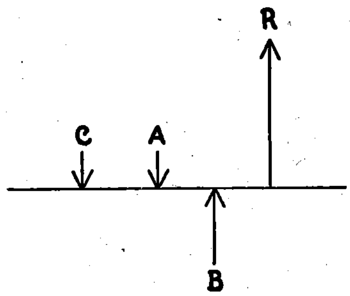

Fig. 1

$$R = \sigma T^4 - R', \tag{2}$$

where T = surface temperature.

Any advective term in the energy-balance will be discarded.

To give an idea of the order of magnitude of the various terms of the energy balance (1), we write down some plausible values for summer nights in moderate latitudes: $B = 0{,}10$, $R = 0{,}15$, $A = 0{,}035$, $C = 0{,}015$ cal/cm²min.

For the atmospheric radiation R' from a clear sky formulas have been proposed, best known of which are the following two:

1. Formula of Brunt [1]):

$$R' = \sigma T^4(a + b\sqrt{e}). \tag{3}$$

2. Formula of Ångström [2]):

$$R' = \sigma T^4 (\alpha - \beta \,.\, 10^{-\gamma e}). \tag{4}$$

In the formulas e denotes the water vapour pressure in the atmosphere at the earth's surface; a and b, or α, β and γ are mostly taken to be constants. According to the theoretical deductions given to justify these formulas [3]), they ought to be functions of temperature and humidity of the air. It may be readily seen, however, that even then, the two formulas have only *statistical* value [4]). The best method to determine R' for a clear sky and a given atmospheric structure is the radiation chart method. For the construction and use of a radiation chart we refer to the papers by Möller, 1932, 1943, 1944 and by Elsasser, 1942, and to section 10 of the present paper (part II).

The radiation from a covered sky depends on the sort and height of the clouds and on the structure of the atmosphere below them. When we denote this radiation by R'_{10}, the radiation from a partially clouded sky, with a cloudiness N/10, can in general be written as follows:

$$R'_N = R' + \frac{N}{10}(R'_{10} - R') = \frac{10 - N}{10} R' + \frac{N}{10} R'_{10},$$

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1) Brunt 1932, 1939.
2) Ångström 1918, 1929.
3) For formula (1) see Pekeris, 1934; for formula (2) see Ramanathan and Ramdas, 1935 and Philipps, 1940.
4) Other formulas have been proposed by Robitsch and by Elsasser.

where R' is the radiation from a clear sky, discussed above. For R'_{10} we may refer to the papers by Ångström, 1919, 1936, Asklöf, 1920, Meinander, 1928, Dufour, 1938, Brunt, 1939, Geiger, 1942 (Ch. 2).

In section 11 we shall return to this practical question. Here we may further remark the following:

As ground frosts are phenomena, which, in the seasons when their prediction is of economical importance, are of a more or less *exceptional* character, they will, in those seasons, only occur when atmospheric conditions are *exceptionally* favourable to their realization (small degree of cloudiness, small wind velocity). For this reason we shall, in the following, assume that these atmospheric conditions are not influenced by a frontal cloud or wind system; for cloudiness, this means, that we assume it to be small and constant during the night.

The left hand side of (1) can also be written as follows:

$$B = -\lambda\left(\frac{\partial T}{\partial z}\right)_0, \tag{5}$$

where λ is the heat conduction coefficient of the soil; the suffix $_0$ means that the value of $\partial T/\partial z$ must be taken at the earth's surface, where $z = 0$; z is counted positive in the upward direction.

The temperature variations in the soil are governed by the equation

$$\frac{\partial T}{\partial t} = \frac{1}{\varrho c}\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right), \tag{6}$$

where ϱ = density; c = specific heat of the soil. When the soil is homogeneous, this becomes:

$$\frac{\partial T}{\partial t} = \varkappa\frac{\partial^2 T}{\partial z^2}, \quad \varkappa = \frac{\lambda}{\varrho c}; \tag{7}$$

$\varkappa$ is called the "thermal diffusivity".

By this equation the temperature function $T(z, t)$ is determined once for all, if only an initial temperature distribution $T(z, 0)$ be given:

$$T(z, 0) = F(z). \tag{8}$$

Now in our problem $T(z, 0)$ is only defined for $z < 0$, because for the time being we have only to deal with ground temperatures (the air temperature variations are governed by different laws). $T(z, t)$ as a solution of (6) or (7) is, therefore, only determined if one more boundary condition for $z = 0$ is given; this boundary condition is, according to (1) and (5), that for all values of t

$$-\lambda\left(\frac{\partial T}{\partial z}\right)_0 = R - A - C. \tag{9}$$

The right hand side of (9) is a function of time, as R, A and C are in general not constant.

The solutions $T(z, t)$ concern us especially as to their behaviour for $z = 0$, that is to say at the earth's surface.

2. *Solutions for special cases.*

In order to find solutions of (6) or (7), satisfying conditions (8) and (9), we can make various simplifying suppositions. To begin with, we shall suppose a homogeneous soil, or at least a soil for which we may operate with a mean value of λ, which may be treated as a constant, so that we have to deal with the simpler equation (7).

For the three following special cases solutions have formerly been given in the literature:

1. $T(z, 0)$ = const (with respect to z), R = const (with respect to t), $A = 0$, $C = 0$.

The solution for this case was put forward by Brunt, 1932. Brunt did not mention the first of the above assumptions explicitly, but made an essential use of it, nevertheless. What he lays stress on, viz. the abrupt beginning of (effective) radiation and of cooling, which is a characteristic feature of his first treatment (compare case 3, below), is not simply equivalent to this assumption. If T_0 denotes the temperature for $t = 0$ (for all values of z, according to the first of the above assumptions), this solution is:

$$T(z, t) = T_0 - \frac{2R}{\lambda\sqrt{\pi}}\left(\sqrt{\varkappa t}\,.\,e^{-\frac{z^2}{4\varkappa t}} + z \int\limits_{-z/2\sqrt{\varkappa t}}^{\infty} e^{-u^2}\,du\right). \tag{10}$$

For $z = 0$ we find for the surface temperature as a function of t:

$$T = T(o, t) = T_0 - \frac{2R}{\sqrt{\pi}\,.\,w}\sqrt{t}, \tag{11}$$

where

$$w = \sqrt{\varrho c \lambda} = \varrho c \sqrt{\varkappa} \tag{12}$$

has been called the "contact coefficient".

2. $T(z, 0) = \text{const}$, $R = \text{const}$, $A = \text{const} \neq 0$, $C = 0$.

The solution for this case was given by Philipps, 1940. The latter started from the supposition, that in the air also a transport of heat takes place obeying an equation of the type (7) [1]), that is to say, that the eddy conductivity η is independent of z (a supposition, which surely does not hold in the atmosphere). He then finds a solution, which can be obtained from Brunt's solution by substituting $w + w_A = \varrho c\sqrt{\varkappa} + \varrho_a c_a \sqrt{\varkappa_A} = \sqrt{\varrho c \lambda} + c_a\sqrt{\varrho_a \eta}$ for w; here ϱ_a, c_a mean the density and the specific heat at constant pressure of the air, respectively; $\varkappa_A = \eta/\varrho_a$ might be called the thermal eddy diffusivity of the air, analogous to the coefficient $\varkappa$ in equation (7). In this way he finds for the surface temperature

$$T = T(o, t) = T_0 - \frac{2R\sqrt{t}}{\sqrt{\pi}(\varrho c\sqrt{\varkappa} + \varrho_a c_a \sqrt{\varkappa_A})} = T_0 - \frac{2R\sqrt{t}}{\sqrt{\pi}\,(w + w_A)}, \tag{13}$$

where

$$w_A = c_a\sqrt{\varrho_a \eta}. \tag{14}$$

3. Brunt (1932) has given still another treatment of the matter, by means of a harmonical analysis of a whole 24 hours cycle of radiational heating and cooling of the ground. He assumes again a constant effective radiation during the interval from sunset to sunrise, but on account of the daytime heat transport in the ground, the latter is now not isothermal at the beginning of the night, as it was in the first case. This makes the temperature variation curve, especially in the beginning of the night, much more acceptible than in the first case, where, for $t = 0$, $dT/dt = -\infty$ (see discussion of this case in the following section). As regards the temperature distribution in the ground, this is of such a nature that it makes the 24 hours mean value of temperature independent of z, or: $\overline{T(z)} = \text{const}$ (mean temperature distribution is isothermal, a simplification, which, of course, although it is a better one than the assumption $T(z, 0) = \text{const}$, does not yet fit the actual state of things, in general). A and C are here again discarded as in case 1 ($A = 0$, $C = 0$). The result is a Fourier development of temperature variation. We shall return to this treatment in section 5 (C).

In a theoretical investigation on this subject, performed during the war, the author has gone beyond these three special cases. One of the results of this investigation has already been published elsewhere (Journal of Meteorology, 1947). In the present paper it is developed still somewhat farther (section 4). Furthermore, we wish to make ourselves free from as many as possible of the simplifying assumptions, referred to above. We shall start with a closer examination of the simplest case.

[1]) Besides, he neglects the rôle of radiation in the heat interchange in the air.

3. Simplest case (Brunt).

$T(z, o) = \text{const}, \; R = \text{const}, \; A = 0, \; C = 0$

Conditions (8) and (9) become here respectively:

$$T(z, o) = T_o \text{ for } z < o, \tag{15}$$

$$-\lambda \left(\frac{\partial T}{\partial z}\right)_{z=o} = R. \tag{16}$$

The solution $T(z, t)$ for this case is given by (10), as we have seen. It is easily obtained by first finding $S = \partial T/\partial z$, which satisfies also equation (7) and for which we have as conditions equivalent to (15) and (16),

$$S(z, o) = 0 \text{ for } z < 0 \tag{17}$$

$$S(o, t) = -R/\lambda. \tag{18}$$

The solution $S(z, t)$ is

$$S = -\frac{2}{\sqrt{\pi}} \frac{R}{\lambda} \int_{-z/2\sqrt{\varkappa t}}^{\infty} e^{-u^2} du. \tag{19}$$

For some fixed values of t its graph is given in fig. 2; the curves are Gauss-integral curves, of which only the right hand parts ($z < o$) have a physical meaning here. $T(z, t)$ (10) is then found by integrating (19) with respect to z and applying (15).

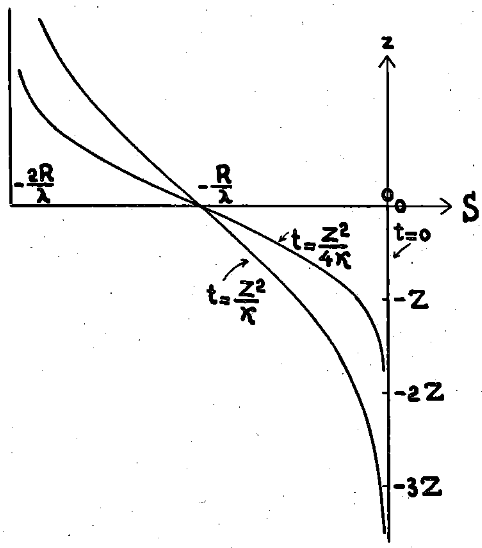

Fig. 2

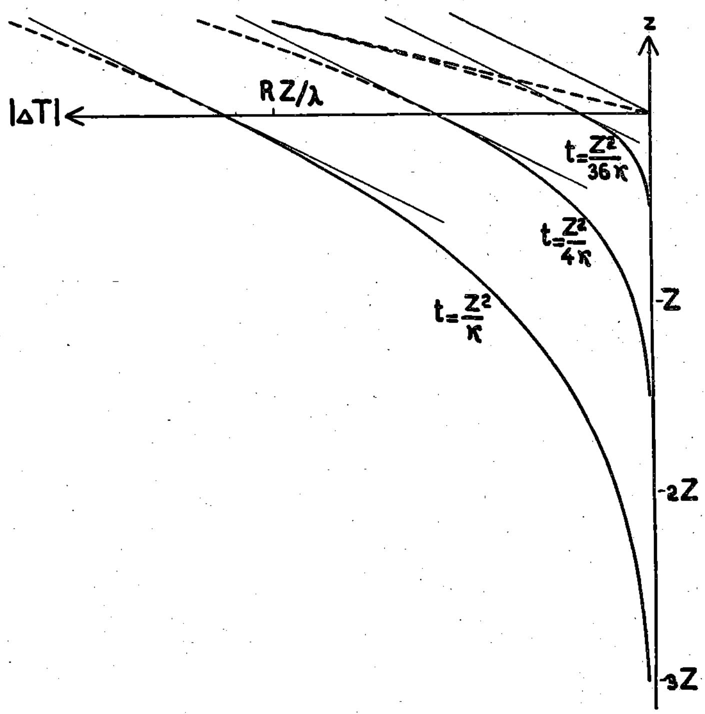

Fig. 3

The temperature distribution in the ground, as given by (10), is illustrated for a set of fixed values of t, in fig. 3. The dashed parts of the curves ($z > o$) have again no physical meaning here.

Characteristic of these temperature distribution curves is that their tangents in $z = o$ all have the same inclination, determined by (16). For $t = o$, properly speaking, this tangent is undetermined, the complete solution showing a discontinuity in $\partial T/\partial z$.

The temperature variation at the surface ($z = o$) is described by (11) and illustrated in fig. 4.

There are two characteristic features of this solution, which claim our attention, viz.:

1. The unlimited temperature fall, when $t \longrightarrow \infty$; this is a consequence of the assumption that $R = \text{const}$ for all values of t.

2. The infinite negative value of $\partial T/\partial t$ for $t = o$; this is a consequence of the discontinuity existing in the upward energy stream E at the earth's surface, for $t = o$, according to our assumption, that for $z = +o$, $E = R$, for $z = -o$, $E = o$, at that moment.

We may also formulate the state of things as follows: for $t = o$, the second derivative of $T(z, t)$ in $z = o$ is infinite, as is shown in fig. 3, curve $t = 0$ [1]); and this means, according to (7), that $\partial T/\partial t = -\infty$.

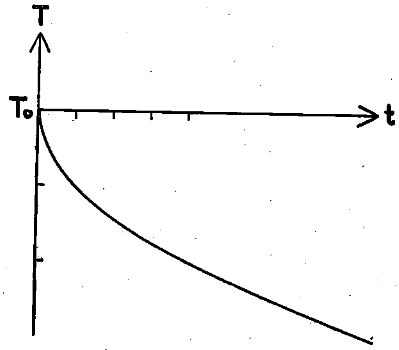

Fig. 4

Finally we shall give a numerical example to illustrate formula (11). For R we take the same value as was given in section 1: $R = 0{,}15$ cal/cm²min $= 9$ cal/cm²h; for ϱ, c, λ, we take some more or less "normal" values, so that $w = \sqrt{\varrho c \lambda} = 3$ cal/cm²h$^{1/2}$ deg.

(We shall return in Part II, sections 11 and 12 to the question of the numerical values of these soil constants under various circumstances).

Formula (11) now becomes:

$$T = T_o - 3{,}3^\circ \left(\frac{t}{\text{hour}}\right)^{1/2},$$

from which the temperature fall in, say, 9 hours is computed to be $T_o - T = 10^\circ$.

4. Variation of R.

$T(z, o) = \text{const}$, $A = o$, $C = o$ (as above); $R \neq \text{const}$.

We have introduced here, as a first corrective to the simplifications of the foregoing case (B r u n t), a characteristic feature of the process of nocturnal cooling, viz. the non-constancy of R. The cause of the variation of R lies in the cooling of the earth's surface and of the lower layers of the atmosphere, causing σT^4 as well as R' to diminish. The decrease of R' being smaller than that of σT^4, R will decrease also, its rate of decrease depending on the steepness of the arising ground inversion. This effect was treated by the author's theoretical paper in the Journal of Meteorology, referred to above. This treatment runs, in brief, as follows. As an approximation we may formally write R, as a function of ground temperature, as follows:

$$R = R(T) = R_o + f(T - T_o) = f(T - T_1), \tag{20}$$

where $f = \partial R/\partial T$ and $T_1 = T_o - R_o/f$ is the value of T, at which R would vanish, if we would extrapolate formula (20) so far.

The boundary condition (16) now becomes

$$-\lambda \left(\frac{\partial T}{\partial z}\right)_{z=o} = R_o + f(T - T_o)_{z=o} = f(T - T_1)_{z=o}. \tag{21}$$

In order to find a solution of (7) satisfying conditions (15) and (21), we first introduce $T - T_1 = u$ as a new variable, instead of T; u satisfies the same differential equation (7) as T, whereas (15) and (21) are transformed into:

$$u(z, 0) = T_o - T_1 = \frac{R_o}{f} = u_o, \text{ for } z < o \quad \text{and} \quad \frac{\partial u}{\partial z} + \frac{f}{\lambda} u = o, \text{ for } z = o, \tag{22}$$

[1]) The upper part of which is the dashed straight line.

respectively. Now, by performing a further transformation, putting $\partial u/\partial z + uf/\lambda = v$, it has been possible to find a solution u, satisfying conditions (22), this solution being

$$T - T_1 = u = \frac{R_o}{f}\left\{\Phi\left(\frac{-z}{2\sqrt{\varkappa t}}\right) + e^{-\frac{fz}{\lambda}+\frac{f^2 t}{\lambda\varrho c}}\left[1 - \Phi\left(\frac{-z}{2\sqrt{\varkappa t}} + f\sqrt{\frac{t}{\lambda\varrho c}}\right)\right]\right\}, \tag{23}$$

where Φ means the Gauss integral function:

$$\Phi\,(x) = \frac{2}{\sqrt{\pi}}\int_o^x e^{-y^2}\,dy.$$

We may give a dimensionless representation of this result in the following form:

$$\frac{T - T_1}{u_o} = \frac{u}{u_o} = \Phi\left(\frac{-z/z_1}{2\sqrt{t/t_1}}\right) + e^{-\frac{z}{z_1}+\frac{t}{t_1}}\left[1 - \Phi\left(\frac{-z/z_1}{2\sqrt{t/t_1}} + \sqrt{\frac{t}{t_1}}\right)\right], \tag{24}$$

where

$$u_o = R_o/f \text{ (a temperature)}, \tag{25}$$

$$t_1 = \lambda\varrho c/f^2 \text{ (a time)}, \tag{26}$$

$$z_1 = \lambda/f \text{ (a length)}; \tag{27}$$

it should be born in mind here, that in the application of (23) and (24) z is negative or zero, $-z$ being the depth considered.

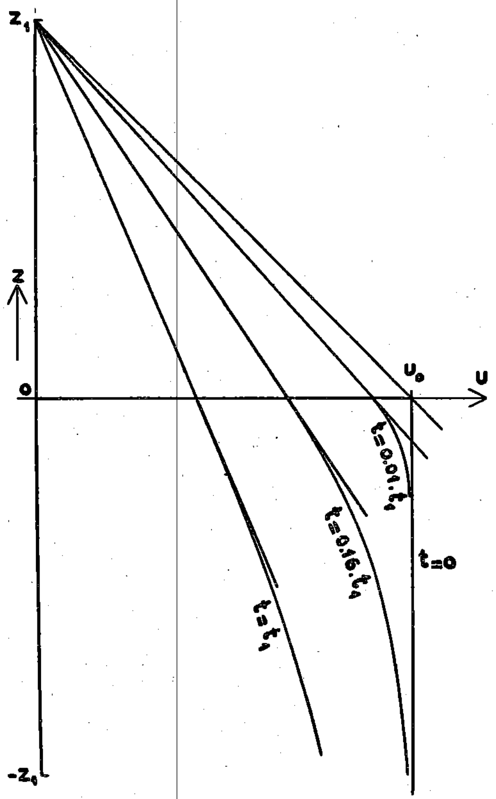

Fig. 5

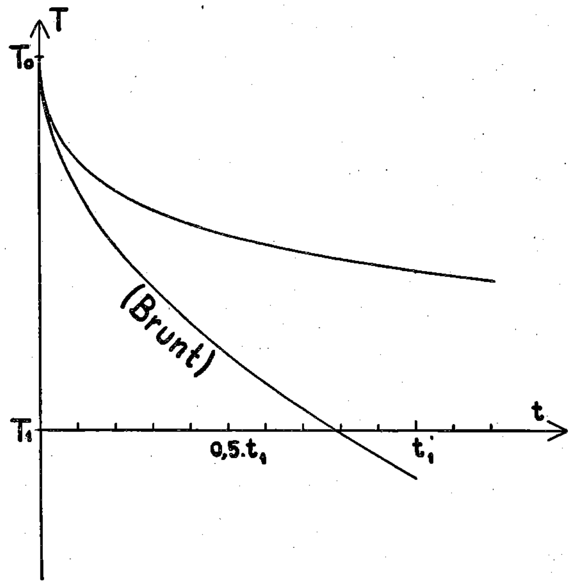

Fig. 6

In fig. 5 we have drawn the temperature curves as given by (23), (24) for a set of fixed values of t.

Unlike the tangents in $z = o$ to the curves of fig. 3, the tangents to the temperature curves in that level are not parallel; they have one common point of intersection at $u = o$, $z = z_1 = \lambda/f$.

For $t = o$ the complete curve shows again a discontinuity in the inclination of its tangent in the point $z = o$.

The temperature variation at the surface may be obtained from (23) or (24) by putting $z = o$; the result is:

$$u = T - T_1 = \frac{R_o}{f} e^{\frac{f^2 t}{\lambda \varrho c}} \left[1 - \Phi\left(f \sqrt{\frac{t}{\lambda \varrho c}}\right)\right], \tag{28}$$

or, in a dimensionless representation:

$$\frac{u}{u_o} = \frac{T - T_1}{u_o} = e^{t/t_1}\left[1 - \Phi\left(\sqrt{t/t_1}\right)\right], \tag{29}$$

as illustrated by fig. 6 (upper curve) and fig. 7.

From fig. 6 we observe, that here again the value of $\partial T/\partial t$ for $t = o$ equals $-\infty$, as in Brunt's solution; the cause of this fact is the same as above, viz. the assumption,

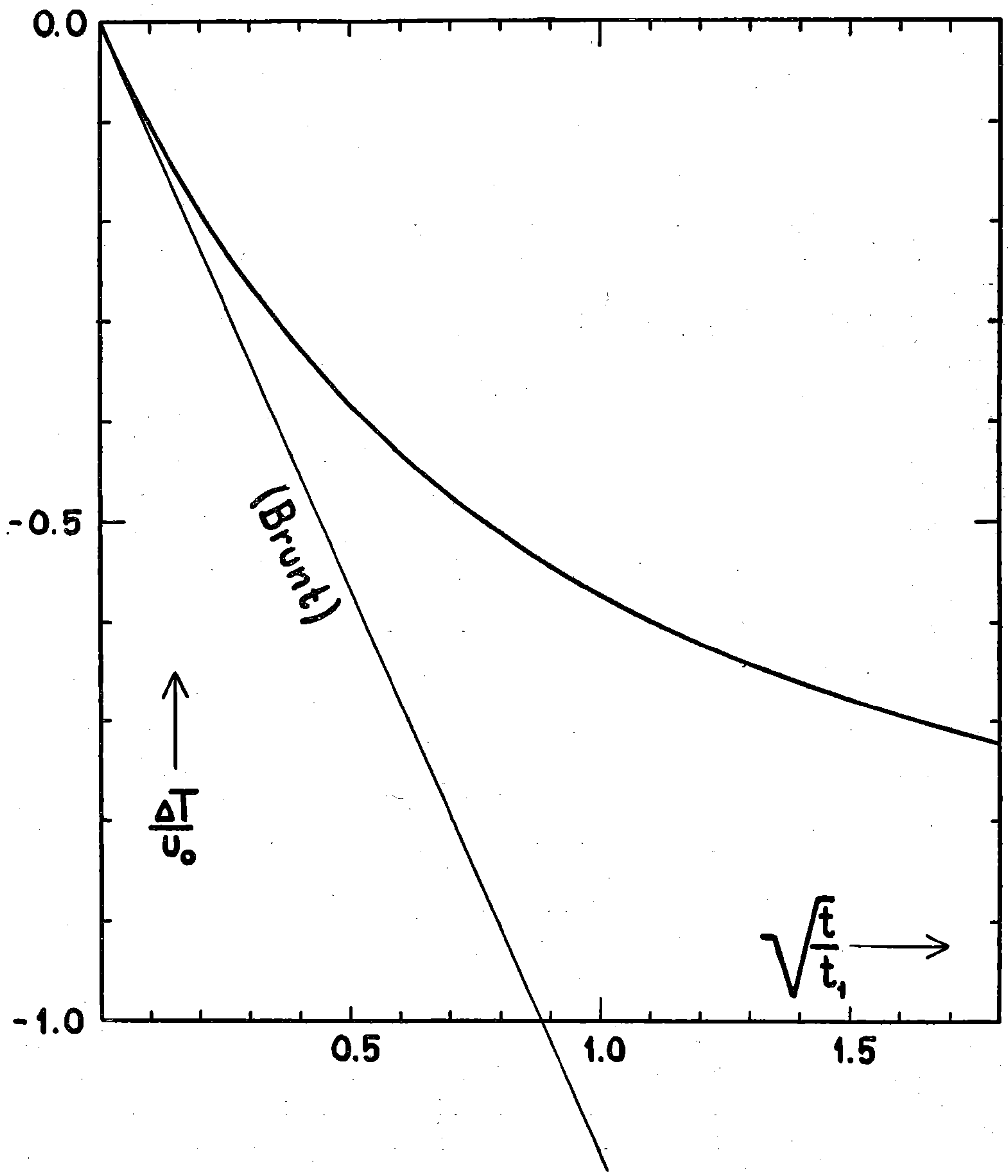

Fig. 7

that, for $t = o$, $\partial T/\partial z = o$, for all values of z, whereas the outgoing radiation $R_o \neq 0$.

The fundamental difference of our formula from that of Brunt, however, is that in the former T approaches to T_1 as a limit, when t increases indefinitely.

The precise connection between B r u n t's result and ours may best be seen from a series development. According tot (28) the temperature fall after a time t is

$$T_0 - T = u_0 - u = \frac{R_0}{f}\left\{1 - e^{\frac{f^2 t}{\lambda \varrho c}}\left[1 - \Phi\left(f\sqrt{\frac{t}{\lambda \varrho c}}\right)\right]\right\}. \qquad (30)$$

By developing this into a Taylor series after ascending powers of $f\sqrt{t/\lambda\varrho c}$ we obtain:

$$T_0 - T = \frac{R_0}{f}\left[\frac{2}{\sqrt{\pi}}\left(f\sqrt{\frac{t}{\lambda\varrho c}}\right) - \left(f\sqrt{\frac{t}{\lambda\varrho c}}\right)^2 + \frac{4}{3\sqrt{\pi}}\left(f\sqrt{\frac{t}{\varrho\lambda c}}\right)^3 - \ldots\right]$$

If we cut off all terms except the first one, we are left with B r u n t's formula (11), if in the latter we put $R = R_0$.

In fig. 7 we give a representation of the computed temperature variation as a function of $\sqrt{t}$. It is clear now, that B r u n t's formula in this representation gives a straight line, touching our curve in the origin, when $R = R_0$. By choosing u_0 and t_1 as units of temperature and of time, respectively, we have obtained in fig. 7 a graph, which corresponds to the dimensionless representation (29) and is therefore independent of the special values of the various constants occurring in (28). For the proper use of this graph we only need to know the quantities u_0 (25) and t_1 (26).

As a numerical example we may substitute the same values of R_0, ϱ, c, λ and f as previously used: $R_0 = 9$ cal/cm²h, $\sqrt{\varrho c \lambda} = 3$ cal/cm²h$^{1/2}$ deg, $f = 0{,}3$ cal/cm²hdeg, by which $u_0 = 30°$, $t_1 = 100$ h.

For $t = 9$ h we find a temperature fall of $0{,}265\ u_0 = 8°$, whereas formula (11) gave 10°, as we have seen.

For large values of t the deviation of B r u n t's curve from ours becomes very large. Our result may therefore be of importance for studying the temperature fall during the polar night, a problem treated by W e x l e r, 1936. W e x l e r used B r u n t's formula, which assumes a constant value of R, for estimating the time necessary for attaining the temperature at which the incoming radiation and the outgoing radiation cancel each other (our temperature T_1), a somewhat inconsistent procedure.

From the asymptotic behaviour of the function Φ for large values of the argument it follows, that for very large t the decrease of $T - T_1$ become proportional to $t^{-\frac{1}{2}}$.

Evaluation of f.

In section 11 a qualitative discussion of the factors influencing the value of f will be given, in connection with the discussion of typical radiation diagram curves. For the present we shall yet try to find a way of estimating this quantity theoretically. To this end we choose a very simple ground inversion model, see fig. 8. We have chosen, here, as the ordinate the quantity

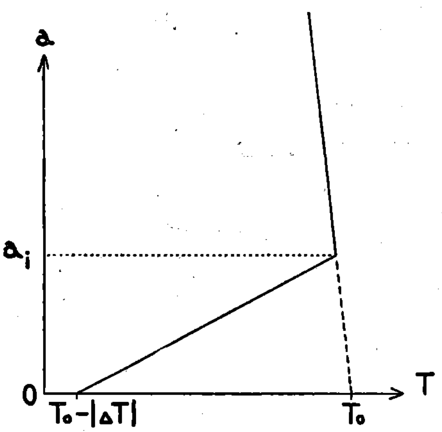

Fig. 8

$$a = \int_0^h \varrho\, dz,$$

where h = height above the earth's surface. The initial temperature distribution may be represented by the function $T(a, o)$; let it be linear with respect to a, near the earth's surface. Let the ground inversion that arises be represented by

$$T(a, t) = T(a, o) - \frac{a_i - a}{\alpha}, \text{ for } a \leq a_i = \alpha\,|\triangle T|, \qquad (32)$$

where $|\triangle T|$ is the temperature fall at the earth's surface; the quantity α (the "flatness" of the inversion) is assumed to be constant while the depth of the inversion grows.

Now, when the ground temperature has fallen an amount $\triangle T$, the heat given off by the air is

$$\int_o^{a_i} c_p \left[T(a,o) - T(a,t)\right] = \tfrac{1}{2} c_p a_i^2 \alpha^{-1} = \tfrac{1}{2} c_p (\triangle T)^2 \alpha = \tfrac{1}{2} c_p \gamma^2 t \alpha, \tag{33}$$

where we have used the approximative formula: $\triangle T = \gamma\sqrt{t}$, according to Brunt.

On the other hand the amount of heat given off by the air to the earth may also be calculated from the heat balance, as it is equal to the time-integral of the sum of (1) the atmospherical counter-radiation R' and (2) the heat conducted downwards by eddy conductivity, A, *minus* (3) the fraction of the earth's radiation absorbed by the air, which is dependent on the total water vapour content (per cm²) of the atmosphere. We may write for the last term: $k\sigma T^4$, where k can be determined from the radiation diagram; it may be treated as a constant within any not too large interval of temperature. For the second term we may write: $A = (r/1+r)\,R$, where $r = A/B$ is assumed to be constant during the night (It is necessary to use some of the results of section 6, here). Thus we obtain:

$$\tfrac{1}{2} c_p \gamma^2 t_1 \alpha = \int_o^{t_1} \left(R' + \frac{rR}{1+r} - k\,\sigma\,T^4\right) dt = \int_o^{t_1} \left(\left[1-k\right]\sigma\,T^4 - \frac{R}{1+r}\right) dt = \int_o^{t_1} R^*\,dt = \overline{R^*}\,t_1. \tag{34}$$

Or:

$$\alpha = \frac{2\overline{R^*}}{c_p \gamma^2}. \tag{35}$$

As in first approximation T varies linearly with $\sqrt{t}$ (formula of Brunt) and the quantities σT^4 and R are considered as linear functions of T in the limited temperature interval we are concerned with, we may, for the time mean of R^*, as defined by (34), take the value corresponding to the temperature $T_0 - {}^2/_3\,|\triangle T\,|$.

For applying this we should first make a preliminary estimate of $\triangle T$ (for instance by means of the formula $|\triangle T\,| = \gamma\sqrt{t}$) and of the variation of R with T. Eventually we may, with help of the value of f found, afterwards repeat the computation and thus find a better value of f. The quantity denoted by γ may be put equal to

$$\gamma = \frac{2R}{(1+r)\sqrt{\pi\lambda\varrho c}}. \tag{36}$$

This follows from Brunt's formula, modified for the effect of eddy conductivity (see section 6); for R we may again take some mean value, here.

The "flatness" α of the inversion to be expected having been computed in this way, we must now calculate f from it. To this end we proceed as follows. The limiting values, between which $f = (\partial/\partial T)\,(\sigma T^4 - R')$ lies, are:

$$f_{\max} = 4\sigma T^3, \qquad f_{\min} = 4\sigma T^3\left(1 - \frac{R_o'}{\sigma T_o^4}\right),$$

the latter value corresponding to the limiting case of $R'/\sigma T^4$ remaining constant, which would be realized approximatively for $\alpha = \infty$ (the temperature curve then shifts parallel to itself; an eventual loss of water vapour by condensation is left out of consideration). Therefore we take the following working form:

$$f = 4\,\sigma\,T^3\left(1 - \beta\frac{R_o'}{\sigma T_o^4}\right), \tag{37}$$

where β is assumed to be a function of α only and to go from 0 to 1; $\beta(o) = 0$, $\beta(\infty) = 1$. The function $\beta(\alpha)$ to be used should now be determined by computing from a large number of examples, plotted on the radiation chart, the values of β for various values of α. This may be done most easily in the following way:

$$1 - \beta \frac{R_o'}{\sigma T_o^4} = \frac{\partial R/\partial T}{4\sigma T^3};$$

hence:

$$\beta = \frac{\sigma T_o^4}{R_o'} \left(\frac{4\sigma T^3 - \partial R/\partial T}{4\sigma T^3} \right) = \frac{\sigma T_o^4}{R_o'} \frac{\triangle R'}{\triangle(\sigma T^4)}. \tag{38}$$

The ratio $\triangle R'/\triangle(\sigma T^4)$, for any value of $\triangle T$, is equal to the ratio of two areas on the radiation diagram which may be easily measured.

Some provisional random tests gave as a result an interdependence of β and α that may rather well be represented by the following *provisional formula:*

$$\beta = \tfrac{1}{2} + \tfrac{1}{2}\Phi\,(0{,}53\ {}^{10}\!\log \alpha - 0{,}07), \tag{39}$$

where Φ is the Gauss integral function, α being expressed in g/cm² deg. Between the values $\alpha = 0{,}1$ and $\alpha = 10$ this formula may rather well be replaced by the simpler one:

$$\beta = 0{,}465 + 0{,}265\ {}^{10}\!\log \alpha. \tag{40}$$

Thus, in order to determine f we have first to determine α from formula (35), then β from (39) or (40) and finally f from (37).

By applying this procedure to the example discussed in section 10 we get $f = 0{,}25$, whereas from Franssila's measurements the value $f = 0{,}30$ (cal/cm² h deg) was derived.

5. *Non-isothermal temperature distribution in the ground for* $t = o$.

A. If we have a solution $T_m\,(z, t)$ of equation (7), satisfying (15) as an initial condition, the function

$$T\,(z, t) = T_m\,(z, t) + mz \tag{41}$$

also satisfies (7), while for $t = o$ we shall have

$$T\,(z, o) = T_o + mz. \tag{42}$$

If, now, we want the solution (41) to satisfy the condition (16) or (21) we obtain

$$-\lambda \left(\frac{\partial T_m}{\partial z} \right)_{z=o} - \lambda m = R,$$

whence

$$-\lambda \left(\frac{\partial T_m}{\partial z} \right)_{z=o} = R + \lambda m,$$

or, according to (20),

$$-\lambda \left(\frac{\partial T_m}{\partial z} \right)_{z=o} = R_o + \lambda m + f\,(T - T_o)_{z=o} = f\,(T - T_1)_{z=o}, \tag{43}$$

where now

$$T_1 = T_o - \frac{R_o + \lambda m}{f}. \tag{44}$$

The solution $T\,(z, t)$ satisfying the system of equations (7) — (42) — (21), can therefore be constructed from (41), where $T_m\,(z, t)$ is the solution, satisfying the system (7) — (15) — (43); the latter solution is known from the preceding section; we have only to substitute $R_o + \lambda m$ for R_o in (23), (25), (28), (30). The effect of the occurrence of the term $+ mz$ in the intial condition (42) upon temperature variation may therefore be said to consist in the substitution of an "apparent radiation" $R + \lambda m$ for the real radiation R.

We observe further that the solution gives a temperature function for $z = 0$ (at the earth's surface),

$$T = T(o, t) = T_m(o, t), \tag{45}$$

which starts again with a $dT/dt = -\infty$ for $t = 0$, except when $R_o + \lambda m = 0$.

There are two reasons why a function $T_o + mz$ (42) will, in many cases, be a better approximation of the initial temperature distribution in the ground, than the isothermal distribution (15) — so long as we restrict ourselves to that layer of ground, which takes part in the diurnal temperature variation. These reasons are:

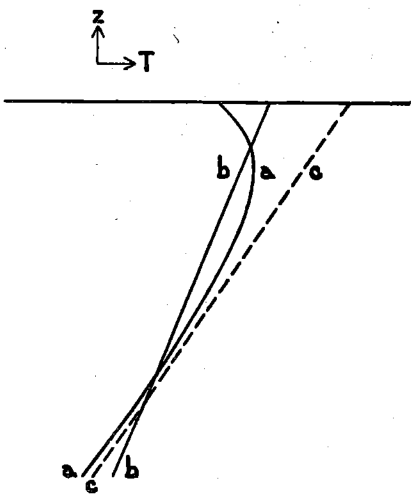

Fig. 9

1. Annual temperature wave in the ground. In the cold season, the effect of this "wave" may be described as a heat store in the ground; in the warm season as a "cold store". As this wave penetrates into a much deeper layer of the ground than the diurnal temperature variation, the 24 hours mean temperature distribution may in the upper layers, where the latter variation is perceptible, be approximated fairly well by a linear function of z.

2. On this linear mean temperature distribution a diurnal temperature wave of much smaller dimensions is superposed. The effect of this wave is a "cold store" in the ground (of a much smaller capacity than the one mentioned above, to which it superposes itself) during the warmer part of the day, i. e. during the afternoon. At the beginning of the night, this "cold store" will, in general, not yet have been compensated. This means that the effect in question will, in general, give a positive contribution to the coefficient m in (42) and (43). (It is clear, that the linear form (42) is only a first approximation, which, however, will often fit the facts better than the isothermal one). One might object that the real temperature distribution at the beginning of the night is rather of the type shown by fig. 9, *a*, instead of a linear distribution, fig. 9, *b* (the seasonal effect has been left out here; or, in other words, the temperature distribution, represented in fig. 9, should be superposed to the 24 hours mean temperature distribution). That is to say: in the upper *centimeters* of the ground we have to expect a negative value of $\partial T/\partial z$, instead of a positive one, even if we should have, across the *whole* layer considered (which is, in general, at least 30 cm deep), a positive value of $\triangle T/\triangle z$. It may be expected, that, in consequence of this small heat store in the uppermost groundlayer, a less steep temperature fall will set in than in the case of an isothermal initial state. This will, however, only be true for the first one or two hours, in general. The *total* temperature fall during the night, if the latter be not too short, will be *larger* than in the isothermal case, as is to be expected if we replace the temperature curve *a* (fig. 9) by a straight line *b*.

This conclusion is confirmed by B r u n t's computation, mentioned above (section 2, case 3), to which we shall return in part C of this section. The result of this computation is a temperature curve giving a total temperature fall during the night, about 10 % *larger* than the temperature fall, computed from an isothermal temperature distribution at the beginning of the night, though the temperature curve *starts less* steeply.

Numerical example. For effect (1) (seasonal effect), we quote a result of P e e r l k a m p, 1944, who measured ground temperatures under various circumstances. For a number of clear summer days, for instance, he found a 24 hours mean temperature difference of 4,3° between the levels $z = o$ and $z = -30$ cm in a grass covered clay ground. This yields a mean temperature gradient $m = 0{,}14°/\text{cm}$ in this layer. With $\lambda = 0{,}15$ cal/cm min deg we obtain $\lambda m = 0{,}021$ cal/cm²min $= 1{,}3$ cal/cm²h. In the numerical example given above (section 4) we had $R_o = 9$ cal/cm²h, so that $R_o + \lambda m$ becomes 10,3 cal/cm²h. This is a increase of 14 %.

Further, for effect (2) we simply refer to the above mentioned computation by B r u n t.

Finally, for the total effect, we quote measurements by L. H e r r, 1936, from which can be decuced a temperature gradient $m = 0{,}24°/\text{cm}$ for the upper 35 cm of the ground, in the evening of a day in July. With the same value of λ as used just now, we obtain $\lambda m = 2{,}2$ cal/cm²h; if R_0 would again be 9 cal/cm²h, we should have an *apparent* increase of effective radiation of 25 %.

We see, therefore, that, at least in summer, the effects discussed in this section may sometimes give an appreciable correction to the computation of the nocturnal temperature fall.

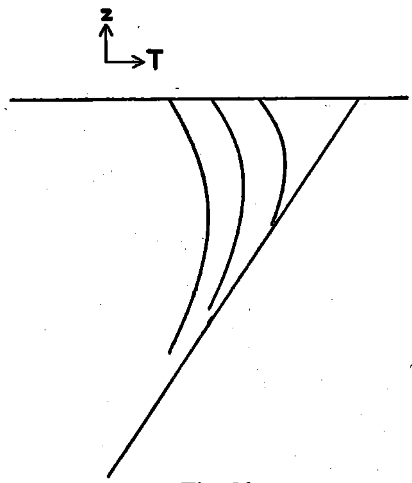

Fig. 10

B. There is another possibility to make our initial condition agree more closely with real temperature distributions.

If, from (41), we construct the temperature curves for a set of values of t, as we can conveniently do with the help of the curves of fig. 5, we obtain, for any positive value of m, a set of curves as shown in fig. 10.

Now, if $T^*(z,t)$ is a solution of (7), satisfying (21) as a boundary condition, the same is true for $T(z,t) \equiv T^*(z, t + t_0)$; if $T^*(z, o)$ is given by the right hand member of (42), the initial state, corresponding to this solution, is given by

$$T(z, o) = T^*(z, t_0) = T_m(z, t_0) + mz. \tag{46}$$

This initial state is represented by one of the curves of fig. 10, if only the right value of m be used. When we now compare fig. 10 with fig. 9, we see, that among the curves of fig. 10, we may in general find one that better fits the curve a of fig. 9 than curve b of fig. 9 does, if only the right value of m be used in fig. 10.

If curve b, fig. 9, is given, this value of m can be estimated by drawing some sort of "asymptote" to curve b (dashed line c in fig. 9), which has to play the rôle of the straight line in fig. 10.

Since $t_0 > 0$, the solution $T(z,t)$ constructed here, gives a temperature variation at the earth's surface, having, for all $t \geq 0$, a *finite* value of its time derivative.

The *temperature fall* after a time is given by

$$|\triangle T| = T(t_0) - T(t + t_0),$$

where $T(t)$ denotes the function, known from (45).

C. It is here the place to discuss briefly B r u n t's computation of a whole 24 hours day's temperature variation, by which he corrected his own computation of the temperature fall during a night. Apart from the expounding of this method we shall give two simple generalizations, of which it is capable.

B r u n t (l. c.) assumed the net *inward* radiation $F(t)$ (= incoming radiation from sun and atmosphere *minus* outgoing terrestrial radiation) to follow a harmonic law during the daytime and to be constant during the night:

$$F(t) = -I \sin qt - R \text{ for } -\pi < qt < o \text{ (daytime)},$$

$$F(t) = -R \text{ (const.) for } o < qt < \pi \text{ (night)}.$$

This function can be developed into a F o u r i e r series as follows:

$$F(t) = \frac{I}{\pi} - R - {}^1/_2\, I \sin qt - 2\frac{I}{\pi}\left({}^1/_3 \cos 2qt + {}^1/_{15} \cos 4qt + \ldots\right).$$

Let the temperature at the surface of the ground be represented by the Fourier series

$$T = T_o + P_1 \cos(qt - \varepsilon_1) + P_2 \cos(2qt - \varepsilon_2) + \ldots \text{ etc.}$$

The corresponding solution $T(z, t)$ of our equation (7) is

$$T(z, t) = T_o + P_1 e^{\mu_1 z} \cos(qt - \varepsilon_1 + \mu_1 z) + P_2 e^{\mu_2 z} \cos(2qt - \varepsilon_2 + \mu_2 z) + \ldots \text{ etc.},$$

where

$$\mu_1 = \sqrt{q/2\varkappa},\ \mu_2 = \sqrt{2q/2\varkappa}, \ldots \text{ etc.}$$

On differentiating and putting $z = 0$ we find that

$$F(t) = \lambda \frac{\partial T}{\partial z} = \sqrt{\varrho c \lambda} \sum_{n=1}^{\infty} P_n \sqrt{nq} \cos\left(nqt - \varepsilon_n + \frac{\pi}{4}\right).$$

As this series must be identical with the one above for $F(t)$, we easily find all coefficients P_n and phase constants ε_n of the temperature function. Furthermore it appears that

$$I/\pi - R = o,$$

in other words, that the net gain or loss of heat by the ground during 24 hours is zero. This is a consequence of the fact that the solution $T(z, t)$ used here is such as to make the 24 hours mean temperature distribution in the ground isothermal.

The characteristics of the temperature curve during the night, computed in this way, have already been mentioned in part A of this section.

This calculation can still easily be improved in two respects. First we can add a term $+ \alpha z$ to the above temperature function $T(z, t)$, the new temperature function $T(z, t) + \alpha z$ still being a solution of (7). The 24 hours mean temperature distribution in the ground is then no longer isothermal and the 24 hours net gain or loss of heat by the ground no longer zero, but:

$$I/\pi - R = \lambda\alpha.$$

The small secular change of ground temperature from one day to the next, which is, in nature, involved in this gain or loss, may of course be neglected when it is only the diurnal variation we are concerned with.

The second modification we might perform is to drop the assumption that R is a constant. We might for instance put $R = R_o - P \sin(qt + \varepsilon)$ (not simply $R = R_o - P \sin qt$, as Brunt suggests).

Still better would be to use:

$$R = R_o + f(T - T_o) = R_o + f\left[P_1 \cos(qt - \varepsilon_1) + \ldots \text{ etc.}\right].$$

The unknown coefficients P_1, P_2 etc. would thereby occur in both Fourier series-representations of $F(t)$, but this would not essentially complicate the calculation.

6. *Conductivity of the air.*

In this section we drop the assumption $A = 0$. Indeed, in general, there will be an eddy conductivity of the air causing a heat stream, which during the night, runs downward towards the radiating surface of the earth. Now to develope a satisfactory theory in order to take into account this very complicated phenomenon is an extremely difficult problem. We shall therefore attack the matter from the empirical side. To this end we have examined the results of measurements of the terms of the energy-balance at the earth's surface, as have been carried out for example by F. Albrecht, 1930, and M. Franssila, 1936. From this examination it has appeared, that the ratio A/B did not vary much during the nights (there seems to be a maximum in the forenight). For illustrating this statement we have here written down two sets of hourly values of this ratio, derived from results of measurements by the above mentioned authors.

F. Albrecht, one clear night, 19—20 VII 1925.

	19—20 h.	20—21 h.	21—22 h.	22—23 h.	23—0 h.	0—1 h.
A/B	0.28	0.32	0.30	0.28	0.26	0.23

M. Franssila, 3 clear or slightly clouded nights in August 1934.

	20—21 h.	21—22 h.	22—23 h.	23—0 h.	0—1 h.	1—2 h.	2—3 h.	3—4 h.	4—5 h.
A/B	0.23	0.37	0.45	0.40	0.33	0.33	0.31^5	0.43	0.34

For our calculations, we may therefore assume, as a *working hypothesis*, this ratio to be *constant*. This means that we assume the total heat stream, compensating the radiative energy loss at the earth's surface, to be distributed over the two conduction heat streams, viz. A and B, in a *constant* ratio — so long as the condensation heat stream C is still neglected. The value of this ratio will depend upon the ratio of the conductivities of the two media (air and soil).

It can be proved theoretically, that this ratio will indeed be a constant if the two following conditions are satisfied:

1. The mechanism of the air heat "conduction" must be such that the air temperature variation obeys a differential equation

$$\frac{\partial T}{\partial t} = \varkappa_a \frac{\partial^2 T}{\partial z^2} \quad (\varkappa_a \text{ being constant}) \tag{7a}$$

of the same type as equation (7), which is valid for the soil.

2. The temperature distributions in the air and in the ground must stand in such a relation to each other, that, for a certain moment $t = t_0$, the temperature curve for the air becomes the counter-image of that for the ground, if the coordinates z of all points in the air are multiplied by a factor $\sqrt{\varkappa/\varkappa_a}$.

It will appear that, if the latter condition is satisfied for $t = t_0$, it is satisfied for all values of t; furthermore, that, even if the first condition be fulfilled, the second one is a sufficient condition, it is true, but *not* a *necessary* one.

We shall denote the temperature of the air, as a function of z and t, by $T_a(z, t)$ $(z \geq 0)$, that of the soil as before, by $T(z, t)$, so that, for continuity, we have

$$T_a(o, t) = T(o, t) \tag{47}$$

If $T(z, t)$ is supposed to be known $(z < 0)$, the temperature distribution in the air for a certain moment, $T_a(z, t_0)$, together with the boundary condition (47), determines the function $T_a(z, t)$ as a solution of (7a) for all values of t.

Now, for $z > o$, we introduce the following transformation of coordinates:

$$z = -Z\sqrt{\varkappa_a/\varkappa}.$$

By substituting this into (7a) we see, that T_a, as a function of Z and t, satisfies the same differential equation as $T(z, t)$. If, therefore, for $t = t_0$, T_a is the same function of Z as T is of z, the same is true for all values of t. Hence

$$\left(\frac{\partial T_a}{\partial Z}\right)_{Z=o} = \left(\frac{\partial T}{\partial z}\right)_{z=o} \quad \text{for all values of } t. \tag{48}$$

Now, the fact that for $t = t_o$, T_a is the same function of Z as T is of z is equivalent to the above condition 2. From (48), on the other hand, follows:

$$\left(\frac{\partial T_a}{\partial z}\right)_{z=o} = -\sqrt{\frac{\varkappa}{\varkappa_a}}\left(\frac{\partial T}{\partial z}\right)_{z=o},$$

or [1])

$$A = \lambda_A\left(\frac{\partial T_a}{\partial z}\right)_{z=o} = B\frac{\lambda_A}{\lambda}\sqrt{\frac{\varkappa}{\varkappa_a}} = rB,$$

where

$$r = \frac{\lambda_A}{\lambda}\sqrt{\frac{\varkappa}{\varkappa_a}} = \frac{\lambda_A}{\sqrt{\varkappa_a \lambda \varrho c}}. \tag{49}$$

In (49) λ_A denotes the eddy conduction coefficient of the air; we may write

$$\lambda_A = \eta c_a,$$

where η is the "eddy conductivity" ("Austausch-coeffizient"), c_a is the specific heat of the air. As to the $\varkappa_a$ in (7a) and in (49), we should bear in mind, that equation (7a) is supposed to describe the total temperature variation in the air and this is not only an effect of eddy conduction, but also of a radiative apparent "conduction". In writing $\varkappa_a = \lambda_a/\varrho_a c_a$, we have therefore, in general: $\lambda_a > \lambda_A$, $\varkappa_a > \eta/\varrho_a$.

Equation (49) states the constancy of $r = A/B$, in so far as λ_A and $\varkappa_a$ may be considered as constants, as supposed, and in addition gives the value of r, if the two conditions underlying its deduction are satisfied. It may easily be seen, however, that, when (7a) is satisfied, there are ∞ possible temperature distributions in the air, for a certain moment $t = t_o$, such that the ratio A/B is constant. Indeed, any given constant value of $A/B = r$ gives a constant value of $\frac{(\partial T_a/\partial z)_{z=o}}{(\partial T/\partial z)_{z=o}}$; that is to say: if, as before, $T(z, t)$ is supposed to be known [2]), not only $T_a(o, t)$ but also $(\partial T_a/\partial z)_{z=o}$ is known and consequently $T_a(z, t)$ is determined as a solution of (7a), for all values of t and z.

We have therefore ∞ solutions, such that A/B is constant.

The suppositions, underlying Philipps' calculation (l. c.), mentioned in section 2 (case 2) are only a special case of the above stated conditions, underlying (49). Indeed, Philipps started from isothermal initial temperature distributions in the air as well as in the soil (so that condition 2 is satisfied); as to the heat conduction in the air, he reckoned only with an eddy conduction; and by assuming the eddy conductivity to be independent of z, he arrived at equation (7a) whereby condition 1 is fulfilled.

We, however, shall not try to give the above deduction, resulting into equation (49) the character of a theoretical "basis" of our calculation, but shall assume the constancy of A/B merely as a more or less empirical suggestion; we have already seen, that the above "deduction" uses certain suppositions, which are *sufficient*, but not at all *necessary* for this constancy, so that in reality it implies assumptions of a much wider scope.

As we have seen, we may have in B a constant term $-\lambda m$ (see preceding section), which is a consequence of a "heat store" or a "cold store" in the ground. As in the atmosphere such "store" effects do not, by far, occur in the same degree, we shall subtract this term from B and from R; so, in this case, what we distribute in a constant ratio over soil conduction and air conduction, is the "apparent" radiation $R + \lambda m$, introduced in the preceding section.

As an illustration, we might easily perform a reconstruction of the above discussed theoretical case so, that the postulated relation between the temperature distributions in

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[1]) As in the lowest layers of the atmosphere, which concern us here, the temperature gradients are in general large compared with the adiabatic lapse-rate, we may, without introducing any appreciable error, use $\partial T/\partial z$ instead of $(\partial\vartheta/\partial z)T/\vartheta$, as, strictly speaking, we ought to do (ϑ = potential temperature of the air).

[2]) As $R = A + B = (1 + r)A$, we have, as a determining boundary condition for $T(z, t)$: $-\lambda\left(\frac{T\partial}{\partial z}\right)_{z=o} = \frac{R}{1+r}$.

the soil and in the air (condition 2) now applies to $T(z,t) - mz = T_m(z,t)$ and $T_a(z,t)$ instead of to $T(z,t)$ and $T_a(z,t)$.

In the following of this section we shall, for brevity, always write R and B, but shall keep in mind, that, when necessary, we shall have to understand by this symbols the "corrected" values $R + \lambda m$, $B + \lambda m$.

If r is given, we can find the solution of our problem in the following way: As $R = A + B = B(1 + r)$, we have $B = R/(1 + r)$. Instead of (23), we obtain therefore, as a boundary condition:

$$-\lambda\left(\frac{\partial T}{\partial z}\right)_{z=o} = \frac{R_o}{1+r} + \frac{f}{1+r}(T - T_o). \tag{50}$$

For the calculation of temperature variations by means of the formulas of section 4, this amounts to the substitution of $R_o/(1+r)$ and $f/(1+r)$ for R_o and f, respectively.

From the measurements by Albrecht and by Franssila, discussed above, we can deduce[1]) mean values of r of 0,28 and of 0,375, respectively; these values would give apparent reductions of R_o and f of about 20 % and 27 %, respectively.

From (25), (26) and (27) it can be seen, that t_1 and z_1 are increased by this reduction, whereas u_o (and therefore also T_1) remains unaltered.

We may of course apply this modification to the calculation by Brunt (section 3) as well as to ours. In doing so, we easily arrive at the formula of Philipps, if only we use special suppositions, underlying his deduction. By neglecting radiative "conduction" of heat in the air, as he did, we obtain $\varkappa_a = \eta/\varrho_a = \lambda_A/\varrho_a c_a$ and thereby, according to (49):

$$r = \sqrt{\frac{\lambda_A \varrho_a c_a}{\lambda \varrho c}} = c_a \sqrt{\frac{\eta \varrho_a}{\lambda \varrho c}}. \tag{51}$$

By substituting $R/(1+r) = R\Big/\left(1 + \sqrt{\frac{\lambda_A \varrho_a c_a}{\lambda \varrho c}}\right)$ for R in Brunt's formula, we obtain Philipps' solution (section 2, case 2). In addition to what has already been said about his treatment of the problem, we may here emphasize, that his discarding the rôle of atmospheric radiation in the interchange of heat within the air is serious, as this rôle is an important one.

In order to apply the result of this section to any practical case, we must know the value of r to be used. This value will largely depend upon wind velocity. The stronger the wind, the larger the eddy conduction in the air and, consequently, the larger $A/B = r$.

Although we have seen that formula (49) had only a very limited validity, it may give an illustration of this dependence. If, for the moment, we write $\varkappa_a = \varkappa_A + \varkappa' = \eta/\varrho_a + \varkappa'$, (49) yields:

$$r = \frac{\eta c_a}{\sqrt{(\eta/\varrho_a + \varkappa')\lambda \varrho c}}.$$

According to this expression, r increases when the eddy conductivity increases; now we know, that η is roughly proportional to the wind velocity. If the latter would become so large, that $\varkappa'$ could be neglected compared with η/ϱ_a, we would again obtain the expression (51); r would then become roughly proportional to v.

We may try formula (51) by using for the various constants, occurring in it, data, quoted from Franssila's investigation, mentioned above: $\lambda = 0{,}123$ cal/cm min deg, $\varrho c = 0{,}59$ cal/cm³ deg, $c_a = c_p = 0{,}24$ cal/g deg, $\varrho_a = 0{,}001275$ g/cm³, $\eta = 0{,}2$ g/cm sec.

This yields: $r = 0{,}11$, whereas his direct measurements of the terms of the energy balance give: $r = 0{,}32$ (see section 11, part II of the present paper; we have here reckoned with a term λm in R and B, so that $r = A/(B + \lambda m) \neq A/B$; in the beginning

[1]) We have not reckoned with a term λm, here.

of the present section r was equal to A/B, so that a slightly different value of r was found).

We observe here first, that the above value of η is a rather uncertain element in the computation; and secondly, that (49) is not a general expression for r. Apart from the fact that the actual temperature interchange in the air does not obey such a simple law as expressed by equation (7a) — neither so far as the eddy conduction, nor, more in particular, so far as the radiative "conduction" in the vicinity of the earth's surface is concerned [1]) — the value of r will not only depend upon the conductivity of the air, but also, as we have already seen, upon the initial temperature distribution in it.

We shall not try to give an elaborate theory of this side of our problem. In those nights, in which there is danger of ground frosts [2]), wind velocity and eddy conductivity will be relatively small, so that the application of the quantity r has only the significance of a correction.

For that reason we shall, in general, be justified in restricting ourselves to an estimate of this quantity. In order to obtain an empirical basis for such an estimate, we need, however, a much more complete material of measurements of the energy balance at the earth's surface onder various circumstances, especially as to wind velocity (see section 11).

7. *Condensation.*

The problem of condensation at the earth's surface is even more difficult than that of eddy conduction. We shall, therefore, again start here by considering some empirical data, viz. those obtained by Franssila, l. c. In section 10, fig. 14, we haven given a graph of some of his measurements of the energy balance at the earth's surface (mean hourly values of 3 days of August). As we see there, the condensation term was about zero at the beginning of the night; it increased during the whole night, the rate of increase diminishing in the course of it.

We may, therefore, take this term to be more or less closely proportional to the temperature fall $T_0 - T$, which shows an similar progress. In order to test this, we have plotted the mean hourly values of the condensation term C, measured by Franssila during three nights (with clear or very slightly clouded sky and only light wind or calm), against the temperature at 1 cm above the ground (which was grown with short cut grass); see fig. 11, where C is expressed in cal/cm² min. In this way we obtain a set of points, which can be seen to lie indeed approximately on a straight line. We put, therefore,

$$C = f'(T_0 - T). \qquad (52)$$

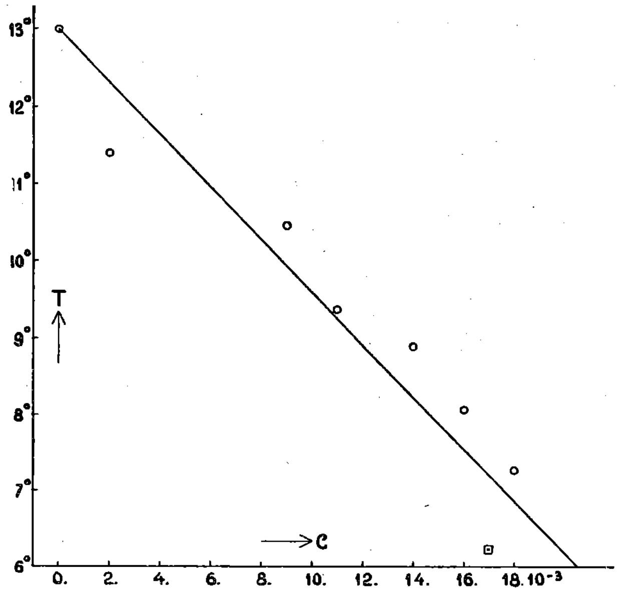

Fig. 11

The equation of the energy balance may now be written as follows:

$$A + B = R - C = R_0 - (f + f')(T_0 - T). \qquad (53)$$

From (53) we can see, that the modification brought about by the term C (52) amounts to the substitution of $f + f'$ for f in the formulas, previously used.

[1]) Compare Brunt, 1939, ch. VI; for the same reason the computation of the radiational heat transport within the air near the earth's surface, presented by J. Kampé de Fériet, La Météorologie, 1942, page 137—148, should not be accepted.

[2]) We confine ourselves to the seasons in which the occurrence of groundfrosts may imply an economical loss.

In the case of fig. 11 the quantity $f' = -\triangle C/\triangle T$ is about 0,175 cal/cm²h deg, whereas f was 0,30 cal/cm² h deg in those nights.

We can try to set up a theory of a simplified model, in order to test the above working hypothesis. To this end we suppose for the moment the water vapour distribution in the air for $t = 0$ to be such that the specific humidity q in the lower layers of the atmosphere is independent of z:

$$q(z)_{t=o} = q_o. \tag{54}$$

As regards the vertical water transport, we suppose, that it can be computed by means of an equation of the type:

$$\frac{\partial q}{\partial t} = k\,\frac{\partial^2 q}{\partial z^2}. \tag{55}$$

Finally, let, for $z = 0$, q be $q_{\max}(T)$, which is a known function of temperature, when the atmospheric pressure is given. If T at the earth's surface is therefore given as a function of the time, q for $z = 0$ is also known as a function of that quantity:

$$q_{z=o} = g(t), \quad g(o) = q_o. \tag{56}$$

The solution of (55) satisfying (54) and (56) is known (see f. i. Webster-Szegö, 1930, page 201):

$$q\,(z, t) = \frac{2}{\sqrt{\pi}}\, q_o \int\limits_o^{z/2\sqrt{kt}} e^{-\xi^2}\, d\xi + \frac{2}{\sqrt{\pi}} \int\limits_{z/2\sqrt{kt}}^{\infty} g\left(t - \frac{z^2}{4k\,\xi^2}\right) e^{-\xi^2}\, d\xi. \tag{57}$$

The water vapour stream, which we are concerned with, is determined by the derivative of $q\,(z, t)$ with respect to z:

$$\frac{\partial q}{\partial z} = -\frac{2}{\sqrt{\pi}} \int\limits_{z/2\sqrt{kt}}^{\infty} \frac{z}{2k\,\xi^2}\, g'\left(t - \frac{z^2}{4k\,\xi^2}\right) e^{-\xi^2}\, d\xi = -\frac{1}{\sqrt{\pi k}} \int\limits_o^t \frac{g'(\zeta)}{\sqrt{t-\zeta}}\, e^{-\frac{z^2}{4k(t-\zeta)}}\, d\zeta,$$

$g'\,(t)$ being the first derivative of $g\,(t)$. For $z = 0$ we obtain

$$\left(\frac{\partial q}{\partial z}\right)_{z=o} = -\frac{1}{\sqrt{\pi k}} \int\limits_o^t \frac{g'(\zeta)}{\sqrt{t-\zeta}}\, d\zeta. \tag{58}$$

Now we can write

$$g'(t) = \frac{dq_{\max}}{dT}\,\frac{\partial T}{\partial t}. \tag{59}$$

As a first approximation, we shall take for $T(t)$ Brunt's solution, corrected, however, in the sense of section 5, B, above (in order to prevent that $\partial T/\partial t$ should become infinite for $t = 0$); so that we write:

$$T_o - T = s(\sqrt{t + t_o} - \sqrt{t_o}), \tag{60}$$

$$\frac{\partial T}{\partial t} = -\frac{s}{2\sqrt{t + t_o}}. \tag{61}$$

Substituting (59) and (61) in (58), we obtain:

$$\left(\frac{\partial q}{\partial z}\right)_{z=o} = \frac{s}{2\sqrt{\pi k}} \int\limits_o^t \left(\frac{dq_{\max}}{dT}\right)_{t=\zeta} \frac{d\zeta}{\sqrt{(\zeta + t_o)\,(t-\zeta)}} = \frac{s}{2\sqrt{\pi k}}\, \overline{\left(\frac{dq_{\max}}{dT}\right)} \int\limits_o^t \frac{d\zeta}{\sqrt{\left(\frac{t+t_o}{2}\right)^2 - \left(\zeta - \frac{t-t_o}{2}\right)^2}} =$$

$$= \frac{s}{2\sqrt{\pi k}}\, \overline{\left(\frac{dq_{\max}}{dT}\right)} \left(\frac{\pi}{2} - \arcsin\frac{t_o - t}{t_o + t}\right). \tag{62}$$

Here $\overline{\left(\frac{dq_{\max}}{dT}\right)}$ is the value of $\frac{dq_{\max}}{dT}$ for a certain definite value of T between T_o

and $T(t)$; this value of T decreases, when t increases, and, as $\frac{dq_{\max}}{dT}$ decreases with temperature, the factor $\left(\frac{dq_{\max}}{dT}\right)$ is not independent of time; it varies rather slowly, however, and we may therefore conclude $\left(\frac{\partial q}{\partial z}\right)_{z=o}$ to be approximately proportional to the function $\frac{\pi}{2}$ — arc sin $\frac{t_o - t}{t_o + t}$, of which a graph is given in fig. 12. For $t = 0$ it is zero, its first derivative being here ∞; for $t > 0$ it increases monotonously; for $t \rightarrow \infty$ it approaches asymptotically the value π. As C is proportional to $\eta \left(\frac{\partial q}{\partial z}\right)_{z=o}$, or, in other words,

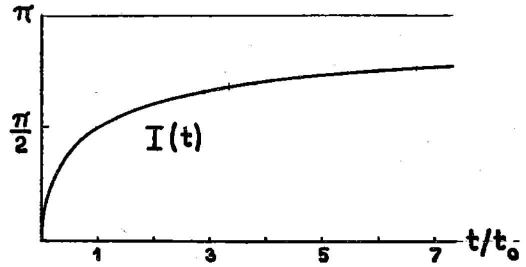

Fig. 12

$$C = l\eta \left(\frac{\partial q}{\partial z}\right)_{z=o}, \tag{63}$$

fig. 12 presents also a picture of the relative variation of C with time, computed for our specialized model. This picture is qualitatively such as we described above. An *exact* proportionality to $|\triangle T|$ could not be expected, of course. We may, however, define a "proportionality factor" f' as a mean value for a certain interval of time (t_1):

$$f' = t_1^{-1} \int_0^{t_1} \frac{C}{T_o - T} dt,$$

where $T_o - T$ and C are given by (60), (63) and (62).

This expression for f' is dependent on t_0; for convenience, in order to get rid of this quantity, we put it equal to zero (the dependence meant is only a slight one) and obtain in this way:

$$f' = l \sqrt{\frac{\pi \eta \varrho_a}{t}} \left(\frac{dq_{\max}}{dT}\right), \tag{64}$$

where l denotes the latent heat of condensation per gram of water; the constant k in (62) and (55) has here been put equal to η/ϱ_a.

The deduction just given starts from a very special initial water vapour distribution (54). Is is easy, however, to obtain a somewhat more general case by adding a term nz to $q(z, t)$, where n is a certain constant (pos. or neg.). This increases C (63) by a constant term $l\varrho_a\eta n = C_o$ (pos. or neg.). In accordance herewith we can put, as our semi-empirical "Ansatz", instead of (52),

$$C = C_o + f'(T_o - T), \tag{65}$$

so that the energy balance now yields:

$$A + B = R_o - C_o + (f + f')(T_o - T). \tag{66}$$

If we apply formula (64) to the case mentioned above, which was studied experimentally by F r a n s s i l a (l. c.), we have to substitute the following values for the constants: $\left(\frac{dq_{\max}}{dT}\right) = 0{,}45 \cdot 10^{-3}$/deg, $t = 7$ h, $\eta = 0{,}2$ g/cm sec, $\varrho_a = 1{,}275 \cdot 10^{-3}$ g/cm^3; then (64) yields: $f' = 0{,}2$ cal/cm^2 h deg, whereas the direct measurements (fig. 11) gave: $f' = 0{,}175$ cal/cm^2 h deg, as we have seen. The agreement is satisfactory.

The replacement of f by $f + f'$ implies an enlargement of the coefficient of $T_o - T$ in the expression for $A + B$; this effect is counteracted by the drying of the air near the ground, caused by the downward transport of water, for the latter process results in a relative increase of the effective radiation of the earth's surface, that is to say in a diminution of its decrease during the night.

Finally we may mention two complicating effects, which occur when the freezing point is attained. First, an isothermal stadium must elapse before the temperature will drop further — at least when the soil is wet. Secondly, from that moment we are concerned with the *heat of sublimation,* set free while the deposition of water goes on, instead of the heat of condensation (the former being larger than the latter).

8. Complications.

In this section we shall devote a few words to two possible complications, which will have to be the object of further empirical and theoretical investigation.

(A) *Layered soil, vegetation.*

In the foregoing sections we have constantly assumed homogeneity of the soil, or at least an inhomogeneity not so large as to exclude the use of mean values for the various constants, entering into our computations.

If, at a certain depth, the inhomogeneity becomes so strong, that we should speak of a layered soil, it is no longer permitted to use such mean values. Now, all depends upon whether the upper layer is deeper than the range of the nocturnal cooling, or not. In the former case we need not take it into account. In the latter case, however, the problem of calculating this cooling becomes essentially more difficult.

If we confine ourselves for the moment to the case of *two* layers, each of which may be treated as homogeneous, we have *two* equations of the type (7), with *different* values of $\varkappa$, which we call $\varkappa_I$ and $\varkappa_{II}$, the suffix I applying to the upper layer, the suffix II to the lower one. We then have to find a solution $T_I(z, t)$ of (7_I) and a solution $T_{II}(z, t)$ of (7_{II}) such, that

$$T_I(z_I, t) = T_{II}(z_I, t), \quad \lambda_I \frac{\partial T_I}{\partial z}(z_I, t) = \lambda_{II} \frac{\partial T_{II}}{\partial z}(z_I, t),$$

where $-z_I$ is the depth of the upper layer and λ_I, λ_{II} are the two heat conduction coefficients for this case. Apart from these two internal boundary conditions $T_I(z, t)$ should still have to satisfy the boundary condition for $z = 0$,

$$\lambda_I \frac{\partial T}{\partial z}(o, t) = R - A - C.$$

We shall not enter further into this mathematical problem[1]). As we have already seen, we are only concerned with the complication in question, when $-z_I$ is larger than the range of the nocturnal cooling. The latter is of course not defined exactly; we may take it to be of the order of $2\sqrt{\varkappa t}$.

When the ground is not bare, but covered by a vegetation, we may often treat the latter as the upper layer of the soil; the level $z = 0$ must then be located at its upper boundary level and the foregoing discussion of a layered soil may be applied to the whole of soil + vegetation. For this purpose we shall need sufficient information about the thermic properties of such layers of vegetation.

(B) *Fog.*

The following is meant as an *outline* of a theoretical treatment of the problem of fog; its working-out asks for still further study. Although this problem is touched upon here in connection with ground frosts, its importance covers a much wider field and its working out is sure to be worth while.

In the preceding section we have calculated the function $q(z, t)$ for a certain simplified "model" of water transport, characterized by equation (55), from a given initial water vapour distribution (54) and the boundary condition (56). We imagine

[1]) Another possible way of treating the whole problem may be found in Richardson's "layer method" (Richardson, l. c.).

that, in principle, this might quite generally be done, even if we replaced equation (55) by a better one, viz.

$$\frac{\partial q}{\partial t} = \frac{1}{\varrho_a} \frac{\partial}{\partial z} \left(\eta \frac{\partial q}{\partial z} \right).$$

Analogously we have for the air temperature T_a an equation determining the function $T_a(z, t)$, if an initial temperature distribution and a boundary condition at $z = 0$ are given; we have already discussed an example of this in section 6.

The functions $q\,(z, t)$ and $T_a(z, t)$ are not independent of each other; as we have seen, $q\,(o, t)$ is determined by $T_a(o, t)$ by the relation

$$q\,(o, t) = q_{\max}\left[T_a(o, t)\right].$$

On the other hand, $T_a(z, t)$ is influenced by $q\,(z, t)$, as in the energy balance at $z = 0$ the term C, determined by $q\,(z, t)$, plays a rôle. As a first approximation, however, we might neglect the latter dependence.

Now the function $T_a(z, t)$ determines a function $q_{\max}\,(z, t)$ not only for $z = 0$, because for a given temperature and pressure a maximum exists of the specific humidity, beyond which condensation sets in spontaneously. This maximum needs not coincide with a relative humidity of 100 %, because it is a known fact, that spontaneous condensation in the atmosphere often needs a relative humidity of over 100 %, dependent on the dust contents of the air (condensation nuclei). If the latter are known (qualitatively and quantitatively) we can imagine the function $q_{\max}\,(z, t)$ to be derived from the function $T_a(z, t)$.

At the ground the value $q_{\max}\,(o, t)$, where condensation (dewing) sets in, will in general be the value belonging to a relative humidity of 100 %, and (when the ground is wet) will coincide with the actual value $q\,(o, t)$ of the specific humidity.

For a set of values of t we may make graphs of the functions $q\,(z, t)$ and $q_{\max}\,(z, t)$ in a q-z-diagram and so obtain two sets of curves, as shown in fig. 13 (full lines $= q$, dashed lines $= q_{\max}$). Now, if from a certain moment the two corresponding curves q and $q_{\max}$ would intersect at still another point than at $z = 0$, the formation of fog will, from that moment, set in (from this same moment the curves q and $q_{\max}$, calculated *without* taking the condensation into account, will begin to deviate more and more from the actual distributions, especially as to their lower parts).

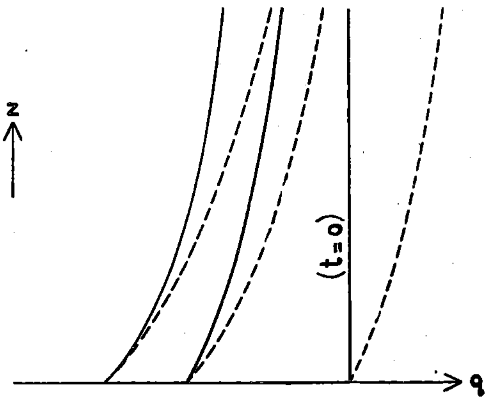

Fig. 13

It will depend upon this moment, whether a ground frost, to be expected if no fog were present during the whole night, is now prevented or not.

9. Summary of results.

On account of the results of the foregoing sections, we may now write down the following general formula to be used in computing the temperature fall at the ground during the night:

$$T(t) - T_o = u_o\{\varphi(t + t_o) - \varphi(t_o)\}, \quad \varphi(x) = e^{\frac{x}{t_1}} \left[1 - \Phi\left(\sqrt{\frac{x}{t_1}}\right)\right]. \tag{67}$$

Here t_o is the "rest time" defined in section 5 *B* — if it is put equal to zero we have $\varphi\,(t_o) = \varphi\,(o) = 1$ and (67) becomes formally the same as (39) —; $T_o = T\,(o)$ is the ground temperature at the beginning of the night;

$$u_o = \frac{R_o + \lambda m + C_o}{f + f'}, \quad t_1 = \left(\frac{1+r}{f+f'}\right)^2 \varrho c \lambda, \tag{68}$$

where the constants R_o and f are defined in section 4, m is defined in section 5 A, r in section 6, C_o and f' in section 7; ϱ, c and λ are the soil contstants introduced in section 1.

A short discussion of the ways for a *determination* of these various constants will be given in section 11 (Part II).

A graph of φ as a function of $\sqrt{\frac{x}{t_1}}$ is found in fig. 7 (section 4). From this graph the temperature fall may immediately be determined, if only u_o, as unit of temperature, t_1, as unit of time, and t_o are fixed.

PART II PRACTICAL PART

The present paper does not pretend to give an exhaustive treatment of the problem of ground frost. Some special cases are either left out of consideration (sloping land) or treated only superficiously (e. g. layered soil).

Apart from this, certain details of the general problem are still waiting for a thorough investigation; this applies especially to the prediction of T_o, the temperature at the beginning of the night; on the other hand, at the very end of the night, when insolation sets in, the temperature curve will begin to deviate from the computed curve, and the minimum will often not coincide with the astronomical sunrise (so that the value of t we should use may not always be the astronomical duration of the night).

Finally our study is rather theoretical. It should be completed by experimental and practical investigations to make it fully efficient.

In the sections 11 and 12 of the present part we shall make a *beginning* of such a practical treatment by giving some directions for the application of the results of Part I in the practice of forecasting.

10. Numerical example.

We have applied the result of Part I (section 9) to a case which was experimentally studied by Franssila l. c.; to this end we have chosen a night of August (6—7 Aug. 1934) of which Franssila gives mean hourly values of the terms of the energy balance at the ground (a short cut grass meadow) and of the temperatures at various heights (table 23 of his paper). The variation of the terms of the energy balance is illustrated by fig. 14, which is a reproduction of fig. 17 of the cited paper.

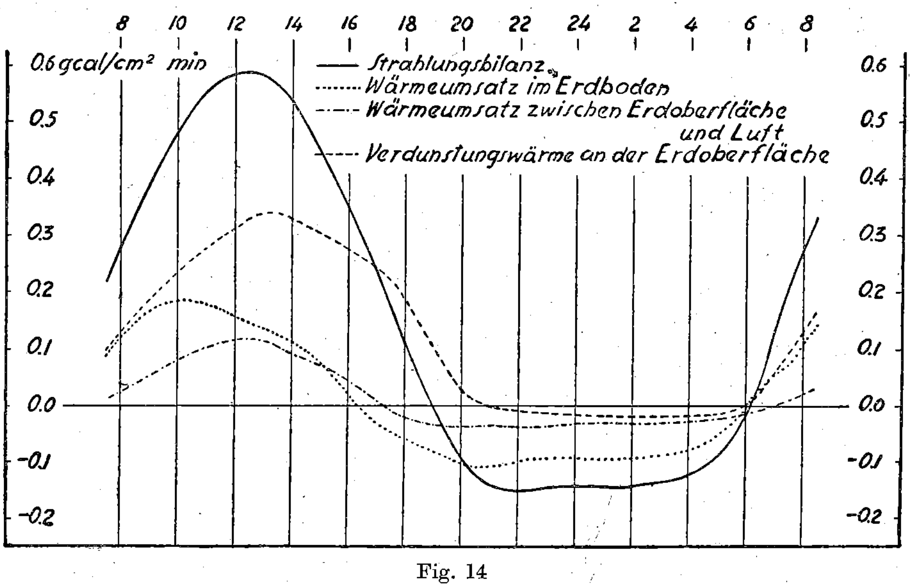

Fig. 14

Most of the constants used for the evaluation of formula (67) have been borrowed or directly derived from Franssila's empirical data. The only exceptions are the constants t_o and m, as Franssila gives no temperatures *in* the ground ($z < o$). For this reason we have put $t_o = o$ and have taken a value of m derived from measurements by Peerlkamp l. c., performed in the same month of the year (August) for clay, overgrown with short cut grass [1]):

$$m = 1^\circ/8 \text{ cm.}$$

1) It may be emphasized here, that *all* constants have been fixed *a priori*; we have *not* chosen them in such a way that the computed curve would fit in best with the facts.

The other constants are:

$R_0 = 9{,}0$ cal/cm²h. $f' = 0{,}15$ cal/cm²h deg. $\varrho c = 0{,}59$ cal/cm³ deg.
$f = 0{,}3$ cal/cm²h deg. $r = 0{,}32$. $\lambda = 7{,}4$ cal/cm h deg.

The two fundamental quantities of our formula, derived from these constants, are now:

$$u_0 = 22°, \quad t_1 = 37\text{h}.$$

By substituting these values in (67) we get our computed temperature curve.

In order to compare it with F r a n s s i l a's measurements, we have derived from this curve mean hourly values of the temperature and have plotted them against t. The result is seen in fig. 15; for the first half of an hour we had to perform an extrapolation of this mean hourly temperature curve (dashed part of the curve). The empirical mean hourly values of the temperature at 1 cm above the ground (the grass was cut short) are marked by small circles. The initial temperature of the computed curve was put equal to the value found at 20—21 h (the astronomical sunset being at about 20.45).

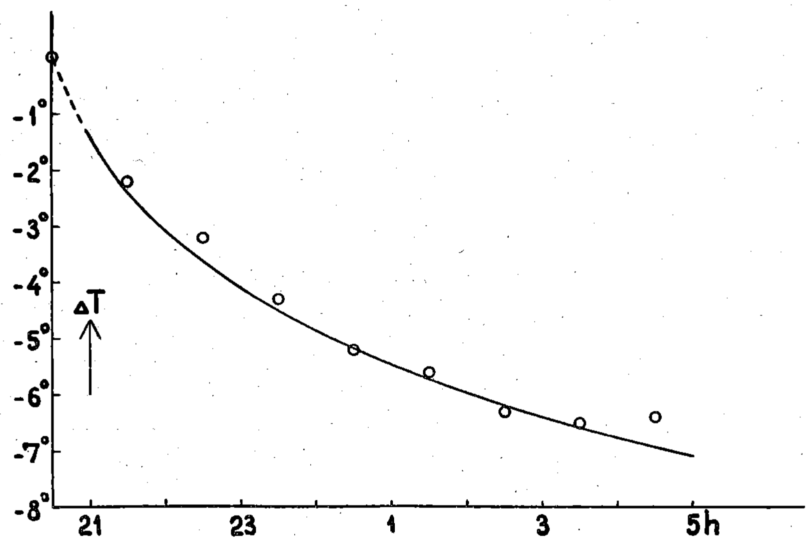

Fig. 15

The agreement is rather good, expecially later on in the night; in the beginning, the computed temperature fall is somewhat too steep [1]), in accordance with the fact, that we have used a linear initial temperature distribution in the ground (fig. 9, *b*; see section 5A). At about 4 o'clock in the morning the deviation resulting from the beginning insolation sets in (the astronomical sunrise was at about 4.10).

It is clear, that the agreement of our computation with the measured temperature variation gives only a demonstration of the purely *physical* value of our formula (67). For its use in practice the meteorologist has even to *predict* some of the fundamental constants, which for the case of this section had been measured.

11. *Determination of constants.*

It is impossible to compute for each piece of land, in its own special circumstances, the temperature variation to be expected. On the other hand, to give only *one* prediction for a whole district will not suffice, in general. The meteorologist may therefore adopt a middle course by giving a small number of „standard" forecasts, relating to certain definite types of soil (including the degree of wetness), both for the open field and for wind sheltered places.

In order to apply formula (67) he should for each type know the various relevant constants. We shall shortly discuss all of them here.

$\mathbf{T_0}$. The ground temperature at the beginning of the night is strongly influenced by local circumstances and local past weather (showers). It will therefore be very useful, that this quantity be measured by the agriculturist, who wants to apply a forecast (the

[1]) This difference would have been larger, if actual values of the temperature had been plotted, instead of mean hourly values.

practical question of the *application* of the forecast will be dealt with in the next section). Nevertheless the meteorologist for his part has still to make himself an estimate of T_o, in order to predict the nocturnal minimum — not only for those agriculturists, who do not measure T_o themselves, but also in order to be able to give his warning in good time; moreover he needs T_o for his prediction of R (see below).

We have already said, that this problem has not yet been studied sufficiently. Further empirical material should be gathered and studied theoretically (compare the treatment by Brunt, discussed in section 5 C; this treatment should be more closely adapted to physical reality, however, by taking into account the eddy conduction of heat in the air).

Constants referring to radiation.

R_o. Two cases should be distinguished:

1. Clear sky.
2. Partially or wholly clouded sky.

1. For a clear sky R_o is determined with the aid of the radiation chart. We shall give a very short explanation of the use of it[1]). (For simplicity the rôle of the CO_2 in atmospheric radiation will not be considered; in reality it is not neglected in modern radiation charts).

In the radiation chart a curve of state of the "water atmosphere" is plotted, so to say. Any point of the atmosphere is characterized by (1) the mass of water pro cm² below it, denoted by w, (2) the temperature T at the point considered. Now in the radiation chart a w-scale is found in the abscis-direction. This w-scale is constructed in a very special manner we shall not enter upon here; we may only remark, that for small w this scale is much more stretched than for large w, in such a way that the line $w = \infty$ is found at a finite distance from the line $w = 0$. Besides, a set of curves $T = \text{const}$ has been drawn on the diagram; $T = 0°$ (abs.) is a straight line, forming the base line of the diagram, all other isotherms are curved lines. *The area contained between an isotherm $T = T_o$ and the base line is proportional to σT_o^4, the total radiation pro cm² of a black body having a temperature T_o*, thereby constituting a measure of this radiation. Let T_o be the temperature of the earth's surface.

When we have plotted the points (T, w) of the "water atmosphere", the area beneath the curve, obtained by joining them, is a measure of the incoming atmospheric radiation at the earth's surface.

In this way the effective radiation R, which is the difference between outgoing (terrestrial) and incoming (atmospheric) radiation, is represented by the difference between these two areas. When the curve of state of the water atmosphere, expected at the beginning of the night, is plotted in this way, we find R_o (see fig. 16*a*, shaded area).

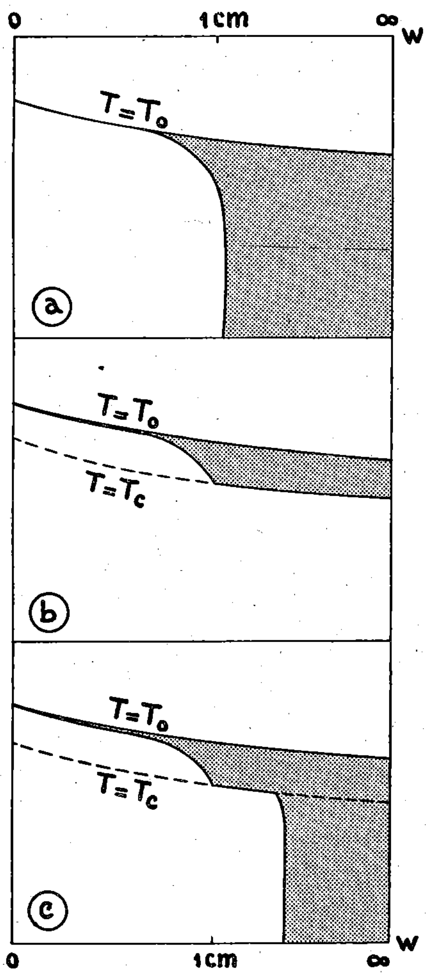

Fig. 16

2. When the sky is partially or wholly clouded, we have an incoming radiation R_N', which, in general, may be represented by the formula

$$R_N' = \frac{10-N}{10} R' + \frac{N}{10} R'_{10},$$

[1]) We have used the Möller form of radiation chart.

already written down in section 1. Now, if the clouds are considered to be black radiators, R'_{10} can easily be seen to be represented on the radiation chart by the area, contained between the lines $T = T_o$, $T = T_c$, $w = \infty$ and the "curve of state" of the water atmosphere, where T_c is the temperature at the cloud base (see fig. 16*b*, shaded area). When the clouds are more our less transparant, however, the matter is less simple. For computing R'_{10} they may then, in principle, be put equivalent to a certain finite mass of water vapour pro cm² and thus be intercalated on the radiation chart between the water beneath them and the water above them (fig. 16*c*).

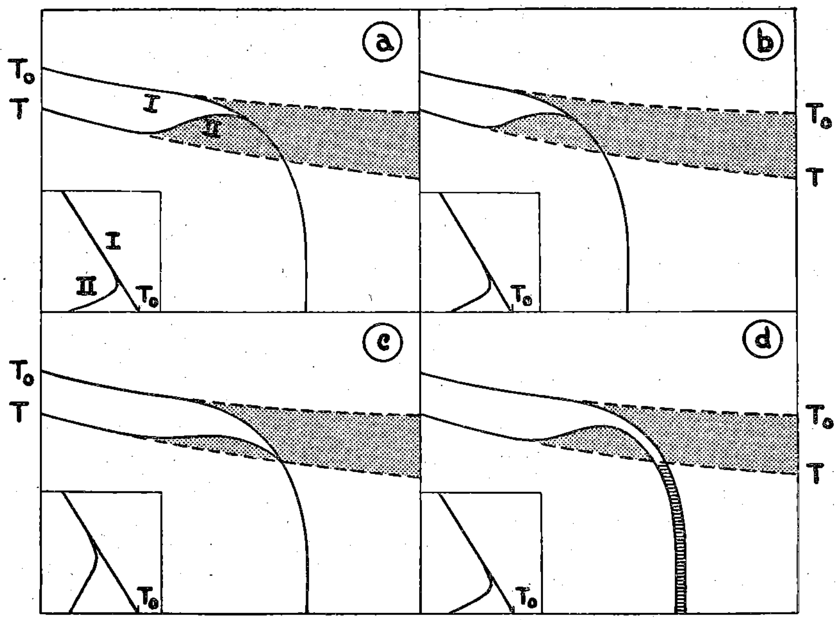

Fig. 17

It is clear, that N and R'_{10} should refer to a certain mean cloudiness during the night. (The variation of R, taken into account for our computation of the temperature fall, is the variation caused by the cooling of the ground and of the atmosphere in its vicinity. When we foresee a gradual variation of R_N' during the night, we might, of course, attempt to take this into account by means of the coefficient f . . . . . .). We have already seen, however, that in the season, in which the prediction of groundfrosts is of economical importance, the cloudiness should be small throughout the night or else no groundgrost will have to be feared, in general.

f. Fig. 17*a* shows a curve of state of the water atmosphere (in the following we shall, for brevity speak of "curve of state") at the beginning of the night (curve I) and another one (curve II) representing the situation after a certain time, when at the ground the temperature has fallen from T_o tot $T = T_o - \triangle T$.

It is easily seen, that the effective radiation R has decreased by an amount $\triangle R$, represented by the shaded area. Part of the curve of state II runs above the isotherm of the ground temperature T; this part corresponds to the strong ground inversion, here present. The inset represents very schematically the ordinary temperature-height-diagrams corresponding to these two atmospheric situations.

In fig. 17*b* (and also in fig. 17*c* and *d*) T_o and T are the same as in fig. 17*a*; the inversion that is established, may have the same form as in the preceding case, but R_o is smaller. As we see, $\triangle R$ (shaded area) is larger, here, than in case *a*.

In fig. 17*c* the initial atmospheric radiation and terrestrial radiation are the same as in case *a*; here, however, the inversion is supposed to be much less sharp than in the two preceding cases (see inset). It is seen that now $\triangle R$ is smaller than in case *a*.

From the foregoing we can conclude, that, $\triangle T$ being the same, $\triangle R$ and therefore also f depend upon two main factors, viz.:

(1) the water contents of the atmosphere, characterized by R_o,

(2) the sharpness of the ground-inversion formed during the night;

these dependences being such as to make f the larger, the smaller R_o and the sharper the ground-inversion.

The latter factor depends upon the eddy conduction (which is principally determined by the *wind velocity*) and the radiative "conduction", on the one side, and, on the other side, upon the thermal properties of the soil, which may be characterized by the quantity $\lambda\varrho c$ (if the same $\triangle T$ is reached in a shorter time, owing to a smaller value of $\lambda\varrho c$ — compare (36) —, the ground inversion will be sharper).

For the present, therefore, we may use as the three main determining quantities R_o, the *wind velocity* and $\lambda\varrho c$. For practical purposes it would be usefull to construct tables or graphs, each giving, for a certain value of $\lambda\varrho c$, f as a function of R_o and the wind velocity. Such tables or graphs should be based either upon empirical data [1]) or upon formulas (35), (37) and (39), the latter being, however, only a provisional one.

One might point out as a third factor influencing the variation of R during the night, the drying of the air by condensation at the earth's surface, as has already been mentioned in section 7. The effect of this process can be seen in fig. 17*d*; curve I is here the same as in fig. 17*a*, curve II, however, meets this time the w-axis in a different point; $\triangle R$ is represented by the shaded area *minus* the hatched area and is therefore smaller than in fig. 17*a*. Now the downward stream of water vapour depends upon the water contents, present in the air, and upon the eddy conductivity, which in its turn depends upon wind velocity. These selfsame elements, however, influence also factors (1) and (2), already discussed, so that this third factor is automatically accounted for, if the tables mentioned are empirically constructed.

Constants, referring to the air.

r. From the expositions of section 6 it is clear, that r depends upon the eddy conductivity (*wind velocity*) and the radiative "conductivity" of the air and upon the quantity $\lambda\varrho c$, referring to the soil; compare formula (49). From the deduction and discussion of this formula, however, we have seen, that the initial temperature distribution also has an influence upon the ratio A/B. This question should be theoretically investigated further. It may be expected that r will be proportional to $1/\sqrt{\lambda\varrho c}$, so that we may write:

$$r = r'/\sqrt{\lambda\varrho c}, \tag{69}$$

where r' depends chiefly upon wind velocity and upon the humidity of the air (upon which depends the radiative heat "conduction"). To begin with, therefore, the meteorologist might use an *empirically* constructed table (or a set of graphs), giving r' as a function of wind velocity and specific humidity. For each soil type r might then be computed from (69).

f'. For the present, this quantity might be computed from formula (64). To this purpose we must know the eddy conductivity η (a mean value for the whole night) in its dependence upon *wind velocity*. On the other hand, it will again be desirable to gather and work up more empirical material, in order to make it possible to estimate f' in any practical case.

From the deduction and discussion of (64) it appears that the initial distribution of water vapour also has an influence upon the variation of the condensation heat stream; it will not be easy, however, to take it into account for the determination of f' (with respect to C_o it plays the principal rôle, of course, but this is an other matter).

C_o. For practical use, C_o may be put equal to

$$C_o = l\varrho_a\eta \frac{q_{200} - q_{\max}(T_o)}{200 \text{ cm}},$$

where q_{200} and $q_{\max}(T_o)$ are the specific humidity at a height of 200 cm at the beginning of the night and the maximum specific humidity belonging to a temperature T_o, respectively; see section 7.

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[1]) Directly, by way of radiation measurements, or indirectly, via calculations with the radiation chart, based upon micro-aerological research.

C o n s t a n t s r e f e r r i n g t o t h e s o i l.

λ, ϱ, c: These constants should be know from measurements, for various soil types and degrees of wetness (past weather!).

m. More empirical material should be gathered. We need very systematical measurements of soil temperatures at various depths for all hours of the day, throughout the year, for various soil types and for climatologically different regions.

We may theoretically expect the part of λm depending upon annual variation of temperature — we shall call it λm_1 — to be roughly proportional to $\sqrt{\varrho c \lambda}$, as can easily be seen.

Indeed, let us represent the annual variation of the 24 hours mean temperature at $z = 0$ by $T = a \sin \omega t$, where $\omega = 2\pi/365$ days^{-1}. Owing to the smoothing effect of the air on surface temperatures (as far as long-periodical variations are concerned), the amplitude will be nearly the same for different types of soil (if not too wet), provided the climatological conditions are the same.

Now the corresponding temperature wave in the ground obeying equation (7) is described by

$$T = a\, e^{\mu z} \sin (\omega t + \mu z),$$

where

$$\mu = \sqrt{\frac{\omega}{2\varkappa}} = \sqrt{\frac{\omega \varrho c}{2\lambda}}.$$

Differentiating this with respect to z, we find

$$\frac{\partial T}{\partial z} = a\mu\sqrt{2}\,.\, e^{\mu z} \sin \left(\omega t + \mu z + \frac{\pi}{4}\right).$$

The mean temperature gradients in the soil will, therefore, be proportional to μ, or to $\sqrt{\varrho c/\lambda}$, and λm_1 turns out to be roughly proportional to $\sqrt{\varrho c \lambda}$.

The other part of λm, depending upon daily variation of temperature (section 5 A, part 2) is more difficult to treat theoretically. In this connection we may once more mention B r u n t's harmonical analysis, discussed in section 5 C (this treatment should be completed, however, by taking into account eddy conduction) and R i c h a r d s o n's layer method (R i c h a r d s o n, l. c.).

Apart from any theoretical treatment of the matter, the practical meteorologist may use for his calculations seasonal "normal" values of λm for each soil type, determined for various climatological and synoptical circumstances.

It should be noticed, besides, that λm is coupled to t_0, so that a different value of λm should be used if t_0 is chosen different from *zero* (section 5 B).

t_0. The above remarks, referring to λm, apply also to the quantity t_0, except for the part of λm depending upon the annual variation of temperature. The conclusion is also the same. The coupling of λm to t_0 makes the choice of t_0 less important than that of the other constants (to begin with, therefore, one might simply put $t_0 = 0$).

12. Practical application of forecasts.

Any agriculturist, who wishes to use the forecasts, will *first* have to know, which of the "standard" forecasts he must choose for his piece of land, considering the type of soil, degree of wetness and wind. The wetness will depend upon local past weather; as to the factor "wind", the meteorologist can only forecast wind direction and wind velocity in the open field, whereas the "user" of the forecasts knows, whether his land is sheltered from the wind or not, if wind direction is given.

Secondly, however, he might moreover apply certain reductions to this standard forecast in so far as the special conditions of his land deviate from the "standard" conditions. These special conditions depend upon (*a*) local past weather, (*b*) soil, (*c*) external conditions.

(*a*) *Local past weather* influences (1) the wetness of the soil and, consequently, the various soil constants, (2) the temperature T_o; for (1), the meteorologist might give forecasts for two or three degrees of wetness, from which a choice could be made — more the user can hardly do; as regards (2), however, he might *measure* T_o *himself*, and so eventually correct the nocturnal minimum, predicted by the forecaster. For this purpose it will be necessary to mention not only the predicted minimum in the forecasts, but also the amount of the temperature fall during the night.

It stands to reason that the placing of a thermometer for measuring temperatures at the ground is a matter of great care, preferably supervised by some climatological service. On the other hand, such an inclusion of agriculture will be of great value for micro-climatological research.

(*b*) *Soil* constants, to be used for the evaluation of formula (67) are: $\varrho c\lambda$ and λm. Even when the soil is *dry*, these constants may, for a certain piece of land, differ from the values used by the meteorologist for the type of soil in question. To correct for a deviation of λm from the standard value will be difficult; in general, however, the influence of such a correction will not be very large. The influence of $\varrho c\lambda$ is much larger.

It can easily be seen, that, if t is not too large, the temperature fall $\triangle T$, computed from (67), is nearly proportional to $1/\sqrt{\varrho c\lambda}$ (if we used Brunt's formula in stead of (67), it would be exactly proportional to this quantity). The user of the forecast might, therefore, correct the predicted value of $\triangle T$ by multiplying it by a constant factor (in general not differing much from unity); this factor should be determined by an expert, the whole under supervision of the meteorological institute.

(*c*) *External conditions* may influence especially R — by screening part of the sky —, r — by diminishing the wind velocity on the spot — and $f + f'$ — also by diminishing the wind velocity and consequently influencing the sharpness of the nocturnal ground-inversion. The effect of a small variation of $f + f'$ may safely be neglected, as, if t is not too large, $\triangle T$ is nearly proportional to $u_o/\sqrt{t_1}$, u_o and $\sqrt{t_1}$ being both inversely proportional to $f + f'$.

For R and r, additional corrections must be applied to $\triangle T$ in the same manner as was exposed with respect to $\varrho c\lambda$. Indeed, it can easily be seen, that $\triangle T$ is nearly proportional to $1/(1 + r)$ and, if, compared with R_o, $m\lambda$ is not too large, to R_o. It should be borne in mind, however, that any local reduction of r will in general depend upon wind *direction* and will furthermore have a larger effect on the factor $1/(1 + r)$ when wind *velocity* is large than when it is small (when $r = 0$ there would be no effect at all).

In concluding, we may state, that, $\triangle T$ being nearly proportional to $\frac{R_o}{(1+r)\sqrt{\varrho c\lambda}}$, for each piece of land, apart from more accidental effects, caused by local past weather, a correction factor could be determined, which would be more or less characteristic of it, if wind conditions are given.

By this factor the predicted value of $\triangle T$ would have to be multiplied in order to obtain a better result.

Its determination, in its dependence on wind conditions, should in any case be performed by an expert (the whole again under supervision of the meteorological institute); this might be done either indirectly, by computing or measuring separately the local deviations, exhibited by the constants R_o, r and $\varrho c\lambda$, or directly by measuring ground temperatures on the spot.

Concluding remarks.

At first sight the expositions of the last section may seem to be of a rather academical sort. In reality, however, any more active participation of agriculture in micrometeorological research will be of the greatest value, both for micrometeorology and for agriculture, especially with respect to the problem of groundfrosts; besides it will be the only way towards a solution of this problem.

As to the practical use of formula (67) it should not be forgotten that several of the constants, occurring in it, only play the rôle of correctives, so that a rather rough estimate of such a constant will often suffice (compare the remark concerning t_0 at the end of section 11).

Nevertheless it is clear, that a lot of experimental work is still to be done, to make theoretical investigations like the present one fully efficient for practical use. As we have seen, however, it will be worth while. Summarizing, therefore, we may write down the following list of requirements in this respect:

1. Systematical information concerning the thermal properties of all types of soil, especially with respect to the quantity $\sqrt{\varrho c \lambda}$ (dependence on watercontents!).

2. Systematical and exhaustive investigation of annual and daily temperature variation *in* the ground, for all types of soil (temperature distribution for each hour of the day and for each month of the year).

3. Data about eddy conductivity in dependence on wind velocity; effect of eddy conductivity upon the sharpness of nocturnal ground inversions and upon the ratio of air heat stream A and soil heat stream B.

4. Further empirical information concerning the processes of condensation under various circumstances.

On the other hand, theoretical investigation will have to proceed in various directions, with a view to several questions we have already touched on shortly in Part I and in the introductory remark of Part II of the present paper.

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