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Hydrostatic dynamics of a layer with constant buoyancy frequency

By W.T.M. VERKLEY*
Royal Netherlands Meteorological Institute, The Netherlands

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SUMMARY

We study the dynamics of a hydrostatic layer of air of which the buoyancy frequency is constant in space and time. Like an isothermal and isentropic layer, a layer with constant buoyancy frequency is autobarotropic, implying that the horizontal velocity is independent of height if this is assumed to be the case at some initial time. From the horizontal velocity inside the layer and the pressure at the upper and lower boundaries of the layer we obtain the profile of vertical velocity. This profile satisfies the continuity equation and, together with the horizontal velocity and the upper and lower boundary pressures, completely specifies the layer's dynamical state. It is pointed out how layers with constant buoyancy frequency can be used to construct atmospheric models.

KEYWORDS: Buoyancy frequency Vertical velocity Layered models

1. INTRODUCTION

It has been emphasized recently by Held (2005) that a hierarchy of climate models is needed to bridge the gap between simulation and understanding in climate modelling. Layered models of the atmosphere, in which the potential temperature has a constant but different value in each of the layers, may be part of such a hierarchy because their states, however idealized, are dynamically realizable solutions of the hydrostatic primitive equations. However, if we require that a model with very few layers still gives an acceptable description of the atmosphere, constant potential temperature in each of the layers is perhaps not the optimal choice. More suitable, perhaps, are layers in which the Brunt-Väisälä frequency, or buoyancy frequency, is constant. This idea is supported by Birner *et al.* (2002) who have remarked that both tropospheric and stratospheric soundings of temperature are consistent with constant but different values of the buoyancy frequency. A two layer troposphere-stratosphere model, with layers in which the buoyancy frequency has two different values, might then be a simple example of such a model. In a previous article (Verkley, 2000) the author has examined the properties of an atmospheric layer in which the potential temperature is constant. Here, with a view on models based on layers with constant buoyancy frequency, we wish to extend this study to the more general case of a layer in which the buoyancy frequency is constant.

Quite generally, a vertical profile of temperature can be characterized by its buoyancy frequency as a function of height. The buoyancy frequency N is given by

$$N^2 = \frac{g}{\theta}\frac{\partial \theta}{\partial z}, \tag{1}$$

where z is height, g is the acceleration of gravity and θ is the potential temperature. The latter is defined by

$$\theta = T(\frac{p_r}{p})^{\kappa}, \tag{2}$$

where p is pressure, T is the absolute temperature, p_r a reference pressure of 1000 hPa and $\kappa = R/c_p$, with R the gas constant and c_p the specific heat at constant pressure, both for dry air.

* Corresponding author: Royal Netherlands Meteorological Institute, P.O. Box 201, 3730 AE De Bilt, The Netherlands. e-mail: verkley@knmi.nl

In the following section we will study the temperature profile of a hydrostatic layer of air that is characterized by a constant value of the buoyancy frequency. Isothermal layers with constant absolute temperature T and isentropic layers with constant potential temperature θ are special cases. Using the mass continuity equation in pressure coordinates we will obtain in section 3 the pressure tendency $\omega = Dp/Dt$ as a function of pressure p, from which the vertical velocity $w = Dz/Dt$ as a function of the height z will be derived by transforming from pressure to height coordinates. The vertical velocity so derived gives a complete dynamical description of a layer with constant buoyancy frequency N, given the horizontal velocity inside the layer, the pressures at the upper and lower boundaries of the layer and possibly prescribed flows of mass through these boundaries. In section 4 we discuss the horizontal dynamical equations and point out how layers with constant buoyancy frequency can be used to construct atmosperic models. Section 5 concludes the paper.

2. A LAYER WITH CONSTANT BUOYANCY FREQUENCY

We will assume that the atmosphere obeys the ideal-gas law and satisfies the equation of hydrostatic balance. It is convenient to use the normalized pressure $\eta = p/p_r$ instead of the pressure p. In terms of η we have for the definition of potential temperature, the ideal-gas law and the equation of hydrostatic balance:

$$\begin{aligned} \theta &= T\eta^{-\kappa}, \\ \rho &= \frac{p_r}{RT}\,\eta, \\ \frac{\partial \eta}{\partial z} &= -\frac{g}{p_r}\rho, \end{aligned} \tag{3}$$

where ρ is density. Using these equations in combination with (1) and noting that (1) can be written alternatively as $N^2 = -g\theta\, \partial/\partial z\, (1/\theta)$, we have

$$N^2 = \frac{g^2}{R}\eta^{1-\kappa}\frac{\partial}{\partial \eta}[\frac{\eta^\kappa}{T}]. \tag{4}$$

From this expression we may check that for an *isentropic* layer, in which $\eta^\kappa/T = 1/\theta = 1/\mathcal{T}$ is constant, the buoyancy frequency is $N^2 = 0$. For an *isothermal* layer, in which $1/T = 1/\mathcal{T}$ is constant, the buoyancy frequency is $N^2 = g^2/(c_p\mathcal{T})$. In both cases the buoyancy frequency is independent of pressure and therefore independent of height.

To find the temperature as a function of pressure for a general profile of N^2 we have to solve the following differential equation

$$\frac{\partial}{\partial \eta}[\frac{\eta^\kappa}{T}] = \frac{RN^2}{g^2}\eta^{\kappa-1}. \tag{5}$$

This equation follows simply from (4) and has, when N^2 is constant, the solution

$$\frac{\eta^\kappa}{T} = \frac{c_p N^2}{g^2}\eta^\kappa + A, \tag{6}$$

where A is independent of pressure. If we replace the parameters N^2 and A by $\mathcal{T}$ and α, defined by

$$\begin{aligned} N^2 &= \frac{g^2}{c_p\mathcal{T}}\,\frac{1}{1+\alpha}, \\ A &= \frac{1}{\mathcal{T}}\,\frac{\alpha}{1+\alpha}, \end{aligned} \tag{7}$$

the solution can be written as

$$T = \mathcal{T} \frac{1+\alpha}{1+\alpha\eta^{-\kappa}}, \tag{8}$$

which expression is recognized as the maximum entropy profile obtained by Verkley and Gerkema (2004, their Eq. (22)). If $\alpha \to \infty$ the isentropic profile is recovered, whereas if $\alpha \to 0$ the isothermal profile is recovered. Note that $\mathcal{T}$ is the temperature at the reference pressure p_r and that the value of the buoyancy frequency N^2 is given by the first expression in (7). In Verkley and Gerkema (2004) this profile was obtained for a hydrostatic column of dry air as the result of a variational principle in which the vertically integrated entropy is maximized with the mass, integrated absolute temperature and integrated potential temperature kept constant. Profiles like (8), characterized by a vertically uniform value of the buoyancy frequency, are expected to give good descriptions of the vertical temperature structure in both the troposphere and the stratosphere (Birner *et al.*, 2002). In Verkley and Gerkema (2004) this was confirmed for the troposphere. Pacheco and Sañudo (2005) have shown that profile (8) also gives a good description of the vertical temperature structure of the mesosphere.

In the foregoing we have found the temperature, and therewith the potential temperature and density, as a function of pressure p. To obtain the temperature, potential temperature and density as a function of height z we need to find the relation between z and p. Combining the ideal-gas law with the hydrostatic relation and then substituting expression (8) for the temperature T we get

$$\frac{\partial z}{\partial \eta} = -\kappa\, H\, (1+\alpha)\, \frac{\eta^{-1}}{1+\alpha\eta^{-\kappa}}, \tag{9}$$

where H is a height scale defined by

$$H = \frac{c_p \mathcal{T}}{g}. \tag{10}$$

It can be checked that the solution of this equation, satisfying the condition that at the lower boundary of the layer we have $z = z_l$ at $\eta = \eta_l$, is given by

$$z = H\, (1+\alpha)\, \ln \frac{\eta_l^\kappa + \alpha}{\eta^\kappa + \alpha} + z_l. \tag{11}$$

Evaluating this expression at the upper boundary of the layer, with height $z = z_u$ and pressure $\eta = \eta_u$, we may obtain

$$z_u - z_l = H(1+\alpha)\, \ln \frac{\eta_l^\kappa + \alpha}{\eta_u^\kappa + \alpha}, \tag{12}$$

which gives the thickness of the layer in terms of the pressures at the upper and lower boundaries.

Expression (11) can be inverted to get η as a function of z, i.e.,

$$\frac{\eta^\kappa + \alpha}{\eta_l^\kappa + \alpha} = \exp \frac{z_l - z}{H(1+\alpha)}, \tag{13}$$

with the help of which the temperature, potential temperature and density can be written as functions of z. To show this, we deduce from (8) that the potential temperature varies with pressure as

$$\theta = \mathcal{T} \frac{1+\alpha}{\eta^\kappa + \alpha}. \tag{14}$$

From this expression we find, by combining it with (8), that the absolute and potential temperature in a layer with constant buoyancy frequency are related by

$$T + \alpha\theta = \mathcal{T}(1+\alpha). \tag{15}$$

To obtain θ and T as functions of z we first note that we have from (14)

$$\theta_l = \mathcal{T}\frac{1+\alpha}{\eta_l^\kappa + \alpha}, \tag{16}$$

where θ_l is the potential temperature at the lower boundary. For the fraction θ/θ_l we thus have

$$\frac{\theta}{\theta_l} = \frac{\eta_l^\kappa + \alpha}{\eta^\kappa + \alpha}. \tag{17}$$

When combined with (13) we find immediately

$$\theta = \theta_l \exp\frac{z - z_l}{H(1+\alpha)}. \tag{18}$$

The latter result can be checked directly from expression (1) of the buoyancy frequency. The temperature T as a function of z is obtained by using (15). For the density ρ one has to invert (13) to obtain η as a function of z and then use the ideal-gas law and the expression for T.

We note that from equations (10) and (11) it can be deduced that

$$gz + c_p\mathcal{T}(1+\alpha)\ \ln\frac{\eta^\kappa + \alpha}{1+\alpha} = gz_l + c_p\mathcal{T}(1+\alpha)\ \ln\frac{\eta_l^\kappa + \alpha}{1+\alpha}. \tag{19}$$

Here the $1+\alpha$ in the argument of the logarithm could be replaced by any other constant value, but this particular choice is convenient when taking the limits $\alpha \to \infty$ and $\alpha \to 0$. This means that in a layer with constant buoyancy frequency N^2 the quantity

$$\Psi \equiv gz + c_p\mathcal{T}(1+\alpha)\ \ln\frac{\eta^\kappa + \alpha}{1+\alpha} \tag{20}$$

is *independent* of pressure. This fact is essential in finding an expression for the vertical velocity and in obtaining an energy invariant, as we will see below.

3. VERTICAL VELOCITY

Equations (3) constitute the diagnostic part of the hydrostatic primitive equations. Using pressure as a vertical coordinate also the mass conservation equation assumes a diagnostic form, in which case case the dynamical part consists of the equations for the thermodynamic energy and the momentum. Using η instead of p and $\dot{\eta} = \omega/p_r$, where ω is the vertical velocity in pressure coordinates, we have for the mass conservation equation, momentum equation and thermodynamic energy equation

$$\begin{aligned}
&\frac{\partial\dot{\eta}}{\partial\eta} + \nabla_\eta \cdot \mathbf{v} = 0,\\
&\frac{D_\eta\mathbf{v}}{Dt} + \dot{\eta}\frac{\partial\mathbf{v}}{\partial\eta} = -f\mathbf{k}\times\mathbf{v} - \nabla_\eta\Phi + \mathbf{F},\\
&\frac{D_\eta\theta}{Dt} + \dot{\eta}\frac{\partial\theta}{\partial\eta} = \frac{\theta}{c_pT}Q.
\end{aligned} \tag{21}$$

In these equations the subscript η added to the differential operators means that the time and horizontal derivatives - the latter with respect to longitude λ and latitude ϕ - are to be taken at constant η. As is common in the hydrostatic primitive equations, in the different metric coefficients the radial coordinate r is replaced by the average radius of the earth a. Furthermore, $\mathbf{v}$ is the horizontal velocity, $f = 2\Omega \sin\phi$ is the Coriolis parameter where Ω is the angular velocity of the earth's rotation, and $\Phi = gz$ is the geopotential. The symbol $\mathbf{F}$ denotes mechanical friction and Q the heating. It will be assumed that the heating Q is such that the layer retains the given value of the buoyancy frequency in the course of time.

Now, for the pressure gradient term in the momentum equation we may write

$$\nabla_\eta \Phi = \nabla_\eta \Psi = \nabla \Psi. \tag{22}$$

The first equality follows from the fact that the second term in the expression of Ψ depends only on pressure. Furthermore, the full expression of Ψ is independent of pressure so that any subscript referring to a vertical coordinate is superfluous when evaluating the horizontal gradient of Ψ. This is expressed by the second equality. As a consequence, we may conclude that the pressure gradient term is also independent of pressure. Because the Coriolis parameter is independent of pressure this then implies that $\mathbf{v}$ is independent of pressure as well if this is assumed to be the case initially. We may thus assume that

$$\frac{\partial \mathbf{v}}{\partial \eta} = 0, \tag{23}$$

which allows us to solve the continuity equation in pressure coordinates in a very straightforward way. Because the horizontal velocity is independent of pressure we have that also the horizontal divergence is independent of pressure and can be written as $\nabla \cdot \mathbf{v}$, i.e., without the subscript η attached to the horizontal divergence operator. The solution of the continuity equation can then simply be written in either of the two following ways.

$$\begin{aligned} \dot{\eta} &= \dot{\eta}_l - (\nabla \cdot \mathbf{v})(\eta - \eta_l), \\ \dot{\eta} &= \dot{\eta}_u - (\nabla \cdot \mathbf{v})(\eta - \eta_u). \end{aligned} \tag{24}$$

For $\dot{\eta}_l$ and $\dot{\eta}_u$ we have

$$\begin{aligned} \dot{\eta}_l &= \frac{D\eta_l}{Dt} - \mathcal{F}_l, \\ \dot{\eta}_u &= \frac{D\eta_u}{Dt} - \mathcal{F}_u, \end{aligned} \tag{25}$$

where D/Dt denotes the horizontal material derivative, written without a subscript η because it is applied to fields that do not depend on the vertical coordinate†. The terms $\mathcal{F}_l$ and $\mathcal{F}_u$ are present if the surfaces are not material so that there is a flow of mass $(p_r/g)\mathcal{F}$ (kgm^{-2}s^{-1}) through the surface, taken positive when upward. If the first of (24) is evaluated at the upper boundary or the second of (24) at the lower boundary of the layer and use is made of (25) we obtain

$$\frac{D}{Dt}(\eta_l - \eta_u) + (\nabla \cdot \mathbf{v})(\eta_l - \eta_u) = \mathcal{F}_l - \mathcal{F}_u, \tag{26}$$

† In Verkley (2000) we retained the subscript on the horizontal material derivative even if this was not strictly necessary. Skipping the subscript from the material derivative does not lead to confusion with the full material derivative when it is applied to a field that is independent of height.

which is the mass conservation equation for the whole layer. For the vertical velocity $\dot{\eta}$ inside the layer we may combine the first expression of (24) with the first expression of (25) to find

$$\dot{\eta} = \frac{D\eta_l}{Dt} - \mathcal{F}_l + (\nabla \cdot \mathbf{v})(\eta_l - \eta). \tag{27}$$

A completely analogous expression, with the subscripts u and l interchanged, is obtained when the second expression of (24) is combined with the second expression of (25).

In order to obtain the vertical verlocity w in height coordinates we use the general expression

$$w = \frac{D_\eta z}{Dt} + \dot{\eta}\frac{\partial z}{\partial \eta}. \tag{28}$$

Substituting (11) into (28) we obtain

$$w = \frac{Dz_l}{Dt} + \kappa\, H\, (1+\alpha)[\frac{\eta_l^{\kappa-1}}{\eta_l^\kappa + \alpha}\,\frac{D\eta_l}{Dt} - \frac{\eta^{\kappa-1}}{\eta^\kappa + \alpha}\,\dot{\eta}]. \tag{29}$$

Then substituting (27) gives

$$\begin{aligned} w = \frac{Dz_l}{Dt} + \kappa\, H\, (1+\alpha)[&\frac{\eta_l^{\kappa-1}}{\eta_l^\kappa + \alpha}\,\frac{D\eta_l}{Dt} - \frac{\eta^{\kappa-1}}{\eta^\kappa + \alpha}\,\frac{D\eta_l}{Dt} + \\ &\frac{\eta^{\kappa-1}}{\eta^\kappa + \alpha}\mathcal{F}_l - \frac{\eta^{\kappa-1}}{\eta^\kappa + \alpha}(\nabla \cdot \mathbf{v})\eta_l + \frac{\eta^{\kappa-1}}{\eta^\kappa + \alpha}(\nabla \cdot \mathbf{v})\eta]. \end{aligned} \tag{30}$$

After rearranging terms we may obtain the following form

$$\begin{aligned} w = &\frac{Dz_l}{Dt} + \kappa\, H\, \frac{1+\alpha}{\eta_l^\kappa + \alpha}\eta_l^{\kappa-1}\mathcal{F}_l + \\ &\kappa\, H\, [\nabla \cdot \mathbf{v}][\frac{1+\alpha}{\eta^\kappa + \alpha}\eta^\kappa - \frac{1+\alpha}{\eta_l^\kappa + \alpha}\eta_l^\kappa] + \\ &\kappa\, H\, [\frac{1+\alpha}{\eta_l^\kappa + \alpha}\eta_l^{\kappa-1} - \frac{1+\alpha}{\eta^\kappa + \alpha}\eta^{\kappa-1}][\frac{D\eta_l}{Dt} + (\nabla \cdot \mathbf{v})\eta_l - \mathcal{F}_l]. \end{aligned} \tag{31}$$

This is our expression for the vertical velocity w. It reduces to the corresponding expression (42) in Verkley (2000) in the case $\mathcal{F}_l = 0$ if the limit $\alpha \to \infty$ is taken. This is verified directly by noting that the second factor between square brackets, in the third term on the right-hand side of the equation above, reduces to the expression of $(z_l - z)/H$ for an isentropic layer, as given by (21) in Verkley (2000). We note that at $z = z_l$ the equation above reduces to

$$w_l = \frac{Dz_l}{Dt} + \kappa\, H\, \frac{1+\alpha}{\eta_l^\kappa + \alpha}\eta_l^{\kappa-1}\mathcal{F}_l = \frac{Dz_l}{Dt} + \frac{1}{\rho_l}\frac{p_r}{g}\mathcal{F}_l, \tag{32}$$

where ρ_l is the density at the lower boundary. By using the layer mass conservation equation (26) in combination with expression (12) for $z_u - z_l$ it can be shown that we have at $z = z_u$

$$w_u = \frac{Dz_u}{Dt} + \kappa\, H\, \frac{1+\alpha}{\eta_u^\kappa + \alpha}\eta_u^{\kappa-1}\mathcal{F}_u = \frac{Dz_u}{Dt} + \frac{1}{\rho_u}\frac{p_r}{g}\mathcal{F}_u, \tag{33}$$

where ρ_u is the density at the upper boundary. The second terms in the expressions just derived are contributions to the vertical velocity w resulting from mass flow through the boundaries. If the boundaries are material surfaces then these contributions are zero. The validity of the two expressions above can alternatively be deduced from the symmetry in the treatment of the upper and lower boundaries. We note that for an isothermal layer, in which $\alpha \to 0$, we have from (31)

$$w = \frac{Dz_l}{Dt} + \kappa\, H\, \eta_l^{-1}\mathcal{F}_l + \kappa\, H\, [\eta_l^{-1} - \eta^{-1}][\frac{D\eta_l}{Dt} + (\nabla \cdot \mathbf{v})\eta_l - \mathcal{F}_l]. \tag{34}$$

For a single, isothermal layer with zero pressure and no mass flow trough the upper boundary (which for an isothermal layer is at infinite height) the layer mass continuity equation (26) implies that the last factor between square brackets is zero. The vertical velocity in a single unbounded isothermal layer is therefore independent of height and equal to the vertical velocity at the lower boundary.

Given the potential temperature as a function of pressure, we may use the last equation of (21) in combination with (27) to find an expression of $\dot{\theta}$, i.e., of the vertical velocity in θ-coordinates and therewith of the heating Q that is necesaary to keep the layer at a fixed value of the buoyancy frequency. The resulting expression is analogous to (31) and reads

$$\begin{aligned} \frac{\dot{\theta}}{\theta} = \frac{1}{\theta_l}\frac{D\theta_l}{Dt} + \kappa\, \frac{\eta_l^{\kappa-1}}{\eta_l^{\kappa} + \alpha}\mathcal{F}_l + \\ \kappa\, [\nabla \cdot \mathbf{v}][\frac{\eta^{\kappa}}{\eta^{\kappa} + \alpha} - \frac{\eta_l^{\kappa}}{\eta_l^{\kappa} + \alpha}] + \\ \kappa\, [\frac{\eta_l^{\kappa-1}}{\eta_l^{\kappa} + \alpha} - \frac{\eta^{\kappa-1}}{\eta^{\kappa} + \alpha}][\frac{D\eta_l}{Dt} + (\nabla \cdot \mathbf{v})\eta_l - \mathcal{F}_l]. \end{aligned} \tag{35}$$

The expression of Q is obtained by multiplying the right-hand side of this equation by c_pT. This heating should be considered as part of the formulation of the layer's dynamics. Note that for an isentropic layer, in which θ_l is constant and $\alpha \to \infty$, the implied heating is zero - as it should.

4. Horizontal dynamics

We will now discuss the horizontal dynamics of a layer with constant buoyancy frequency. The horizontal velocity field $\mathbf{v}$ changes in time according to the momentum equation in (21). If we use (22) and (23) we may write

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} + \nabla\Psi = \mathbf{F}. \tag{36}$$

This equation can also be written as

$$\frac{\partial \mathbf{v}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{v} + \nabla(\Psi + \frac{1}{2}\mathbf{v}^2) = \mathbf{F}, \tag{37}$$

where $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v}$ is the vorticity of the horizontal velocity field. By defining $\mathcal{F} = \mathcal{F}_l - \mathcal{F}_u$ and $\mu = \eta_l - \eta_u$, the mass conservation equation (26) reads

$$\frac{D\mu}{Dt} + \mu(\nabla \cdot \mathbf{v}) = \mathcal{F}. \tag{38}$$

This equation can also be written as

$$\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu\mathbf{v}) = \mathcal{F}. \tag{39}$$

The momentum equation and the mass conservation equation, together with expressions for Ψ and the different forcing terms, completely determine the time-evolution of a layer with constant buoyancy frequency. We repeat here that the heating Q in the third equation of (21) is assumed to be given by c_pT times (35), so that the buoyancy frequency of the layer is kept fixed at the given value.

By applying the operators $\mathbf{k}\cdot\nabla\times$ and $\nabla\cdot$ to (37) we obtain equations for the vorticity ζ and the divergence $\mathcal{D}=\nabla\cdot\mathbf{v}$

$$\begin{aligned}\frac{\partial\zeta}{\partial t}+\nabla\cdot[(f+\zeta)\mathbf{v}]&=\mathbf{k}\cdot\nabla\times\mathbf{F},\\ \frac{\partial\mathcal{D}}{\partial t}+\nabla\cdot[(f+\zeta)(\mathbf{k}\times\mathbf{v})+\nabla(\Psi+\frac{1}{2}\mathbf{v}^2)]&=\nabla\cdot\mathbf{F}.\end{aligned}\tag{40}$$

The vorticity equation can be rewritten as

$$\frac{D}{Dt}(f+\zeta)+(f+\zeta)(\nabla\cdot\mathbf{v})=\mathbf{k}\cdot\nabla\times\mathbf{F}\tag{41}$$

and the latter equation can be combined with the mass conservation equation (38) to eliminate the horizontal divergence. This gives an equation for the potential vorticity P of the layer

$$\frac{DP}{Dt}=\mathcal{S},\qquad P=\frac{f+\zeta}{\mu},\tag{42}$$

where the source $\mathcal{S}$ is given by

$$\mathcal{S}=\frac{1}{\mu}[\mathbf{k}\cdot\nabla\times\mathbf{F}-P\mathcal{F}].\tag{43}$$

This equation can be used, e.g., as a starting point in deriving a quasi-geostrophic approximation of the layer's dynamics.

We will now consider the total energy of the layer. The total energy per unit area is $(p_r/g)\mathcal{E}$, where $\mathcal{E}$ is given by

$$\begin{aligned}\mathcal{E}&=\eta_l\,gz_l-\eta_u\,gz_u+c_p\int_{\eta_u}^{\eta_l}T\,d\eta+\mu\frac{1}{2}\mathbf{v}^2=\\ &\eta_l\,gz_l-\eta_u\,gz_u+c_p\mathcal{T}\int_{\eta_u}^{\eta_l}\frac{1+\alpha}{1+\alpha\eta^{-\kappa}}\,d\eta+\mu\frac{1}{2}\mathbf{v}^2,\end{aligned}\tag{44}$$

and where in the last equality we substituted (8) for the temperature T. Because, as remarked above, a heating Q is assumed to keep the layer at the fixed value of the buoyancy frequency there is no reason to assume that this expression of the energy is conserved, even in the absense of the forcing terms $\mathcal{F}$ and $\mathbf{F}$. We will show, however, that there is an expression of the energy that is conserved when $\mathcal{F}$ and $\mathbf{F}$ are zero. To find this expression we replace the third term on the right-hand side of the last equality by $\chi(\eta_l)-\chi(\eta_u)$, where the function χ is to be determined. This gives for $\mathcal{E}$ the following expression.

$$\mathcal{E}=\eta_l\,gz_l-\eta_u\,gz_u+\chi(\eta_l)-\chi(\eta_u)+\mu\frac{1}{2}\mathbf{v}^2.\tag{45}$$

Taking the time-derivative of $\mathcal{E}$ we obtain

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial t}&=\frac{\partial\eta_l}{\partial t}\,(gz_l)+\eta_l\,\frac{\partial(gz_l)}{\partial t}-\frac{\partial\eta_u}{\partial t}\,(gz_u)-\eta_u\,\frac{\partial(gz_u)}{\partial t}+\\ &\frac{d\chi(\eta_l)}{d\eta_l}\frac{\partial\eta_l}{\partial t}-\frac{d\chi(\eta_u)}{d\eta_u}\frac{\partial\eta_u}{\partial t}+\frac{\partial\mu}{\partial t}\,\frac{1}{2}\mathbf{v}^2+\mu\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}.\end{aligned}\tag{46}$$

By using the momentum equation and the mass conservation equation in the form of (37) and (39), respectively, we have

$$\begin{gathered}\frac{\partial \mu}{\partial t}\,\frac{1}{2}\mathbf{v}^2 + \mu \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial t} = \\ \mathcal{F}[\Psi + \frac{1}{2}\mathbf{v}^2] + \mu\mathbf{v}\cdot\mathbf{F} - \nabla\cdot[\mu\mathbf{v}(\Psi + \frac{1}{2}\mathbf{v}^2)] - \Psi\frac{\partial \mu}{\partial t}.\end{gathered} \tag{47}$$

We thus get

$$\begin{gathered}\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial \eta_l}{\partial t}\,[g z_l + \frac{d\chi(\eta_l)}{d\eta_l} - \Psi] - \frac{\partial \eta_u}{\partial t}\,[g z_u + \frac{d\chi(\eta_u)}{d\eta_u} - \Psi] + \\ \eta_l\frac{\partial (g z_l)}{\partial t} - \eta_u\frac{\partial (g z_u)}{\partial t} + \mathcal{F}[\Psi + \frac{1}{2}\mathbf{v}^2] + \mu\mathbf{v}\cdot\mathbf{F} - \nabla\cdot[\mu\mathbf{v}(\Psi + \frac{1}{2}\mathbf{v}^2)].\end{gathered} \tag{48}$$

The last term in the equation above is a divergence. When integrated over the whole horizontal area, assuming suitable horizontal boundary conditions, this term vanishes. The fifth and sixth term are due to the forcing terms and vanish when this forcing vanishes. The third and fourth term are exchange terms between the atmosphere below and the atmosphere above the layer in question. With a fixed orography and a zero upper boundary pressure, these terms cancel out. Now, using that Ψ as given by (20) is independent of pressure and can be evaluated at either the lower and upper boundary of the layer we can write

$$\begin{aligned} g z_l + \frac{d\chi(\eta_l)}{d\eta_l} - \Psi &= \frac{d\chi(\eta_l)}{d\eta_l} - c_p\mathcal{T}(1+\alpha)\ln\frac{\eta_l^\kappa + \alpha}{1+\alpha}, \\ g z_u + \frac{d\chi(\eta_u)}{d\eta_u} - \Psi &= \frac{d\chi(\eta_u)}{d\eta_u} - c_p\mathcal{T}(1+\alpha)\ln\frac{\eta_u^\kappa + \alpha}{1+\alpha}.\end{aligned} \tag{49}$$

This implies that, if we take for χ a function such that the expressions on the right-hand sides of the equations above are zero, we have an energy invariant. A function $\chi(\eta)$ that satisfies

$$\frac{d\chi}{d\eta} = c_p\mathcal{T}(1+\alpha)\ln\frac{\eta^\kappa + \alpha}{1+\alpha} \tag{50}$$

gives us this energy invariant. The solution of the equation above can be expressed in terms of the hypergeometric function $F(a, b; c; z)$, see Chapter 15 of Abramowitz and Stegun (1965). In the appendix it will be checked that we have, apart from an arbitrary integration constant,

$$\chi(\eta) = c_p\mathcal{T}(1+\alpha)\,\eta\,[-\kappa + \kappa\,F(\frac{1}{\kappa}, 1; 1+\frac{1}{\kappa}; -\frac{\eta^\kappa}{\alpha}) + \ln\frac{\eta^\kappa+\alpha}{1+\alpha}]. \tag{51}$$

For the true energy of the layer the function χ would be defined by (50) with a right-hand side given by c_pT, where T is given by (8). The integral of this function, denoted by $\xi(\eta)$, is given by

$$\xi(\eta) = c_p\mathcal{T}(1+\alpha)\,\eta\,[1 - F(\frac{1}{\kappa}, 1; 1+\frac{1}{\kappa}; -\frac{\eta^\kappa}{\alpha})]. \tag{52}$$

This is also verified in the appendix. The latter expression might be used to calculate the flow of energy involved in keeping the layer at the fixed value of the buoyancy frequency. This is, in fact, an alternative to integrating (35), after multiplying by c_pT, over the layer's depth.

We conclude this section by demonstrating how models of the atmosphere may be constructed on the basis of N layers with constant buoyancy frequency. We will assume that the layers are numbered from 0 to $N-1$ and the boundaries from 0 to N, counting from top to bottom. This means that we use the convention that the upper boundary of layer i is denoted by the index i and the lower boundary of the layer by the index $i+1$‡. We also assume that the pressure at the upper boundary of the uppermost layer is zero ($\eta_0 = 0$) and that the height of the lower boundary of the lowermost layer is fixed by the orography ($z_N = z_B$). The momentum and mass conservation equations in the layer i, $i = 0, ..., N-1$ are

$$\begin{aligned} \frac{D\mathbf{v}^i}{Dt} + f\mathbf{k} \times \mathbf{v}^i + \nabla\Psi^i = \mathbf{F}^i, \\ \frac{D\mu^i}{Dt} + \mu^i(\nabla \cdot \mathbf{v}^i) = \mathcal{F}^i. \end{aligned} \tag{53}$$

For the potentials Ψ^i we have

$$\begin{aligned} \Psi^i &= gz_{i+1} + c_p\mathcal{T}^i(1+\alpha^i)\ \ln \frac{\eta_{i+1}^\kappa + \alpha^i}{1+\alpha^i}, \\ g(z_i - z_{i+1}) &= c_p\mathcal{T}^i(1+\alpha^i)\ \ln \frac{\eta_{i+1}^\kappa + \alpha^i}{\eta_i^\kappa + \alpha^i}, \end{aligned} \tag{54}$$

in which expressions η_i and η_{i+1} can be obtained from

$$\mu^i = \eta_{i+1} - \eta_i. \tag{55}$$

The potential Ψ^{N-1} might be calculated directly from the expression for $i = N-1$, the height $z_N = z_B$ being fixed by the orography. The potential Ψ^{N-2} might then be obtained by the expression for $i = N-2$ in which the height z_{N-1} is evaluated in terms of z_N by using the expression for the height difference for $i = N-1$. This can done consecutively for all the other layers. Together with the boundary conditions already mentioned this system is a closed dynamical system. The structure inside the layers in terms of temperature, pressure, density and vertical velocity as functions of height z can be obtained from the formulas given above.

5. Conclusions

We have analysed the dynamics of a hydrostatic layer of air which is assumed to have a spatially uniform value of the Brunt-Väisälä frequency or buoyancy frequency N. The heating Q in the prognostic equation for the potential temperature is assumed to keep the air in the layer fixed at the given value of the buoyancy frequency. We have analyzed the relation between temperature T and pressure p in a constant N^2 layer and found it to be idential to the maximum entropy profile obtained by Verkley and Gerkema (2004). From the relation between T and p also the relation between z and p can be found. From this relation it follows that the potential Ψ, defined in (20), is independent of height in a layer with constant buoyancy frequency. This implies that the pressure gradient term in the horizontal momentum equation is independent of height and therefore, as direct consequence, the horizontal velocity can be assumed

‡ In Verkley (2000) we used the same convention for the boundaries but numbered the layers from 1 to N, in which case the upper boundary of layer i is denoted by $i-1$ and the lower boundary by i.

independent of height if this is assumed to be the case at some initial time. This allows us to obtain the pressure tendency ω as a function of pressure p and, using the relation between p and z, the vertical velocity w as a function of height z. The derivation given here is simpler and more general than the derivation given by Verkley (2000). It is not limited to the specal case of an isentropic layer and account is taken of the possible presence of mass fluxes through the upper and lower boundaries of the layer. We have studied the horizontal dynamics of a layer with constant buoyancy frequency and identified an energy invariant. We have outlined how an atmospheric model can be constructed on the basis of layers with constant buoyancy frequency.

Appendix

Mathematical details

Eqs. (51) and (52) were found by using the website http://integrals.wolfram.com/. In the following we will check the validity of the resulting expressions. The central identity that we need in verifying both (51) and (52) is

$$F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; z) + \kappa z \frac{dF}{dz}(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; z) = \frac{1}{1 - z}. \tag{A.1}$$

This identity can be proved by calculating dF/dz by using (15.2.1) from Abramowitz and Stegun (1965), then transforming both F and dF/dz by applying (15.3.3) and, finally, using (15.2.12) in combination with (15.1.8).

To verify (51) we need to prove the validity of

$$\frac{d}{d\eta}\, \eta[-\kappa + \kappa F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; -\frac{\eta^\kappa}{\alpha}) + \ln \frac{\eta^\kappa + \alpha}{1 + \alpha}] = \ln \frac{\eta^\kappa + \alpha}{1 + \alpha}. \tag{A.2}$$

Carrying out the differentiation we find, defining $z = -\eta^\kappa/\alpha$,

$$\begin{aligned} \frac{d}{d\eta}\, \eta[-\kappa + \kappa F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; -\frac{\eta^\kappa}{\alpha}) + \ln \frac{\eta^\kappa + \alpha}{1 + \alpha}] = \ln \frac{\eta^\kappa + \alpha}{1 + \alpha} + \\ \kappa[\frac{-1}{1 - z} + F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; z) + \kappa z \frac{dF}{dz}(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; z)], \end{aligned} \tag{A.3}$$

which leads to the desired result upon substituting (A.1). To verify (52) we need to prove the validity of

$$\frac{d}{d\eta}\, \eta[1 - F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; -\frac{\eta^\kappa}{\alpha})] = \frac{1}{1 + \alpha\eta^{-\kappa}}. \tag{A.4}$$

Carrying out the differentiation we find

$$\begin{aligned} \frac{d}{d\eta}\, \eta[1 - F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; -\frac{\eta^\kappa}{\alpha})] = 1 - \\ F(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; z) - \kappa z \frac{dF}{dz}(\frac{1}{\kappa}, 1; 1 + \frac{1}{\kappa}; z). \end{aligned} \tag{A.5}$$

Substituting (A.1), using $z = -\eta^\kappa/\alpha$, gives the desired result.

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