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Comment on "Complete Eulerian-mean tracer equation for coarse resolution OGCMs" by M. S. Dubovikov and V. M. Canuto

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Dubovikov and Canuto (Dubovikov, M.S. and Canuto, V.M., Complete Eulerian-mean tracer equation for coarse resolution OGCMs. *Geophys. Astrophys. Fluid Dyn.*, 2006, **100**, 197–214), after averaging the tracer conservation equation in density coordinates and transforming to height coordinates, concluded that present ocean models are missing important terms in their mean tracer equations. Here we point out some errors made by Dubovikov and Canuto (2006) in their isopycnal averaging procedure. We draw on the temporal-residual-mean (TRM) theory and show that when the isopycnal averaging and coordinate transformation are performed correctly, the tracer equations of present ocean circulation models are recovered. We therefore conclude that present ocean circulation models are not neglecting the leading order terms identified by Dubovikov and Canuto (2006).

Keywords: Mixing; Turbulence

1. Introduction

Dubovikov and Canuto (2006) (hereinafter DC06) and Canuto and Dubovikov (2007) examined the Reynolds-averaged tracer equation (X being the flux divergence or source term due to unresolved processes)

$$\bar{\tau}_t + \bar{\mathbf{V}} \cdot \nabla_H \bar{\tau} + \bar{w}\bar{\tau}_z = \bar{X} - \nabla \cdot \overline{\mathbf{U}'\tau'}, \tag{1}$$

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and adopting the eddy-induced velocity $\mathbf{U}^* = \mathbf{V}^* + \mathbf{k}w^*$ (where γ is a suitable density variable such as a locally-referenced potential density)

$$\mathbf{V}^* = \left(-\frac{\overline{\mathbf{V}'\gamma'}}{\bar{\gamma}_z}\right)_z, \quad \text{and} \quad w^* = -\nabla_H \cdot \left(-\frac{\overline{\mathbf{V}'\gamma'}}{\bar{\gamma}_z}\right), \tag{2}$$

they rearranged the tracer equation into the form [their equation (7a)]

$$\bar{\tau}_t + \left(\bar{\mathbf{V}} + \mathbf{V}^*\right) \cdot \nabla_H \bar{\tau} + (\bar{w} + w^*)\bar{\tau}_z = \nabla \cdot (\mathbf{S}\nabla\bar{\tau}) - \Sigma^{\tau}_z. \tag{3}$$

Here the symmetric diffusion tensor $\mathbf{S}$ includes both the small-scale (diabatic) turbulence and the epineutral mesoscale and sub-mesoscale eddy tracer diffusion as is often parameterized by the down-gradient Redi (1982) tensor. They then point out that the vertical tracer flux Σ^{τ} is too large to ignore (we agree with this) and that present OGCMs are in error because they do not employ a specific parameterization for Σ^{τ} (we do not agree with this). Dubovikov and Canuto (2005), Canuto and Dubovikov (2006) and Canuto and Dubovikov (2007) make similar claims about the mean density equation in OGCMs. In McDougall *et al.* (2007) we refute these claims that the *density equation* of ocean models is missing an important diapycnal flux term. Here we disprove the claim of DC06 that the *tracer equations* of present OGCMs are missing leading order terms. Also, while the source term $-\Sigma^{\tau}_z$ in DC06's equation (7a) is given as the divergence of a vertical flux, we derive this source term in appendix A and show that it contains many more terms than appears in DC06 and in Canuto and Dubovikov (2007), and that it cannot be represented in terms of a purely vertical flux. The corrected version of (3) is to be found in appendix A as equations (A.1)–(A.2).

Because diapycnal mixing is so weak in the ocean compared with the rather energetic mesoscale eddy field, it has proven very valuable to average the conservation equations in density coordinates in order to preserve the fidelity of the small effects of diapycnal mixing. When the equations are averaged in different coordinates, not only are extra "eddy-induced" velocities introduced, but the physical interpretations of all the scalar variables also change. The concept of averaging along density surfaces motivated the Gent and McWilliams (1990) and Gent *et al*. (1995) papers and, particularly, the Temporal-Residual-Mean (TRM) approach of McDougall and McIntosh (2001) (hereinafter called MM01) which accurately achieves the desired clean separation between epineutral and dianeutral processes. The major benefit to ocean models from implementing the Gent *et al.* (1995) parameterization of the quasi-Stokes streamfunction is that the "adiabatic" property is achieved. That is, with this implementation of the TRM theory, the component of the total velocity through the model's density surfaces is caused only by the explicit diapycnal diffusivity. Before the paper of Gent and McWilliams (1990) this was not possible; rather, unphysical diapycnal diffusion and advection occurred as a result of the unavoidable horizontal diffusion. We consider it of paramount importance that this "adiabatic" property be achieved by eddy parameterizations. Unfortunately, as shown by McDougall *et al.* (2007), the averaging process and the parameterization scheme of Canuto and Dubovikov (2006), Canuto and Dubovikov (2007), Dubovikov and Canuto (2005) and DC06 does not achieve this "adiabatic" property.

Despite averaging in density coordinates, DC06 did not adopt the thickness-weighted tracer $\hat{\tau}$ of density-coordinate averaging; rather they continued to cast their equations in terms of the Eulerian-mean tracer $\bar{\tau}$. Also, their total velocity $\bar{\mathbf{U}} + \mathbf{U}^*$ that appears in (3) is not the same as the total three-dimensional velocity that results from averaging in density coordinates. In addition to their choice of a mean tracer definition that was not consistent with their stated aim of averaging in density coordinates, DC06 dropped leading-order terms in their equations (1e) and (4c), and chose an erroneous balance of terms in their buoyancy variance equation (11b). Their assumption that the perturbation velocity at constant density is the same as that at constant depth [their (1e) and (4c)] is probably their most serious error. The same error is made at equation (4a) of Canuto and Dubovikov (2007) and invalidates the results of that paper.

In section 5 of MM01, the tracer equation has been carefully averaged in density coordinates, and then expressed in *z*-coordinates. Here we briefly review these results and show that, contrary to the conclusions of DC06, the tracer equations of present OGCMs are not missing important terms.

2. Isopycnal averaging

We write the conservation equations for density γ and tracer τ after averaging over small-scale turbulent mixing processes but before averaging over the mesoscale perturbations as

$$\gamma_t + \mathbf{V} \cdot \nabla_H \gamma + w\gamma_z = Q = e\gamma_z \tag{4}$$

and

$$\tau_t + \mathbf{V} \cdot \nabla_H \tau + w\tau_z = X, \tag{5}$$

which serves to define the diapycnal velocity *e* and the source terms *Q* and *X*. When the tracer equation is averaged in density coordinates it becomes (see equation (36) of MM01)

$$\left(\frac{\hat{\tau}}{\tilde{\gamma}_z}\right)_t + \nabla_{\tilde{\gamma}} \cdot \frac{\hat{\mathbf{V}}\hat{\tau}}{\tilde{\gamma}_z} + (\tilde{e}\hat{\tau})_{\tilde{\gamma}} = \frac{\hat{X}}{\tilde{\gamma}_z} - \overline{Q''\tau''/\gamma_z} - \nabla_{\tilde{\gamma}} \cdot \overline{(\mathbf{V}''\tau''/\gamma_z)}, \tag{6}$$

where the long over-bar and the over-tilde both represent temporal averaging at constant density, the double primes are perturbations at constant density, and the hat means the thickness-weighted temporal average at constant density. The traditional down-gradient diffusion of tracer in the vertical and along density surfaces means that the right-hand side of (6) is represented in *z*-coordinates as $1/\tilde{\gamma}_z$ times $\nabla \cdot (\mathbf{S}\nabla\hat{\tau})$ where $\mathbf{S}$ is a symmetric diffusion tensor.

Having thus performed the averaging of the tracer conservation equation in density coordinates, one needs to express the three terms on the right of (6) in terms of perturbations at fixed height (which are denoted by a single prime). These expressions can be found in equations (28), (40) and (41) of MM01. They contain more terms (three terms, four terms and four terms respectively) than retained by DC06 because MM01 did not discard leading-order terms. In particular, DC06 discard all but the leading term in (28) of MM01 and so miss the fact that the tracer that appears after averaging in density coordinates $\hat{\tau}$ is different from the Eulerian-mean tracer $\bar{\tau}$. Also, because of the assumption [at their equations (1e) and (4c)] that the perturbation velocity in density coordinates is the same as the velocity perturbation in z-coordinates, the last two of the four terms in (41) of MM01 are discarded. These two discarded terms appear in equation (A.4) of appendix A as the four missing fluxes that each involve the mean vertical shear $\bar{\mathbf{V}}_z$. These approximations have led both DC06 and Canuto and Dubovikov (2007) to incorrect conclusions regarding the epineutrally-averaged tracer conservation equation.

3. The TRM tracer equation

The thickness-weighted tracer that results from averaging in density coordinates is given in terms of the Eulerian-averaged tracer (up to second order in perturbation quantities) by

$$\hat{\tau} = \bar{\tau} + \left(-\frac{\overline{\tau'\gamma'}}{\bar{\gamma}_z} + \frac{\bar{\tau}_z \bar{\phi}}{\bar{\gamma}_z^2} \right)_z + \mathrm{O}(\alpha^3), \tag{7}$$

where γ' is the density perturbation at fixed height and $\bar{\phi}$ is half the mean density variance at fixed height. Using (7), MM01 rearranged the Eulerian-mean tracer equation (1) into the form [see their (44)–(46) and (53)]

$$\hat{\tau}_t + \bar{\mathbf{V}} \cdot \nabla_H \hat{\tau} + \bar{w}\hat{\tau}_z = \nabla \cdot (\mathbf{S}\nabla\bar{\tau}) - \nabla \cdot \mathbf{F}^{M\tau} + \mathrm{O}(\alpha^3), \tag{8}$$

where the modified flux of tracer $\mathbf{F}^{M\tau}$ is given by the rather long expression

$$\begin{aligned}
\mathbf{F}^{M\tau} \equiv\; & \overline{\mathbf{U}'\tau'} - \bar{\mathbf{U}}\big(-\overline{\tau'\gamma'}/\bar{\gamma}_z + \bar{\tau}_z\bar{\phi}/\bar{\gamma}_z^2\big)_z \\
& - \big(\overline{\mathbf{V}'\tau'} - \bar{\tau}_z\overline{\mathbf{V}'\gamma'}/\bar{\gamma}_z - \bar{\mathbf{V}}_z\overline{\tau'\gamma'}/\bar{\gamma}_z + 2\bar{\mathbf{V}}_z\bar{\tau}_z\bar{\phi}/\bar{\gamma}_z^2\big) \\
& + \mathbf{k}\Big\{-\big(-\overline{\tau'\gamma'}/\bar{\gamma}_z + \bar{\tau}_z\bar{\phi}/\bar{\gamma}_z^2\big)_t - \overline{X'\gamma'}/\bar{\gamma}_z + \bar{X}_z\bar{\phi}/\bar{\gamma}_z^2\Big\} \\
& + \mathbf{k}\big\{-\overline{Q'\gamma'}/\bar{\gamma}_z + \bar{\tau}_z\overline{Q'\gamma'}/\bar{\gamma}_z^2 + \bar{Q}_z\overline{\tau'\gamma'}/\bar{\gamma}_z^2 - 2\bar{Q}_z\bar{\tau}_z\bar{\phi}/\bar{\gamma}_z^3\big\} \\
& + \mathbf{k}\big\{\big(\overline{\mathbf{V}'\tau'} - \bar{\tau}_z\overline{\mathbf{V}'\gamma'}/\bar{\gamma}_z - \bar{\mathbf{V}}_z\overline{\tau'\gamma'}/\bar{\gamma}_z + 2\bar{\mathbf{V}}_z\bar{\tau}_z\bar{\phi}/\bar{\gamma}_z^2\big) \cdot \nabla_H\bar{\gamma}/\bar{\gamma}_z\big\}.
\end{aligned} \tag{9}$$

The first line of the modified tracer flux expression (9) contains three terms that have both horizontal and vertical components, the second line contains four horizontal fluxes and the remainder of (9) contains twelve vertical fluxes. We emphasize that there have been no approximations made in arriving at this TRM tracer equations (8) and (9): it is accurate up to and including second order in perturbation quantities. The divergence of $\mathbf{F}^{M\tau}$ represents the non-diffusive (ie advective) effects of the stirring by mesoscale eddies on the mean tracer equation (8). The stated aim of DC06 was "to find the optimal decomposition of the adiabatic component of the mesoscale tracer flux in the z-coordinate mean tracer equation". This is in fact achieved by the TRM tracer conservation equations (8) and (9), and DC06 would have arrived at the same equations (8) and (9) above, or the form (13) or (A.1)–(A.2) below) if they had not dropped leading order terms from several of their equations.

MM01 used the conservation equations for density variance and for $\overline{\tau'\gamma'}$ to show that the modified tracer flux $\mathbf{F}^{M\tau}$ (9) is equal to

$$\mathbf{F}^{M\tau} = \hat{\tau}\mathbf{U}^{+} + \mathbf{M}^{\tau} + \mathrm{O}(\alpha^3), \tag{10}$$

where $\mathbf{M}^{\tau}$ is a non-divergent flux, the eddy-induced velocity is

$$\mathbf{U}^{+} = \mathbf{V}^{+} + \mathbf{k}w^{+} \equiv \nabla \times (\mathbf{\Psi} \times \mathbf{k}) = \Psi_z - \mathbf{k}(\nabla_H \cdot \mathbf{\Psi}) \tag{11}$$

and the quasi-Stokes streamfunction is defined by

$$\mathbf{\Psi} \equiv \overline{\int_z^{z+z'} \mathbf{V}\mathrm{d}z''} \approx -\frac{\overline{\mathbf{V}'\gamma'}}{\bar{\gamma}_z} + \frac{\bar{\mathbf{V}}_z}{\bar{\gamma}_z}\left(\frac{\bar{\phi}}{\bar{\gamma}_z}\right). \tag{12}$$

The quasi-Stokes streamfunction $\mathbf{\Psi}$ is the contribution of mesoscale perturbations to the horizontal volume flux of water that is denser than $\tilde{\gamma}$, the density surface whose average height is z. Using (10), the tracer conservation equation (8) can be written as

$$\hat{\tau}_t + \left(\bar{\mathbf{V}} + \mathbf{V}^{+}\right) \cdot \nabla_H \hat{\tau} + \left(\bar{w} + w^{+}\right)\hat{\tau}_z = \nabla \cdot (\mathbf{S}\nabla\hat{\tau}) + \mathrm{O}\left(\alpha^3\right). \tag{13}$$

MM01 and Jacobson and Aiki (2006) have shown that if the exact expression is used for the quasi-Stokes streamfunction in (12), then the tracer conservation equation (13) is exact so that the $\mathrm{O}(\alpha^3)$ expression can be deleted from (13).

As explained by MM01, the Gent *et al.* (1995) technique for parameterizing mesoscale eddies is in fact a scheme to parameterize the quasi-Stokes streamfunction (12) and that the tracers in OGCMs should be regarded as the thickness-weighted versions of these tracers. This now-standard explanation of the tracer equations in OGCMs contrasts with the claim by DC06 and by Canuto and Dubovikov (2007) that OGCMs are missing an important term in their tracer equations (compare (13) with the corrected version of (3), namely (A.1)–(A.2) below).

This interpretation of the TRM tracer equation relies on the horizontal momentum equations being written in terms of the Eulerian-mean horizontal velocity. In this regard McDougall and McIntosh (1996) and MM01 showed that the mean horizontal pressure gradient cannot be accurately estimated in a coarse-resolution ocean model for several reasons. The two main reasons are (i) a lack of knowledge of temperature variance and its influence on the mean horizontal density gradient through the cabbeling nonlinearity of the equation of state [see appendix B of McDougall and McIntosh (1996)]; and (ii) the mean horizontal gradient of pressure is more naturally expressed in terms of the Eulerian-mean salinity and potential temperature rather than in terms of the thickness-weighted salinity and potential temperature (see appendix B of MM01). These effects cause uncertainty in the horizontal pressure gradient of order 1–3%, but the errors spatially average to zero over regions of high eddy activity. Such errors do not seem large in comparison with the uncertainty in parameterizing the Reynolds stresses.

Any parameterization is an imperfect approximation of the quantity that is being parameterized. In particular, the horizontal momentum equations will always contain a variety of errors which limit their ability to deliver the Eulerian-mean velocity to the tracer equations. The key feature of the TRM approach (13) is that the tracer equations and therefore the density equation can be made adiabatic, a feature that is lacking with the DC06 approach. That is, by adopting the Gent and

McWilliams (1990) scheme, ocean models only exhibit diapycnal flow through their resolved-scale isopycnals to the extent that small-scale mixing is imposed in the model. This "adiabatic" attribute of the TRM approach applies whether or not errors of order 3% are present in the horizontal momentum equations.

4. The density variance equation

DC06 and Dubovikov and Canuto (2005) claim that, in the density variance equation,

$$\bar{\phi}_t + \bar{\mathbf{U}} \cdot \nabla\bar{\phi} = \overline{Q'\gamma'} - \overline{\mathbf{U}'\gamma'} \cdot \nabla\bar{\gamma} + \mathrm{O}(\alpha^3), \tag{14}$$

the advection of density variance $\bar{\mathbf{U}} \cdot \nabla\bar{\phi}$ is negligible by comparison with the other terms [see section 4 of DC06 and their equation (11b)]. They then criticize Treguier *et al.* (1997) for applying the density variance balance to a stationary flow and claim that the production of potential energy is local. This is incorrect. Unlike the zonal averaging case, the advection of density variance $\bar{\mathbf{U}} \cdot \nabla\bar{\phi}$ is a leading order term in (14), as is demonstrated in figure 1 of McDougall and McIntosh (1996) and in figure 2(a) and (b) of MM01; see also the discussion around equation (25) of MM01.

5. The expression for E

While MM01 emphasized the need to interpret the tracer variables of coarse resolution ocean models as the thickness-weighted values, they also wrote the tracer conservation equation (13) in terms of the Eulerian-mean tracer $\bar{\tau}$ as

$$\bar{\tau}_t + \left(\bar{\mathbf{U}} + \mathbf{U}^+\right) \cdot \nabla\bar{\tau} = \nabla \cdot (\mathbf{S}\nabla\bar{\tau}) - \nabla \cdot \mathbf{E} + \mathrm{O}\left(\alpha^3\right), \tag{15}$$

where the flux $\mathbf{E}$ is given by their (55), namely

$$\mathbf{E} = \bar{\mathbf{U}}\left[-\frac{\overline{\tau'\gamma'}}{\bar{\gamma}_z} + \frac{\bar{\tau}_z\bar{\phi}}{\bar{\gamma}_z^2}\right]_z + \mathbf{k}\left[-\frac{\overline{\tau'\gamma'}}{\bar{\gamma}_z} + \frac{\bar{\tau}_z\bar{\phi}}{\bar{\gamma}_z^2}\right]_t. \tag{16}$$

The individual components of $\mathbf{E}$ can be recognized in the first and third lines of the expression (9) for the modified tracer flux $\mathbf{F}^{M\tau}$. MM01 made the point that the terms in (16) did not need to be parameterized since the tracer variable in coarse resolution OGCMs is best regarded as $\hat{\tau}$. DC06 wrote an expression for $\nabla \cdot \mathbf{E}$ [their equation (22)] as the vertical derivative of $\mathbf{k}\Sigma^\tau$ which is shown as equation (A.3) of our appendix A. Equations (16) and (A.3) are quite different, indicating that equation (22a–b) of DC06 is incorrect.

6. Conclusions

The stated aim of DC06 was "to find the optimal decomposition of the adiabatic component of the mesoscale tracer flux in the *z*-coordinate mean tracer equation". Unfortunately DC06 neglected leading order terms and so did not achieve this aim.

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When the tracer equation is accurately averaged in density coordinates and then accurately transformed into z-coordinates one finds that

- the total velocity is $\bar{\mathbf{U}} + \mathbf{U}^{+}$ of (11)–(13) rather than $\bar{\mathbf{U}} + \mathbf{U}^{*}$ of (3) as assumed by DC06,
- the appropriate average tracer is the thickness-weighted form $\hat{\tau}$ rather than the Eulerian-mean tracer $\bar{\tau}$ assumed by DC06,
- contrary to the claim by DC06, the advection of density variance by the three-dimensional velocity is a leading term in the density variance equation (16),
- contrary to the claim by DC06, the tracer equations carried by present ocean general circulation models are not deficient by leading-order terms,
- rather, the tracer equation of OGCMs is given by (13), with the Gent *et al.* (1995) streamfunction being a parameterization for the quasi-Stokes streamfunction (12).
- Apart from the need to parameterize the down-gradient diffusion [vertical diffusion and the Redi (1982) tensor], the TRM approach only requires the quasi-Stokes streamfunction (12) to be parameterized whereas in DC06's approach each of the three terms on the top line of (A.3) [as well as the additional fluxes in (A.4)] and **V***of (2) all need to be parameterized,
- the TRM approach naturally achieves the adiabatic property that the diapycnal component of the total TRM velocity $\bar{\mathbf{U}} + \mathbf{U}^{+}$ is only due to real small-scale diapycnal diffusion,
- by contrast, the approach of DC06 does not achieve this adiabatic property [see McDougall *et al.* (2007) for a discussion of this issue],
- in both the TRM and DC06 approaches, the horizontal momentum equations are regarded as yielding an estimate of the Eulerian-averaged mean velocity [see McDougall *et al.* (2007) for a discussion of this issue].

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{TANDF_REV}GGAF/GGAF_I_102_03/GGAF_A_263653.3d (GGAF) [Revised Proof]

Appendix A: the corrected equation (7a) of DC06

DC06 choose to advect the Eulerian-mean tracer $\bar{\tau}$ with the velocity $\bar{\mathbf{U}} + \mathbf{U}^*$ [see (2) and (3)]. They allow for epineutral and vertical diffusion and arrive at their conservation equation (7a) which has a source term $-\Sigma_z^\tau$ as described in (3) above. Because of the neglect of leading order terms in the derivation of their (7a), this source term is not correct. Here we derive the correct source term for this type of conservation equation in which $\bar{\mathbf{U}} + \mathbf{U}^*$ advects $\bar{\tau}$.

We start from the accurate conservation equation (8) and subtract $\nabla \cdot \mathbf{E}$ [see (15)] from both sides so that it is written as a conservation equation for $\bar{\tau}$ rather than for $\hat{\tau}$. Then $\mathbf{V}^* \cdot \nabla_H \bar{\tau} + w^* \bar{\tau}_z$ is added to both sides using (2), giving

$$\bar{\tau}_t + (\bar{\mathbf{V}} + \mathbf{V}^*) \cdot \nabla_H \bar{\tau} + (\bar{w} + w^*)\bar{\tau}_z = \nabla \cdot (\mathbf{S}\nabla\bar{\tau}) - \nabla \cdot \mathbf{F}^{\Sigma\tau} + \mathrm{O}(\alpha^3) \tag{A.1}$$

where the source term is now minus the divergence of the flux

$$\begin{aligned} \mathbf{F}^{\Sigma\tau} = {} & \bar{\mathbf{V}}_z \overline{\tau'\gamma'}/\bar{\gamma}_z - 2\bar{\mathbf{V}}_z \bar{\tau}_z \bar{\phi}/\bar{\gamma}_z^2 \\ & + \mathbf{k}\left\{\overline{w'\tau'} + \overline{\mathbf{V}'\gamma'} \cdot \nabla_H \bar{\tau}/\bar{\gamma}_z - \overline{X'\gamma'}/\bar{\gamma}_z + \bar{X}_z \bar{\phi}/\bar{\gamma}_z^2\right\} \\ & + \mathbf{k}\left\{-\overline{Q'\gamma'}/\bar{\gamma}_z + \bar{\tau}_z \overline{Q'\gamma'}/\bar{\gamma}_z^2 + \bar{Q}_z \overline{\tau'\gamma'}/\bar{\gamma}_z^2 - 2\bar{Q}_z \bar{\tau}_z \bar{\phi}/\bar{\gamma}_z^3\right\} \\ & + \mathbf{k}\left\{\left(\overline{\mathbf{V}'\tau'} - \bar{\tau}_z \overline{\mathbf{V}'\gamma'}/\bar{\gamma}_z - \bar{\mathbf{V}}_z \overline{\tau'\gamma'}/\bar{\gamma}_z + 2\bar{\mathbf{V}}_z \bar{\tau}_z \bar{\phi}/\bar{\gamma}_z^2\right) \cdot \nabla_H \bar{\gamma}/\bar{\gamma}_z\right\}. \end{aligned} \tag{A.2}$$

This flux contains fourteen terms, of which the two terms on the first line are horizontal fluxes. This is the flux whose divergence should have appeared as the source term in DC06's tracer conservation equation (7a) and in equation (5b)–(c) of Canuto and Dubovikov (2007). In those equations the extra flux in (A.1) is not $\mathbf{F}^{\Sigma\tau}$, but rather $\mathbf{k}\Sigma^\tau$ defined by

$$\begin{aligned} \mathbf{k}\Sigma^\tau & = \mathbf{k}\left\{\overline{\mathbf{U}'\tau'} \cdot \nabla\bar{\gamma}/\bar{\gamma}_z + \overline{\mathbf{U}'\gamma'} \cdot \nabla\bar{\tau}/\bar{\gamma}_z - \bar{\tau}_z \overline{\mathbf{U}'\gamma'} \cdot \nabla\bar{\gamma}/\bar{\gamma}_z^2\right\} \\ & = \mathbf{k}\left\{\overline{w'\tau'} + \overline{\mathbf{V}'\gamma'} \cdot \nabla_H \bar{\tau}/\bar{\gamma}_z + \left(\overline{\mathbf{V}'\tau'} - \bar{\tau}_z \overline{\mathbf{V}'\gamma'}/\bar{\gamma}_z\right) \cdot \nabla_H \bar{\gamma}/\bar{\gamma}_z\right\}, \end{aligned} \tag{A.3}$$

so that the tracer flux missing from their equations is

$$\begin{aligned} \mathbf{F}^{\Sigma\tau} - \mathbf{k}\Sigma^\tau = {} & \bar{\mathbf{V}}_z \overline{\tau'\gamma'}/\bar{\gamma}_z - 2\bar{\mathbf{V}}_z \bar{\tau}_z \bar{\phi}/\bar{\gamma}_z^2 \\ & + \mathbf{k}\left\{-\overline{X'\gamma'}/\bar{\gamma}_z + \bar{X}_z \bar{\phi}/\bar{\gamma}_z^2\right\} \\ & + \mathbf{k}\left\{-\overline{Q'\gamma'}/\bar{\gamma}_z + \bar{\tau}_z \overline{Q'\gamma'}/\bar{\gamma}_z^2 + \bar{Q}_z \overline{\tau'\gamma'}/\bar{\gamma}_z^2 - 2\bar{Q}_z \bar{\tau}_z \bar{\phi}/\bar{\gamma}_z^3\right\} \\ & + \mathbf{k}\left\{\left(-\bar{\mathbf{V}}_z \overline{\tau'\gamma'}/\bar{\gamma}_z + 2\bar{\mathbf{V}}_z \bar{\tau}_z \bar{\phi}/\bar{\gamma}_z^2\right) \cdot \nabla_H \bar{\gamma}/\bar{\gamma}_z\right\}. \end{aligned} \tag{A.4}$$

Apart from the six terms in (A.4) due to the sources Q and X, there are also four terms in the vertical shear of the mean horizontal velocity $\bar{\mathbf{V}}_z$. These four terms all arise from the neglect of a leading order term when the perturbation horizontal velocity at constant density is equated to that at constant z as, for example, in equation (1e) of DC06 and equation (4a) of Canuto and Dubovikov (2007).