Obliquely incident surface waves in shallow water of slowly varying depth

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The propagation properties of obliquely incident, weakly nonlinear surface waves in shallow water of varying depth are studied analytically. The depth changes slowly in a direction that makes a constant angle with the propagation direction of the incident wave, initially travelling in a region of uniform depth. In the adjacent inhomogeneous region, depth variations are relatively slow. On the other hand, it is assumed that these occur on a scale shorter than that on which the wave evolves. As a consequence, the problem can be reduced to an evolution equation with constant coefficients. Since weak three-dimensional effects are also taken into account, this equation is related to the KP equation (Kadomtsev & Petviashvili 1970). Based on these results, the mechanism of mass transfer is studied. In a subsequent analysis, devoted to the case of a normally incident wave, the problem of describing the leading-order mass balance is solved. A normally incident solitary wave breaks up into a finite number of separate solitons if certain specific conditions are satisfied, such as the condition that this wave enters a region of smaller depth. In the more general case of an obliquely incident solitary wave, it is shown that this phenomenon can also occur, although the conditions are different.

1 Introduction

In the present paper, the effects of varying depth on the development of a long, obliquely incident, weakly nonlinear surface wave in shallow water are investigated theoretically. The rate of change of the depth (specified below) is relatively large. The fluid motions are assumed to be inviscid, incompressible and irrotational. At the free surface, we assume that the pressure is constant.

To describe the configuration, we choose a fixed Cartesian coordinate system $Ox_*y_*z_*$, where the asterisk denotes a dimensional variable. The z_*-axis points vertically upwards, and the undisturbed free surface is located at $z_* = 0$. The depth varies in the x_*-direction, but is independent of y_*. We introduce the dimensionless coordinates $x = x_*/L$, $y = y_*/L$, $z = z_*/H$ and the dimensionless time $t = Vt_*/L$. Here L is a characteristic length of the surface wave, H is a characteristic depth, and $V = (\mathrm{g}H)^{1/2}$ is a reference speed, with g the gravitational acceleration. The surface elevation η_* is described by $\eta_* = a\eta$, where a is the wave amplitude. The depth is described by $z_* = -Hh(\zeta)$, with $\zeta = x_*/L_0$ (where L_0 is the scale of the depth variation.)

The relevant dimensionless parameters, given by

$$\epsilon = \frac{a}{H}, \quad \delta = \frac{H}{L}, \quad \mu = \frac{L}{L_0}, \tag{1.1}$$

are assumed to be small. Since the parameter ϵ is a measure of the effect of nonlinearity, this implies that the problem is weakly nonlinear. Accordingly, the potential ψ_* is scaled as $\psi_* = \epsilon LV\psi$. The parameter δ controls the effect of dispersion. In the case of a uniform depth, the problem can be reduced to the Korteweg-de Vries (KdV) equation if $\delta^2 = O(\epsilon)$, which expresses a leading-order balance of nonlinearity and dispersive effects; see e.g. Whitham (1974). In what follows, we take $\delta^2 = \epsilon$ for convenience, obtained by a suitable choice of the horizontal length scale, L. The parameter μ controls the degree of inhomogeneity, induced by the slowly varying depth $h = h(\mu x)$.

We consider the case of a relatively high degree of inhomogeneity, namely,

$$\epsilon \ll \mu \ll 1. \tag{1.2}$$

This order of magnitude of μ describes the situation in which the depth change occurs on a scale shorter than that associated with the spatial evolution of the wave.

Now assume a depth change $\triangle H$ in a region of length L_0, which corresponds to an average bed slope $\alpha = \triangle H/L_0$. Then the initial scaling (1.1), (1.2) implies that the α-values are limited to the range

$$(a/H)^{3/2}\triangle H/H \ll \alpha \ll (a/H)^{1/2}\triangle H/H, \tag{1.3}$$

where $\triangle H/H$ is the fractional change of depth.

For $\mu = \epsilon$ the problem can be reduced to a KdV equation with varying coefficients (Johnson 1973). At a lower degree of inhomogeneity ($\mu \ll \epsilon$), the scale on which the depth varies is much longer than the scale on which the wave evolves. This case has been studied extensively; see e.g. Peregrine (1967), Grimshaw(1970, 1971), Kakutani (1971), Leibovich & Randall (1973), Kaup & Newell (1978), Karpman & Maslow (1979), Candler & Johnson (1981). For the case of a relatively high degree of inhomogeneity ($\mu \gg \epsilon$) we refer to Benilov (1992), van Groesen & Pudjaprasetya (1993) and Johnson (1994). Benilov considers the case of periodic or random depth variations.

The majority of the previous studies is devoted to the case of a normally incident plane wave, i.e., the propagation direction of the wave is aligned with the direction of changing depth. For the more general case of an obliquely incident wave, the reader is referred to e.g. Ryrie & Peregrine (1982) and Peregrine & Ryrie (1983). The present paper is also devoted to the case of oblique incidence. The study includes an investigation of weak three-dimensional effects, which slightly distort the incident plane wave. At a sufficient degree of inhomogeneity, it is found that the problem can be reduced to an evolution equation with constant coefficients, which proves to be related to the so-called KP equation (Kadomtsev & Petviashvili 1970). A solitary wave, described by such an equation, varies slowly in a direction perpendicular to the direction of propagation. For a review the reader is referred to Akylas (1994).

In §2 the governing equations are derived, based on a degree of inhomogeneity given by (1.2). The actual evolution equation, which is related to the KP equation, describes the development of the uni-directional primary wave, initially propagating in a region of uniform depth. The adjacent region of variable depth is characterized by a transition to another region of uniform depth. We also study the case of formation of a caustic, and derive equations governing higher-order phenomena, induced by the primary wave as it enters the inhomogeneous region. The occurrence of reflection of the wave was already described by Peregrine (1967), starting from a set of linearized long-wave equations. In §3 a different approach is adopted, and extended to the case of an obliquely incident wave. This approach allows a detailed study of the problem of mass transfer from the primary waves to the right-going and reflected shelves. The results differ significantly from the case of a slower change of depth.

A normally incident solitary wave eventually breaks up into a finite number of solitons if this wave enters a region of rapidly decreasing depth; see Tappert & Zabusky (1971), Johnson (1973) and the review by Miles (1980). In §4 we describe the occurrence of this phenomenon, known as fission, in the case of an obliquely incident solitary wave. Due to the presence of weak three-dimensional effects, a detailed description of this phenomenon requires the construction of the N-soliton solution of the KP equation, which is known to exist (Satsuma 1976). Rather than follow Satsuma's original analysis, however, we prefer to show that this solution can be derived directly from the known N-soliton solution of the KdV equation by a simple transformation of the independent variables. It is shown that the conditions for fission also depend on the angle of incidence. Finally, in §5 the principal results are summarized.

2 The evolution equation for the primary wave

In section §1 we described the problem of a weakly nonlinear, obliquely incident surface wave (with amplitude ϵ) in shallow water of slowly varying depth $h = h(\mu x)$; the small parameters ϵ and μ are given by (1.1) and (1.2). All variables are dimensionless, and it is assumed that nonlinearity and dispersion are of equal importance, i.e. $\delta^2 = \epsilon$.

It is tempting to consider first the case of two-dimensional depth variations $h(x, y)$, with $h_x = O(1)$ and $h_y = O(1)$, i.e. the scales of these variations are of the same order of magnitude as the wavelength.

The Laplace equation

$$\epsilon \nabla^2 \psi + \psi_{zz} = 0, \tag{2.1}$$

with $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, has a solution of the form

$$\psi(x,y,z,t) = \phi(x,y,t) - \epsilon[(z+h)^2\nabla^2\phi/2 + (z+h)(\nabla h.\nabla\phi)] + o(\epsilon), \tag{2.2}$$

which satisfies the bottom boundary condition

$$\psi_z = -\epsilon(\nabla h.\nabla\psi), \qquad (z=-h). \tag{2.3}$$

Bernoulli's equation is given by

$$2\eta = -2\psi_t - \epsilon(\psi_x^2 + \psi_y^2) - \psi_z^2, \qquad (z=\epsilon\eta). \tag{2.4}$$

Substitution of (2.2) into (2.4), and differentiation with respect to t, yields

$$\eta_t = -\phi_{tt} + \epsilon[(h^2/2)\nabla^2\phi_{tt} - \phi_x\phi_{xt} - \phi_y\phi_{yt} + h(\nabla h.\nabla\phi)_{tt}] + o(\epsilon). \tag{2.5}$$

Substitution of (2.2) into the equation for mass conservation, given by

$$\eta_t + \frac{\partial}{\partial x}\int_{-h}^{\epsilon\eta}\psi_x \mathrm{d}z + \frac{\partial}{\partial y}\int_{-h}^{\epsilon\eta}\psi_y \mathrm{d}z = 0, \tag{2.6}$$

leads to the result

$$\eta_t = -(h\phi_x)_x - (h\phi_y)_y + \epsilon[(\phi_t\phi_x)_x + (\phi_t\phi_y)_y + \nabla^2(h^3\nabla^2\phi)/6 + \nabla^2(h^2(\nabla h.\nabla\phi))/2] + o(\epsilon), \tag{2.7}$$

where use is made of (2.4), which implies that $\eta = -\phi_t + o(1)$.

Elimination of η_t between (2.5) and (2.7) leads to the equation

$$\phi_{tt} - (h\phi_x)_x - (h\phi_y)_y + \epsilon[2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + \phi_t\nabla^2\phi + \nabla^2(h^3\nabla^2\phi)/6$$
$$-(h^2/2)\nabla^2\phi_{tt} + \nabla^2(h^2(\nabla h.\nabla\phi))/2 - h(\nabla h.\nabla\phi)_{tt}] = o(\epsilon), \tag{2.8}$$

expressed in terms of the potential. Equation (2.8) is a generalisation of a result derived previously by Newell (1985). This author considered the case $h_y = 0$, $h_x = O(\epsilon)$.

Next we revert to the case where the depth depends on x only, with $h_x \ll 1$. Using the relation $\phi_{tt} = h\nabla^2\phi + o(1)$, equation (2.8) reduces to

$$\phi_{tt} - (h\phi_x)_x - h\phi_{yy} + \epsilon[2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + \phi_t\nabla^2\phi - (1/3)h^3\nabla^4\phi] = o(\epsilon). \tag{2.9}$$

As a solution of equation (2.9) we consider an uni-directional primary wave, propagating to the right, i.e. in the direction of increasing x. Accordingly, we introduce the characteristic coordinate

$$\theta = \int_0^x k(\mu\chi)\mathrm{d}\chi + \ell y - t, \tag{2.10}$$

which corresponds to propagation in the direction of the vector (k, ℓ), with $k = k(\mu x)$, and $\ell=$ const. The relevant slow variables are given by

$$\xi = \epsilon^{1/2}(-\ell \int_0^x \frac{\mathrm{d}\chi}{k(\mu\chi)} + y), \qquad \zeta = \mu x, \qquad \vartheta = \epsilon \int_0^x (1/6k(\mu\chi))\mathrm{d}\chi. \tag{2.11}$$

The variable ξ is introduced to include weak three-dimensional effects, which slightly distort the wave. In fact, ξ is a slowly varying coordinate along lines of constant phase; cf. section 3.4.5 in Johnson(1997). The variables ζ and ϑ are multiple-scale variables in the x-direction.

Expressing the potential and the surface elevation in terms of the new variables θ, ζ, ξ, ϑ, the partial derivatives transform as follows

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial \theta} + \mu\frac{\partial}{\partial \zeta} - \epsilon^{1/2}(\ell/k)\frac{\partial}{\partial \xi} + (\epsilon/6k)\frac{\partial}{\partial \vartheta}, \qquad \frac{\partial}{\partial y} = \ell\frac{\partial}{\partial \theta} + \epsilon^{1/2}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \theta}.$$

Strictly speaking, the small parameter μ, specified by (1.2), cannot be considered as an independent parameter. As will be shown below (2.17), the appropriate scale of the depth variation in terms of ϵ is given by

$$\mu = \epsilon^{1/2}. \tag{2.12}$$

Accordingly, we introduce the asymptotic expansions

$$\phi = \phi_0 + \epsilon^{1/2}\phi_1 + \epsilon\phi_2 + o(\epsilon), \qquad \eta = \eta_0 + \epsilon^{1/2}\eta_1 + \epsilon\eta_2 + o(\epsilon). \tag{2.13}$$

Then Bernoulli's equation, given by $\eta = -\phi_t + O(\epsilon)$, implies that

$$\eta_0 = \phi_{0\theta}, \qquad \eta_1 = \phi_{1\theta}. \tag{2.14}$$

Substitution of (2.10)-(2.14) into (2.9) leads to a hierarchy of equations, obtained by equating terms of the same order of magnitude as $\epsilon \to 0$. Routine application of the method of multiple scales leads, to lowest order, to the equation

$$h(\ell^2 + k^2) = 1, \tag{2.15}$$

which determines $k(\zeta)$. Furthermore, (2.15) implies that the condition for the asbsence of a caustic is given by $h_m \ell^2 < 1$, where h_m denotes the maximum depth. When rewritten in terms of dimensional variables, (2.15) corresponds to $c^2 = \mathrm{g}h$, where c is the speed of the linear wave. The nonlinear correction to this speed (which includes the wave amplitude), is determined by the solution of the evolution equation for the surface elevation, derived below.

At higher order we obtain the equations

$$\frac{\partial}{\partial \zeta}(hk\eta_0^2) = 0, \tag{2.16}$$

and

$$(h/3)\eta_{0\vartheta} + (h/3)\eta_{0\theta\theta\theta} + (3/h)\eta_0\eta_{0\theta} + (1/k^2)\phi_{0\xi\xi} + 2(hk)^{1/2}\frac{\partial}{\partial \zeta}((hk)^{1/2}\eta_1)$$
$$+\frac{\partial}{\partial \zeta}(h\phi_{0\zeta}) - 2(h\ell/k)^{1/2}\frac{\partial}{\partial \zeta}((h\ell/k)^{1/2}\phi_{0\xi}) = 0, \tag{2.17}$$

derived by equating the terms proportional to $\epsilon^{1/2}$ and ϵ, respectively. Equation (2.16) corresponds to Green's law, extended to the case of oblique incidence. To derive equation (2.17), we also used (2.15).

The previous results were derived by introducing the scaling (2.12). To demonstrate that this is the appropriate one, we introduce the asymptotic expansions

$$\phi \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mu^m \sigma^n \phi_{mn}, \quad \eta \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mu^m \sigma^n \eta_{mn}, \quad \mu\sigma = \epsilon, \quad \mu, \sigma, \epsilon \to 0,$$

with μ given by (1.2). Choosing $\mu \ll \epsilon^{1/2}$ or $\mu \gg \epsilon^{1/2}$, the terms with $\phi_{0\zeta}$ and $\phi_{0\xi}$ in equation (2.17) drop out. Apparently, the maximum number of terms of the same order of magnitude is obtained by taking $\mu = \epsilon^{1/2}$. According to this "principle of the richest equation" (Kevorkian & Cole 1981), the scaling (2.12) seems to be the proper one if the restriction (1.2) is taken into account.

The equations (2.16) and (2.17) can be simplified considerably by writing

$$\eta_0 = (hk)^{-1/2} S_0. \tag{2.18}$$

Then equation (2.16) reduces to

$$S_{0\zeta} = 0, \tag{2.19}$$

which implies that $S_0 = S_0(\theta, \vartheta, \xi)$.

The evolution equation for the surface elevation is derived from equation (2.17). Differentiation of (2.17) with respect to θ, and substitution of the expressions (2.14) and (2.18) for η_0, leads to the equation

$$2\frac{\partial}{\partial\zeta}((hk)^{1/2}\eta_{1\theta}) = -(1/3k)(S_{0\vartheta} + S_{0\theta\theta\theta})_\theta - (3/h)(hk)^{-3/2}(S_0 S_{0\theta})_\theta - (1/hk^3)S_{0\xi\xi}$$

$$+\frac{\partial}{\partial\zeta}((\ell/k^2)S_{0\xi}) - (hk)^{-1/2}\frac{\partial}{\partial\zeta}(h\eta_{0\zeta}). \quad (2.20)$$

The depth is uniform in the region $x < 0$. In particular, we take

$$h = 1, \qquad k = k_0 = \text{const.} \quad (2.21)$$

In this region, all variables are independent of ζ, and equation (2.20) reduces to

$$(S_{0\vartheta} + S_{0\theta\theta\theta})_\theta + 9k_0^{-1/2}(S_0 S_{0\theta})_\theta + (3/k_0^2)S_{0\xi\xi} = 0, \quad (2.22)$$

which is related to the KP equation (Kadomtsev & Petviashvili 1970).

Initially, the primary wave propagates to the right in the direction of the vector (k_0, ℓ), with $\ell^2 + k_0^2 = 1$; cf. (2.15) and (2.21). Writing

$$k_0 = \cos\varphi, \qquad \ell = \sin\varphi, \quad (2.23)$$

this means that the wavefront of the incident wave moves in the direction that makes an angle φ with the positive x-axis. In the region of varying depth, (2.23) should be rewritten as $kh^{1/2} = \cos\varphi$, $\ell h^{1/2} = \sin\varphi$, which determines $\varphi(\zeta)$.

In the inhomogeneous region, $x > 0$, the depth varies according to

$$h(\zeta) \to h_1 \quad \text{as} \quad \zeta \to \infty, \quad (2.24)$$

where h_1 is a constant. This corresponds to $k(\zeta) \to k_1$ as $\zeta \to \infty$, say. In fact, $h(\zeta)$ describes a transition to another region of uniform depth as ζ increases.

To proceed, we consider a class of bounded functions of the variable ζ. For $f(\zeta)$ belonging to this class we introduce the decomposition

$$f(\zeta) = \overline{f} + f_p(\zeta), \qquad \overline{f} = \lim_{\zeta\to\infty}[\frac{1}{\zeta}\int_0^\zeta f(s)\mathrm{d}s]. \quad (2.25)$$

The overbar denotes averaging on the interval $[0,\infty)$; the index p denotes the perturbation to the mean part, with $\overline{f_p}=0$. If $f(\zeta)\to f_1$ as $\zeta\to\infty$, say, the definition (2.25) implies that $\overline{f}=f_1$. Furthermore, if $g(\zeta)\to g_1$ as $\zeta\to\infty$, we have $\overline{fg}=f_1g_1$.

It will be convenient to omit three-dimensional effects for the time being. Using (2.19) and (2.25) equation (2.20) then has a solution of the form

$$2(hk)^{1/2}\eta_1=-\int_0^{\zeta}[(1/3k_1)(S_{0\vartheta}+S_{0\theta\theta\theta})+(3/h_1)(h_1k_1)^{-3/2}S_0S_{0\theta}]\mathrm{d}\zeta$$

$$-\int_0^{\zeta}[(1/3k)_p(S_{0\vartheta}+S_{0\theta\theta\theta})+((3/h)(hk)^{-3/2})_pS_0S_{0\theta}]\mathrm{d}\zeta-\int_0^{\zeta}(hk)^{-1/2}(h\phi_{0\zeta})_{\zeta}\mathrm{d}\zeta, \quad (2.26)$$

with

$$\phi_{0\zeta}=-\int_{\theta}^{\infty}\eta_{0\zeta}\mathrm{d}\theta. \quad (2.27)$$

In what follows, we suppose that the primary wave, η_0, vanishes as $\zeta\to\infty$. Then the last term in (2.26) is bounded in this limit. The term η_1 represents the right-going shelf, induced by the primary wave as it enters the region of varying depth. The requirement that this shelf is bounded as $\zeta\to\infty$ necessarily leads to the equation

$$S_{0\vartheta}+S_{0\theta\theta\theta}+(9/h_1^2)(h_1k_1)^{-1/2}S_0S_{0\theta}=0, \quad (2.28)$$

obtained by averaging (2.26).

When weak three-dimensional effects are incorporated, it is easily verified that the resulting equation is of the form

$$(S_{0\vartheta}+S_{0\theta\theta\theta})_{\theta}+(9/h_1^2)(h_1k_1)^{-1/2}(S_0S_{0\theta})_{\theta}+(3/h_1k_1^2)S_{0\xi\xi}=0, \quad (2.29)$$

To sum up, it is observed that the development of the wave is governed by the equations (2.22) and (2.29), valid in the homogeneous region $x<0$ and the inhomogeneous region $x>0$, respectively. Note that both equations have constant coefficients, and that either of them is related to the KP equation. In what follows, it is assumed that the depth is continuous throughout.

Since the governing equations (2.22) and (2.29) have different coefficients, one might expect that this gives rise to a reflected wave with amplitude of $O(1)$ with respect to that of the incident wave. To demonstrate that this is not the case, it is noted that both S_0 and $S_{0\theta}$ are constant along the characteristics θ=constant. Since the depth varies continually, it is found that $\eta_{0x} = (hk)^{-1/2} S_{0\theta} + O(\epsilon^{1/2})$ also varies continually along these characteristics, which necessarily implies that η_{0x} is continuous at $x = 0$.

The varying depth gives rise to modulation of the primary wave. Using (2.15), (2.18), (2.19) and (2.23), this is epressed by the equation

$$\eta_{0\zeta} = (1/4) h^{-5/4} (1 - h \sin^2 \varphi)^{-5/4} (-1 + 2h \sin^2 \varphi) h'(\zeta) S_0(\vartheta, \theta), \tag{2.30}$$

which reveals that an obliquely incident wave can increase in height as it propagates into deeper water. This occurs in the region where $2h \sin^2 \varphi > 1$. If the angle of incidence exceeds 45°, the wave height increases as soon as the region of increasing depth is entered (remember that $h = 1$ initially). If the angle of incidence is smaller than 45°, the wave height decreases at first, followed by a transition to growth.

The present analysis also applies to the case of periodic depth variations in the region $x > 0$. For simplicity we consider the case of normal incidence, and weak three-dimensional effects are not taken into account. On averaging equation (2.20), with $\ell = 0$ and $\partial/\partial\xi = 0$, and using (2.15), (2.19) and (2.25), this leads to the equation

$$\overline{h^{1/2}} (S_{0\vartheta} + S_{0\theta\theta\theta})_\theta + 9\overline{h^{-7/4}} (S_0 S_{0\theta})_\theta = (3/16) \overline{h^{-3/2} (h'(\zeta))^2} S_0. \tag{2.31}$$

which agrees with a result derived previously by Benilov(1992).

The term proportional to S_0 on the right-hand side of equation (2.31) is of crucial importance in describing specific effects inherent to periodic depth variations (Benilov 1992). This again justifies the choice of the scaling (2.12). Indeed, if a different scaling were chosen, the resulting equation is again of the form (2.31), with the exception that the term proportonal to S_0 vanishes identically.

Finally, we briefly comment on the case where the incident wave enters a region of steadily increasing depth. Then a caustic is located along the line $h\ell^2 = 1$, in view of (2.15). The present analysis, based on ray theory, ceases to be valid near the caustic and needs to be modified. With regard to this, the following points are relevant. First, the distance travelled by the incident wave in the inhomogeneous region to reach the caustic, is an order of magnitude shorter than the scale on which nonlinear effects modify the wave. This limits the effect of nonlinearity on the wave in the region ahead of the caustic. Secondly, it seems reasonable to suppose that beyond the caustic the effect of nonlinearity is actually negligible due to rapid decay of the wave. Clearly, it suffices to consider the linearized version of equation (2.9) in this case. This leads to the conclusion that linear theory gives a reflection with the appropriate change of phase.

3 Mass conservation

3.1. General

In what follows, we briefly comment on the mechanism of mass transfer between the primary wave and the wave-induced components, known as the shelves. Three-dimensional effects will not be taken into account. Then the governing equations for the primary wave are given by (2.22), with $\partial/\partial\xi = 0$, and (2.28).

Consider an incident solitary wave of the form

$$\eta_0 = A\mathrm{sech}^2\beta\,\tilde{\theta}, \qquad \beta^2 = 3A/4, \tag{3.1}$$

with constant amplitude A; the nonlinear characteristic coordinate $\tilde{\theta}$ is of the form $\tilde{\theta} = \theta - 4\beta^2\vartheta$, with θ and ϑ defined by (2.10) and (2.11), respectively.

We shall restrict the analysis to a solitary wave centered at $x = 0$ initially. In addition, the nonlinear correction in $\tilde{\theta}$ will be neglected in the region $x > 0$ because we only consider the further development of the primary wave in a region that is an order of magnitude smaller than the scale on which nonlinear effects modify the wave. Accordingly, the characteristic $\tilde{\theta} = 0$ reduces to the equation

$$\int_0^\zeta k(\chi)\mathrm{d}\chi + \ell\xi - \tau = 0, \tag{3.2}$$

with $\zeta = \epsilon^{1/2}x$, $\xi = \epsilon^{1/2}y$, $\tau = \epsilon^{1/2}t$.

The primary wave (3.1) propagates along rays ,$y = y(x)$, determined by the equation $\mathrm{d}y/\mathrm{d}x = \ell/k$. In deriving an expression for the mass carried by this wave along a ray, we choose the ray through the origin. This is given by

$$\xi = \ell\int_0^\zeta k^{-1}(\chi)\mathrm{d}\chi. \tag{3.3}$$

The intersection of this ray, denoted by C_+, and the characteristic (3.2) is described by the equation

$$\int_0^\zeta (h(\chi)k(\chi))^{-1}\mathrm{d}\chi - \tau = 0, \tag{3.4}$$

with a solution $\zeta = \overline{\zeta}(\tau)$, say.

The mass carried by the primary wave along C_+ is given by

$$\int_{C_+} \eta_0 \mathrm{d}s = m_0(\overline{k}/k_0)^{-1/2} = m(\tau), \tag{3.5}$$

with $\overline{k} = k(\overline{\zeta}(\tau)$ and $m(0) = m_0$. In deriving this result, it is noted that the dominant part of the integral (3.5) arises from the immediate neighbourhood of the intersection of (3.2) and (3.3); cf Johnson (1994).

As expressed by (3.5), the mass carried by the primary wave decreases if a region of decreasing depth is entered. Since the mass flow, induced by this wave, is aligned with the rays, we conclude that mass is transferred to the shelves.

We first consider the right-going shelf, generated by the primary wave as this enters the region of varying depth. Using (2.26)-(2.28) this leads to the expression

$$\eta_1 = \frac{1}{2}(hk)^{-1/2} \int_0^\zeta (hk)^{-1/2}[h((hk)^{-1/2})']'\mathrm{d}\zeta \int_\theta^\infty S_0 \mathrm{d}\theta - \frac{1}{2} S_0 S_{0\theta}(hk)^{-1/2} \int_0^\zeta P \mathrm{d}\zeta, \tag{3.6}$$

with $P(\zeta) = 3[(1/kh_1^2)(h_1 k_1)^{-1/2} - (1/h)(hk)^{-3/2}]$. The prime denotes differerentiation with respect to ζ. In the case of normal incidence, the expession (3.6) agrees with a result derived previously by Johnson (1994).

In general, the contribution of the shelf (3.6) to mass transport is $O(1)$, see e.g. Miles 1979), Knickerbocker & Newell (1980, 1985) and Johnson (1994). The shelf stretches between the point to which infinitesimal disturbances would have travelled from the position at which the depth change first occurs, and the center of the solitary wave. This implies that inclusion of this wave component is not enough to compensate the mass loss of the primary wave.

In order to accommodate an exact mass balance, we must add another wave component. As is well known, the effect of a reflected shelf on mass transfer should also be taken into account. However, a more general approach is to seek a solution of equation (2.9) of the form $\phi = \hat{\phi}(\theta_+, \theta_-) + O(\epsilon^{1/2})$, with the slow characteristic coordinates θ_+ and θ_- defined by

$$\theta_+ = \int_0^\zeta k(\chi)\mathrm{d}\chi + \ell\xi - \tau, \qquad \theta_- = -\int_0^\zeta k(\chi)\mathrm{d}\chi + \ell\xi - \tau. \tag{3.7}$$

The potential $\hat{\phi}(\theta_+, \theta_-)$ corresponds to a shelf of the form $\eta_c = \epsilon^{1/2}\hat{\eta}(\theta_+, \theta_-)$, with an amplitude of order $\epsilon^{1/2}$ with respect to the primary wave.

Substitution of (3.7) into (2.9) leads to the equation

$$\hat{\eta}_{\theta_+\theta_-} = (4hk)^{-1}[(hk)_{\theta_+} - (hk)_{\theta_-}](\hat{\eta}_{\theta_+} - \hat{\eta}_{\theta_-}), \tag{3.8}$$

the normal form of an equation of hyperbolic type, with characteristics $\theta_+=$ const. and $\theta_-=$ const.

In the region of varying depth, the solution of equation (3.8) consists of right-going and left-going wave components. The amplitudes of these coupled components are of the same order of magnitude. Accordingly, this solution will be called a composite shelf. The cumulative effect of nonlinearity is taken into account by adding a term $-4\epsilon^{1/2}\beta^2\vartheta$ to the expression for θ_+; see below (3.1).

3.2. The case of normal incidence

In what follows, we shall restrict ourselves to the case of a normally incident wave, i.e. $k_0 = 1$, $hk^2 = 1$, $\ell = 0$ and $\partial/\partial\xi = 0$. Then equation (3.8) reduces to

$$\hat{\eta}_{\theta_+\theta_-} = (8h)^{-1}[h_{\theta_+} - h_{\theta_-}](\hat{\eta}_{\theta_+} - \hat{\eta}_{\theta_-}). \tag{3.9}$$

The solution $\hat{\eta}(\theta_+, \theta_-)$ should satisfy the initial condition

$$\hat{\eta}(0,0) = 0. \tag{3.10}$$

The conditions imposed on the solution of equation (3.9) are prescribed by the requirement that mass is conserved. In order to derive these, we reconsider the equation for mass conservation, given by (2.6), and note that (to leading order) the wave components $\hat{\eta}$, $\hat{\phi}$ satisfy a similar equation, namely,

$$\hat{\eta}_\tau + (h\hat{\phi}_\zeta)_\zeta = 0. \tag{3.11}$$

In this equation, the term $h\hat{\phi}_\zeta$ (denoted by F) should be interpreted as a mass flux, induced by the composite shelf.

Integration of (3.11) over ζ from $\theta_- = 0$ to $\theta_+ = 0$, corresponding to an interval with end points $\zeta = -\tau$ and $\zeta = \overline{\zeta}(\tau)$, respectively, leads to the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\tau}^{\overline{\zeta}(\tau} \hat{\eta}\mathrm{d}\zeta + [h\hat{\phi}_\zeta - h^{1/2}\hat{\eta}]_+ - [\hat{\phi}_\zeta + \hat{\eta}]_- = 0. \tag{3.12}$$

Here $[..]_+$ and $[..]_-$ denote evaluation along the characteristics $\theta_+ = 0$ and $\theta_- = 0$, respectively.

In the region $\zeta < 0$ the right-going component of $\hat{\eta}$ should vanish, i.e. $\hat{\eta}_{\theta_+} = 0$. This implies that $[\hat{\eta}]_-=$ const. From (3.10) it then follows that

$$[\hat{\eta}]_- = 0. \tag{3.13}$$

Furthermore, the mass flux induced by the shelf should vanish along the characteristic $\theta_- = 0$, i.e. $[F]_- = 0$; see below (3.11). This leads to the result that the third term in (3.12) vanishes identically. Expressing the resulting equation in terms of the variables θ_+ and θ_- yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\tau}^{\overline{\zeta}(\tau} \hat{\eta}\mathrm{d}\zeta - 2[h^{1/2}\hat{\phi}_{\theta-}]_+ = 0. \tag{3.14}$$

To emphasize the main features, the contribution of the right-going shelf (3.6) to mass transfer will be ignored. This seems justified as long as the effect of nonlinearity on the development of the primary wave can be neglected; see below (3.6). Then the requirement that mass is conserved leads to the equation

$$\int_{-\tau}^{\overline{\zeta}(\tau)} \hat{\eta}\mathrm{d}\zeta = m_0(1 - \overline{h}^{1/4}), \tag{3.15}$$

with $\overline{h} = [h]_+$; cf. the expression (3.5) for the mass carried by the primary wave.

From (3.14) and (3.15) it follows that

$$2[h^{1/2}\hat{\phi}_{\theta_-}]_+ = -m_0[\frac{\mathrm{d}}{\mathrm{d}\tau}(h^{1/4})]_+, \tag{3.16}$$

which reveals that mass transfer to the composite shelf is quenched as the primary wave enters a new region of uniform depth.

Assuming that the contribution of the right-going shelf (3.6) to mass transfer is negligible, it follows from equation (3.14) that the total mass carried by the primary wave and the composite shelf is conserved if the second term satisfies (3.16). However, the composite shelf should satisfy the initial condition (3.10).

Using (3.7), equation (3.16) can be rewritten as

$$[\hat{\phi}_{\theta_-}]_+ = -m_0\frac{\mathrm{d}}{\mathrm{d}\theta_-}(h^{-1/4}). \tag{3.17}$$

The equation for the potential, derived from (2.9) by using the transformations (3.7), is given by

$$\hat{\phi}_{\theta_+\theta_-} = (8h)^{-1}[h_{\theta_+} - h_{\theta_-}](\hat{\phi}_{\theta_+} - \hat{\phi}_{\theta_-}). \tag{3.18}$$

¿From (3.13) it follows that $[\hat{\phi}_{\theta_+} + \hat{\phi}_{\theta_-}]_- = 0$. Furthermore, $[F]_- = 0$ implies that $[\hat{\phi}_{\theta_+} - \hat{\phi}_{\theta_-}]_- = 0$. Then $[\hat{\phi}_{\theta_+}]_- = 0$, i.e. $[\hat{\phi}]_-=$ const. Choosing the initial condition $\hat{\phi}(0,0) = 0$, this implies that

$$[\hat{\phi}]_- = 0. \tag{3.19}$$

From (3.17) and (3.19) it follows that

$$[\hat{\phi}]_+ = m_0(1 - h^{-1/4}), \qquad h = h(\zeta(\theta_-)). \tag{3.20}$$

The expressions (3.19) and (3.20) prescribe the potential along the characteristics.

For $\theta_+ = 0$, i.e. along the θ_--axis, equation (3.18) converts into an ordinary differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}\theta_-}\hat{\phi}_{\theta_+}(\theta_-, 0) + (4h)^{-1}\frac{\mathrm{d}h}{\mathrm{d}\theta_-}\hat{\phi}_{\theta_+}(\theta_-, 0) = (4h)^{-1}\frac{\mathrm{d}h}{\mathrm{d}\theta_-}\hat{\phi}_{\theta_-}(\theta_-, 0),$$

with $\hat{\phi}_{\theta_-}(\theta_-, 0)$ given by (3.17). Solving this equation, and using (3.19), yields

$$[\hat{\eta}]_+ = m_0[-\frac{\mathrm{d}}{\mathrm{d}\theta_-}(h^{-1/4}) + (h^{-1/4}/16)\int_0^{\theta_-}(h^{-1}\frac{\mathrm{d}h}{\mathrm{d}s})^2\mathrm{d}s. \tag{3.21}$$

The expressions (3.13) and (3.21) prescribe $\hat{\eta}$ along the characteristics.

The mass carried by the reflected shelf in the region $\zeta < 0$ is easily calculated as follows. It is recalled that $\hat{\eta}_{\theta_+} = 0$ in this region. Furthermore, $[F]_- = 0$; see below (3.13). Then (3.13) implies that $\hat{\phi}_{\theta_+} = 0$ for $\zeta < 0$, and integration of (3.11) over ζ from $-\tau$ to 0 yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\tau}^{0}\hat{\eta}\mathrm{d}\zeta - [\hat{\phi}_{\theta_-}]_0 = 0, \tag{3.22}$$

where the second term in (3.22) is evaluated at $\zeta = 0$. Now, in the region $\zeta < 0$ the potential is given by $\hat{\phi} = \Phi(\theta_-)$, say, where (3.19) implies that $\Phi(0) = 0$. Furthermore, $\hat{\phi} = \Phi(-\tau)$ at $\zeta = 0$, which implies that $\mathrm{d}\hat{\phi}/\mathrm{d}\tau = -\mathrm{d}\hat{\phi}/\mathrm{d}\theta_-$. This leads to the result

$$\int_{-\tau}^{0}\hat{\eta}\mathrm{d}\zeta = -\Phi(-\tau). \tag{3.23}$$

Thus, in order to calculate the mass carried by the reflected shelf in the region $\zeta < 0$, it suffices to determine the solution of the equation for the potential along the curve $\theta_+ = \theta_-$.

As a specific example we consider the depth profile

$$h(\zeta) = (1 - \alpha\zeta)^2, \tag{3.24}$$

with $\alpha =$ const. and positive, and $h(\zeta) = h_1 = (1-\alpha)^2$ for $\zeta > 1$. Thus, at $\zeta = 1$ the primary wave enters a new region of uniform depth. The equation (3.18) for the potential reduces to an equation with constant coefficients, namely,

$$\hat{\phi}_{\theta_+\theta_-} = -\frac{\alpha}{4}(\hat{\phi}_{\theta_+} - \hat{\phi}_{\theta_-}). \tag{3.25}$$

The right-hand side is non-zero only in the region of varying depth, $0 < \zeta < 1$. Mass transfer is confined to this region.

From (3.7), (3.19), (3.20) and (3.24) we obtain

$$[\hat{\phi}]_- = 0, \qquad [\hat{\phi}]_+ = m_0(1 - \exp(-\alpha\theta_-/4)), \qquad 0 < \zeta < 1. \tag{3.26}$$

Equation (3.16) implies that $[\hat{\phi}]_+=$ const. for $\zeta > 1$. Requiring that $[\hat{\phi}]_+$ is continuous, we find

$$[\hat{\phi}]_+ = m_0(1 - (1-\alpha)^{-1/2}), \qquad \zeta > 1. \tag{3.27}$$

Using standard methods (Garabedian 1964), it is found that equation (3.25) has an implicit solution of the form

$$\hat{\phi}(\theta_-, \theta_+) = [\hat{\phi}]_+ + \frac{\alpha}{4}\int_0^{\theta_+} \mathrm{d}y \int_y^{\theta_-} (\hat{\phi}_x(x,y) - \hat{\phi}_y(x,y))\mathrm{d}x. \tag{3.28}$$

The curve $\zeta=$ const., $0 < \zeta < 1$, in the (ζ, τ)-plane corresponds to the curve $\theta_+ = \theta_- - (2/\alpha)\log(1-\alpha\zeta)$ in the (θ_-, θ_+)-plane; cf. (3.7) and (3.24). Thus, the region of varying depth is mapped on a region bounded by the curves $\theta_+ = \theta_-$ and $\theta_+ = \theta_- + \theta_0$, with $\theta_0 = -(2/\alpha)\log(1-\alpha) > 0$. The physically relevant region is confined to the third quadrant in the (θ_-, θ_+)-plane, bounded by the characteristics $\theta_- = 0$ and $\theta_+ = 0$. Curves of constant τ are mapped on the curves $\theta_+ = -\theta_- - 2\tau$. At increasing τ these move away from the origin.

The integral equation (3.28) is solved iteratively by writing

$$\hat{\phi}(\theta_-,\theta_+) = \sum_{n=0}^{\infty} \mu^n \hat{\phi}_n(\theta_-\theta_+), \qquad \hat{\phi}_0(\theta_-,\theta_+) = [\hat{\phi}]_+, \tag{3.29}$$

with $\mu = \alpha/4$. If μ is relatively small, (3.29) can be considered as an asymptotic expansion. Note, for instance, that $h_1 = 0.5$ corresponds to $\mu = 0.075$. On the other hand, the validity of this expansion is restricted to $\theta_- = O(1)$, $\theta_+ = O(1)$.

Substitution of (3.29) into (3.28) leads to the recurrence relation

$$\hat{\phi}_{n+1}(\theta_-,\theta_+) = \int_0^{\theta_+} \mathrm{d}y \int_y^{\theta_-} (\hat{\phi}_{nx} - \hat{\phi}_{ny})\mathrm{d}x, \qquad (n = 0, 1, 2, \ldots). \tag{3.30}$$

Substitution of the first two terms of (3.29) into (3.23) yields

$$\int_{-\tau}^{0} \hat{\eta}\mathrm{d}\zeta = m_0(-1 + \exp(\mu\tau)) + O(\mu^2), \tag{3.31}$$

valid for $\tau < \tau_0$, with $\tau_0 = -(1/2\mu)\log(1-4\mu)$. Thus, initially, the mass carried by the reflected shelf in the region $\zeta < 0$ increases exponentially. For $\tau > \tau_0$ the right-hand side of (3.31), with τ replaced by τ_0, is constant, which corresponds to stationary mass transport in this region. It is easily verified that the mass carried by the relected shelf is almost the same as the total mass loss of the primary wave.

4 Fission of a solitary wave

4.1. Preliminaries

Consider an incident solitary wave of the form

$$S_0 = A\mathrm{sech}^2(\alpha\vartheta + \beta\theta + \gamma\xi), \qquad \beta^2 = (3/4k_0^{1/2})A, \qquad \alpha\beta + 3\gamma^2/k_0^2 + 4\beta^4 = 0, \tag{4.1}$$

propagating to the right in the homogeneous rgion $x < 0$. As this wave enters the region of varying depth, the further development is governed by equation (2.29). The solution of this equation is written as

$$S_0 = -\frac{2}{3}\beta^2 h_1^2 (h_1 k_1)^{1/2} u(\chi, Y, \tau), \qquad \chi = \beta\theta, \qquad Y = h_1^{1/2} k_1 \beta^2 \xi, \qquad \tau = \beta^3 \vartheta. \tag{4.2}$$

Then the new independent variable $u(\chi, Y, \tau)$ satisfies a KP equation of the form

$$(u_\tau - 6uu_\chi + u_{\chi\chi\chi})_\chi + 3u_{YY} = 0. \tag{4.3}$$

The initial condition, prescribed by (4.1), is given by

$$u(\chi, Y, 0) = -2k_0^{1/2} h_1^{-2} (h_1 k_1)^{-1/2} \mathrm{sech}^2(\chi + pY), \tag{4.4}$$

with

$$p = \gamma / h_1^{1/2} k_1 \beta^2. \tag{4.5}$$

Introducing the transfomation

$$X = \chi + pY - 3p^2\tau, \qquad T = \tau, \tag{4.6}$$

equation (4.3) reduces to

$$(u_T - 6uu_X + u_{XXX})_X = 0. \tag{4.7}$$

Thus, if $u(X,T)$ is a solution of the KdV equation

$$u_T - 6uu_X + u_{XXX} = 0, \tag{4.8}$$

(4.6) implies that the KP equation (4.3) has a solution of the form $u(\chi + pY - 3p^2\tau, \tau)$. In other words, any known solution of the KdV equation can be transformed into a solution of the KP equation with similar properties. This result will be useful in deriving sufficient conditions for fission when weak three-dimensional effects are taken into account.

The initial condition

$$u(X, 0) = -\sigma \mathrm{sech}^2 X, \tag{4.9}$$

uniquely determines the solution of equation (4.8). Applying the transformation (4.6), this corresponds to the initial condition (4.4) if

$$k_0^{-1/2} (h_1 k_1)^{1/2} h_1^2 = 2/\sigma. \tag{4.10}$$

It is recalled that $h_1(\ell^2 + k_1^2) = 1$ in the region $\zeta \gg 1$; cf. (2.15) and (2.24). Then (2.23) implies that (4.10) can be rewritten as

$$(\cos\varphi)^{-1/2} (1 - h_1 \sin^2\varphi)^{1/4} h_1^{9/4} = 2/\sigma, \tag{4.11}$$

where φ denotes the angle of incidence in the region of uniform depth. Thus, in view of (4.4), the solution of the KP equation is uniquely determined by φ and h_1.

If the angle of incidence and the depth h_1 are such that

$$\sigma = N(N+1), \quad N = 1, 2, \ldots, \tag{4.12}$$

the KdV equation (4.8) has an N-soliton solution. Introducing the variable

$$\xi_n = X - 4n^2 T, \tag{4.13}$$

the asymptotic behaviour of the solution of this equation, as $T \to \infty$, ξ_n fixed, is of the form

$$u(X,T) \sim -2n^2 \text{sech}^2(n\xi_n - \varphi_n), \quad n = 1, 2, \ldots, N, \tag{4.14}$$

where φ_n is a constant phase (Drazin & Johnson 1989).

The solution (4.14) represents N separate solitons, ordered according to their speeds as $T \to \infty$. If (4.11) and (4.12) are satisfied, the KP equation (4.3) also has an N-soliton solution. The asymptotic behaviour of this solution is again of the form (4.14), but with (4.13) replaced by $\xi_n = X + pY - (3p^2 + 4n^2)\tau$. The solution of equation (2.29) is obtained from (4.2).

4.2. Analysis of results

Denoting the left-hand-side of equation (4.11) by $q(\varphi, h_1)$, the substitution (4.12) leads to the relation

$$q(\varphi, h_1) = 2/(N(N+1)), \tag{4.15}$$

which is the condition for fission of the solitary wave (4.1).

It is important to note that the formation of separate solitons does not necessarily imply that N is a positive integer. In fact, (4.15) can serve to define a value of N, where the integer part of N determines whether or not solitons can form. Nevertheless, (4.15) is usually called the condition for fission. Note that this is independent of weak three-dimensional effects.

In the case of normal incidence, the condition for fission reduces to

$$h_1 = \left(\frac{2}{N(N+1)}\right)^{4/9}. \tag{4.16}$$

This implies that fission only occurs if the incident wave enters a region of smaller depth. The values of h_1, determined by (4.16), are called the eigendepths.

The result (4.16) was previously derived by Tappert & Zabusky (1971) and Johnson (1973), based on methods that differ from the present one. There is some evidence that (4.16) is confirmed by the numerical work of Madsen & Mei (1969), satisfactorily compared by them with experimental results; cf. the discussion in Johnson (1973).

In the case of oblique incidence, equation (4.15) applies. Then it is found that $q > 0$ within the range $0 < h_1 < 1/\sin^2\varphi$. This corresponds to the condition $h_1 < h_c$, where $h_c = 1/\sin^2\varphi$ is the depth of the caustic. Furthermore, q has a maximum at $h_1 = 9/10\sin^2\varphi$, and the zeroes are given by $h_1 = 0$ and $h_1 = 1/\sin^2\varphi$. For fixed N and φ, with $\varphi \neq 0$, this implies that (4.15) has precisely two distinct roots. Denoting these by $h_{1\alpha}$ and $h_{1\beta}$, with $h_{1\alpha} < h_{1\beta}$, it follows that

$$h_{1\alpha} < 9/10\sin^2\varphi, \qquad 9/10\sin^2\varphi < h_{1\beta} < 1/\sin^2\varphi. \tag{4.17}$$

In what follows, we restrict ourselves to the range $\sin^2\varphi < 9/10$, which limits the angle of incidence. Then (4.17) implies that $h_{1\beta} > 1$. To determine the order of magnitude of $h_{1\alpha}$ more precisely, it is noted that $\partial q/\partial h_1 > 0$ within the range $0 < h_1 < 9/10\sin^2\varphi$ (for fixed φ), which includes the point $h_1 = 1$. From (4.15), with $q(\varphi, 1) = 1$, it then follows that $h_{1\alpha} < 1$ if $N > 1$.

The result $h_{1\alpha} < 1$ implies that fission can occur if the incident wave enters a region of smaller depth. The result $h_{1\beta} > 1$, on the other hand, suggests that fission can also occur if the incident wave enters a region of larger depth. However, it turns out that $h_{1\beta} \approx 1/\sin^2\varphi$, valid within a high degree of accuracy. In view of (2.15) this corresponds to $h_{1\beta} \approx 1/\ell^2$, which implies that the depth $h_{1\beta}$ is located quite close to the depth $h_c = 1/\ell^2$ of the caustic. This leads to the conclusion that the present theory does not predict that fission will occur if the wave enters a region of larger depth.

Equation (4.15) implies that, for fixed N,

$$h_{1\varphi} = -\frac{2h_1(1-h_1)\tan\varphi}{(9 - 10h_1\sin^2\varphi)}. \tag{4.18}$$

Combined with the preceding results it then follows from (4.18) that the eigendepths $h_{1\alpha}$ are largest in the case of normal incidence. It should be noted, however, that the

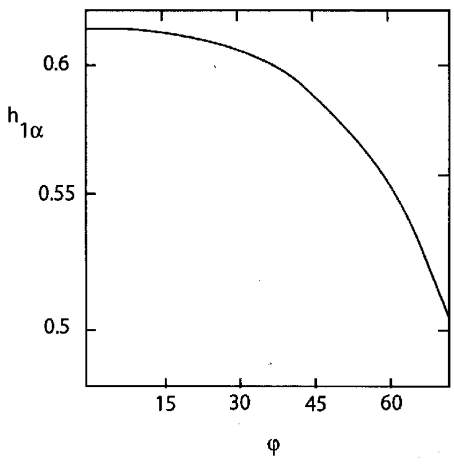

Figure 1: Variation of the eigendepth with the angle of incidence; $N = 2$.

present theory breaks down if the direction of wave propagation is almost parallel to the depth contours. In figure 1 the dependence of the eigendepth $h_{1\alpha}$ on the angle of incidence is depicted for $N = 2$.

As remarked below (4.8), we only derived a sufficient condition for fission if weak three-dimensional effects are taken into account. It will now be shown that this condition is both sufficient and necessary for $N = 2$.

The two-soliton solution of equation (4.3) is of the form (Johnson 1997)

$$u(\chi, Y, \tau) = -2\frac{\partial^2}{\partial \chi^2} \log(1 + E_1 + E_2 + DE_1E_2), \tag{4.19}$$

where $E_i = \exp[-(k_i + \ell_i)\chi + (k_i^2 - \ell_i^2)Y + 4(k_i^3 + \ell_i^3)\tau + \alpha_i]$ and

$$D = \frac{(k_1 - k_2)(\ell_1 - \ell_2)}{(k_1 + k_2)(\ell_1 + \ell_2)}. \tag{4.20}$$

The initial condition (4.4), written as $u(\chi, Y, 0) = -2M\text{sech}^2(\chi + pY)$, is uniquely determined by the incident solitary wave. Substitution of this condition into (4.19) for $Y = 0$, $\tau = 0$ leads to the equation

$$\alpha_0(\cosh \chi)^M \exp(\alpha_1 \chi) = 1 + E_1 + E_2 + DE_1E_2, \tag{4.21}$$

where α_0 and α_1 are constants. With E_1 and E_2 defined below (4.19) this necessarily implies that $M = 3$, which just corresponds to $N = 2$ in the condition for fission (4.15).

5 Summary

We derived the governing equations for an obliquely incident surface wave in shallow water of slowly varying depth. Based on the assumption that the depth variations occur on a scale shorter than that on which the wave evolves, and incorporating weak three-dimensional effects, it is found that the development of the wave is governed by an evolution equation with constant coefficients, which is related to the KP equation.

When three-dimensional effects are not taken into account, the evolution equation is of KdV-type. The present theory also applies to the case of periodic depth variations, and in the case of normal incidence there is agreement with results derived previously by Benilov (1992).

In the case of varying depth the KdV equation fails to satisfy the local conservation of mass flow. Subject to the condition of relatively fast change of depth, we studied the problem of mass transfer from the obliquely incident primary wave to the shelves. In a subsequent more detailed analysis, devoted to the case of a normally icident wave, we were able to solve this problem systematically. The main features were derived without the need to specify the depth-variation profile.

A principal result we derived is that a so-called composite shelf is generated, which consists of both right-going and reflected wave components. In the region of varying depth these components are coupled, and the amplitudes are of the same order of magnitude. Coupling occurs because the width of the composite shelf and the scale of the depth variation are of the same order of magnitude. In fact, the right-going component of the shelf is generated by the reflected component in the region of varying depth, a phenomenon known as re-reflection.

The conditions imposed on the solution of the governing equation for the composite shelf along the characteristics are prescribed by the requirement that mass is conserved. These conditions were systematically derived, based on an alternative equation for the composite shelf, which takes the form of a local mass conservation equation. The theory also applies to the case of slower depth variations.

Finally, we derived sufficient conditions for break up of an obliquely incident solitary wave into a finite number of solitons. As in the case of normal incidence, this requires that the depth change should occur on a sufficiently short scale, and that the wave should propagate into a region of decreasing depth. In addition, the angle of incidence proves to be a relevant parameter. In particular, it is found that the eigendepths are largest in the case of normal incidence. Weak three-dimensional effects do not affect the conditions for fission.

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