

A Balanced Approximation of the One-Layer Shallow-Water Equations on a Sphere

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(Manuscript received 16 May 2008, in final form 11 November 2008)

ABSTRACT

A global version of the equivalent barotropic vorticity equation is derived for the one-layer shallow-water equations on a sphere. The equation has the same form as the corresponding beta plane version, but with one important difference: the stretching (Cressman) term in the expression of the potential vorticity retains its full dependence on f^2 , where f is the Coriolis parameter. As a check of the resulting system, the dynamics of linear Rossby waves are considered. It is shown that these waves are rather accurate approximations of the westward-propagating waves of the second class of the original shallow-water equations. It is also concluded that for Rossby waves with short meridional wavelengths the factor f^2 in the stretching term can be replaced by the constant value f_0^2 , where f_0 is the Coriolis parameter at $\pm 45^\circ$ latitude.

1. Introduction

The quasigeostrophic system of equations, systematically derived by Charney (1948), is the main conceptual framework of dynamic meteorology. The basic physical principle of the theory is stated in section 2 of Charney's article: "the motion of large-scale atmospheric disturbances is governed by the laws of conservation of potential temperature and absolute potential vorticity, and by the conditions that the horizontal velocity be quasigeostrophic and the pressure quasi-hydrostatic." Phillips (1990) remarks that these words must be considered among the most effective meteorological statements of the 20th century. Indeed, Charney's theory has withstood the test of time and expositions of his theory are an integral part of modern textbooks on geophysical fluid dynamics and dynamic meteorology (see Gill 1982; Pedlosky 1987; Salmon 1998; Holton 2004; Vallis 2006).

Quasigeostrophic theory is usually developed in the context of the midlatitude beta plane approximation and the geostrophic relationship is defined as a near-balance between the Coriolis force and the pressure gradient force. A common approach is to assume that the Coriolis parameter f can be replaced by a constant value f_0 in the geostrophic relationship, an approach that Blackburn (1985) refers to as the "theoretician's

geostrophy." If we wish to generalize quasigeostrophic theory to a spherical domain then the theoretician's geostrophy is not a valid option because there is no value of f_0 other than zero that would be representative for the whole sphere. Keeping the full variation of f in the geostrophic relationship, referred to by Blackburn (1985) as the "synoptician's geostrophy," is no option either because, by evaluating the divergence, it can be easily shown that it would constrain the geostrophic meridional velocity to be zero at the equator.¹ The equator would thus be turned into an impenetrable wall for geostrophic flow, which, except for special flows such as equatorial Kelvin waves, would be quite unrealistic.

An alternative that avoids these problems is given by Daley (1983), who proposes a nondivergent horizontal velocity field in combination with the linear balance equation as the natural extension of the geostrophic relationship to a global domain. The linear balance equation is an integral part of the spherical geostrophic model developed by Lorenz (1960) and can be simplified further to what Daley (1983) calls the "simplest form of the geostrophic relationship"—a nondivergent horizontal velocity field in combination with a geopotential of which the deviation from a uniform reference field (corresponding to the state of rest) is given by f times the

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¹ Using pressure coordinates and denoting the divergence by \mathcal{D} , it follows that $v = -a\mathcal{D}\tan\phi$, where v is the meridional component of the geostrophic velocity, a is the radius of the earth, and ϕ is the latitude.

streamfunction of the velocity field. The author (Verkley 2001) has used this balance as the mass-velocity constraint in a Hamiltonian balanced approximation of a one-layer isentropic model of the atmosphere. It served its purpose very well although the resulting balanced model is rather difficult to use in theoretical studies.

In the present article we will combine Daley's (1983) simplest form of the geostrophic relationship with a linearized expression of the potential vorticity to formulate a much simpler balanced model. We will do this in the context of a global one-layer shallow-water model of the atmosphere, introduced in section 2. In section 3 we show how our simplifying procedure leads to an equation that is identical in form to the equivalent barotropic vorticity equation (Cressman 1958) except for a full f^2 dependence of the Cressman (stretching) term. To investigate the consequences of this f^2 dependence, the frequencies and spatial structure of linear Rossby waves are discussed in section 4. It is seen that these waves are a rather accurate approximation of the westward-propagating waves of the second class (comprising the mixed planetary-gravity wave and the planetary waves) as calculated by Longuet-Higgins (1968) for the full underlying shallow-water model. In section 4 it is also shown that linear Rossby waves in the limit of small meridional wavelengths behave as if f^2 were a constant f_0^2 , where f_0 is the Coriolis parameter at $\pm 45^\circ$ latitude. The results are summarized in section 5.

2. One-layer shallow-water equations

The model system that forms the basis of this article consists of a single hydrostatic layer of fluid with uniform density ρ , moving adiabatically and frictionless over the surface of a rotating spherical earth with mean radius a and angular velocity of rotation Ω . The lower boundary of the layer has a fixed height $z_l = \eta_B$, where η_B is the height of the orography. The upper boundary has a variable height $z_u = H_A + \eta$, where H_A is a uniform average height and η is a variable surface elevation. Horizontal positions are denoted by the longitude λ and the latitude ϕ . The shallow-water model is generally considered to be a useful benchmark model of atmospheric and oceanic fluid flow although, of course, for the atmosphere the assumption of a uniform density is rather unrealistic. However, by replacing a uniform density by a uniform potential temperature one obtains a somewhat more realistic model with essentially similar dynamics (see Verkley 2001).

In the shallow-water model the horizontal velocity \mathbf{v} is independent of height and satisfies the momentum equation

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} + \nabla\varphi = 0, \quad (1)$$

where D/Dt is the horizontal material derivative and $f = 2\Omega \sin \phi$ is the Coriolis parameter. The geopotential φ is given by

$$\varphi = g\eta, \quad (2)$$

with g being the gravity acceleration, and is the deviation of the geopotential $\Phi = gz_u = g(H_A + \eta)$ from the state of rest gH_A . The dynamics of the geopotential φ is given by the conservation of mass, which, due to the columnar motion of the fluid, reduces to the following equation for the total fluid depth H :

$$\frac{DH}{Dt} + H\mathcal{D} = 0, \quad (3)$$

where $\mathcal{D} = \nabla \cdot \mathbf{v}$ is the divergence of the horizontal velocity field and $H = z_u - z_l$; that is,

$$H = H_A + \eta - \eta_B. \quad (4)$$

Equations (1), (2), (3), and (4) are the shallow-water equations as derived and discussed extensively by Pedlosky (1987). It is assumed that the average height of the fluid is much smaller than the radius of the earth so that this radius may be substituted for the distance from the earth's center in the metric terms of the differential operators. These operators are purely horizontal and can be written in terms of the geographic coordinates λ and ϕ and the unit vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , where \mathbf{k} points vertically upward.

It can be shown (see the appendix) that the shallow-water system conserves the mass \mathcal{M} ,

$$\mathcal{M} = \rho \int dS\eta, \quad (5)$$

and the energy \mathcal{E} ,

$$\mathcal{E} = \rho \int dS \left(\frac{1}{2} H\mathbf{v}^2 + \frac{1}{2} g\eta^2 \right). \quad (6)$$

Here, $dS = a^2 \cos \phi d\lambda d\phi$ denotes a spherical area element. We note that the mass \mathcal{M} is the difference between the actual mass and the mass in the state of rest. Conservation of \mathcal{M} implies that if the integrated surface elevation η is zero at some initial time, it will remain zero thereafter. The potential energy in the expression of \mathcal{E} is the difference between the actual potential energy and the potential energy at the state of rest (i.e., the available potential energy). As is well known (Pedlosky

1987), the shallow-water system also has a material invariant: the potential vorticity. This invariant is the basis of the balanced approximation, to be discussed in the next section.

3. The balanced approximation

Following the approach of Charney (1948), the balanced approximation to be derived is based on the material conservation of potential vorticity and the assumption that the atmosphere stays close to a form of balance. We first consider the material conservation of potential vorticity and then discuss the balance condition that we will adopt: a simplification of linear balance that Daley (1983) calls the simplest form of the geostrophic relationship.

a. Potential vorticity

We first note that we have for the material derivative of the horizontal velocity \mathbf{v}

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + \zeta(\mathbf{k} \times \mathbf{v}) + \nabla\left(\frac{\mathbf{v} \cdot \mathbf{v}}{2}\right), \quad (7)$$

where $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v}$ is the vorticity of the horizontal velocity field. The momentum equation [(1)] can then be written in the form

$$\frac{\partial\mathbf{v}}{\partial t} + (f + \zeta)(\mathbf{k} \times \mathbf{v}) + \nabla\left(\varphi + \frac{\mathbf{v} \cdot \mathbf{v}}{2}\right) = 0. \quad (8)$$

If the vertical curl and horizontal divergence $\mathbf{k} \cdot \nabla \times$ and $\nabla \cdot$ are applied to this form of the momentum equation, we obtain the following equations for the vorticity and the divergence:

$$\frac{\partial\zeta}{\partial t} + \nabla \cdot [(f + \zeta)\mathbf{v}] = 0, \quad \text{and} \quad (9a)$$

$$\frac{\partial\mathcal{D}}{\partial t} + \nabla \cdot \left[(f + \zeta)(\mathbf{k} \times \mathbf{v}) + \nabla\left(\varphi + \frac{\mathbf{v} \cdot \mathbf{v}}{2}\right) \right] = 0. \quad (9b)$$

The equation for the vorticity can be written in the same form as the mass conservation equation [(3)]:

$$\frac{D\xi}{Dt} + \xi\mathcal{D} = 0, \quad (10)$$

where $\xi = f + \zeta$ is the absolute vorticity. Eliminating the divergence \mathcal{D} from Eqs. (3) and (10), we obtain

$$\frac{DP}{Dt} = 0, \quad (11)$$

where

$$P = \frac{\xi}{H}. \quad (12)$$

This is the material conservation of potential vorticity.

b. Balance condition

In deriving a closed dynamical system from the material conservation of potential vorticity, we need a form of balance that relates the velocity field to the geopotential. To avoid the problems that we mentioned in the introduction, we use Daley's (1983) simplest form of the geostrophic relationship. The basic assumption underlying this form of balance is that the divergence of the horizontal velocity field is so small that it can be taken to be zero to a first approximation. By dividing the mass conservation equation [(3)] by the total depth H we see that, for the shallow-water system, a small divergence would be a direct consequence of the assumption (which we will adopt) that the free surface elevation η and the orography η_B are small compared to H_A . The horizontal velocity field \mathbf{v} can then be expressed in terms of a streamfunction ψ :

$$\mathbf{v} = \mathbf{k} \times \nabla\psi. \quad (13)$$

Furthermore, the condition that the divergence remains small reduces the divergence equation [(9b)] to the balance equation:

$$\nabla \cdot \left[-(f + \zeta)\nabla\psi + \nabla\left(\varphi + \frac{\nabla\psi \cdot \nabla\psi}{2}\right) \right] = 0. \quad (14)$$

This equation will be simplified by ignoring terms that are quadratic in the dynamical variables (i.e., by linearization), giving

$$\nabla \cdot (-f\nabla\psi + \nabla\varphi) = 0, \quad (15)$$

which is known as the linear balance equation. The linear balance equation was used by Lorenz (1960) in his derivation of a balanced system of equations in spherical geometry. According to Daley (1983), it is the natural generalization of geostrophy from a beta plane to spherical geometry. We will make a further approximation, to which end the linear balance equation is rewritten as

$$\nabla^2(-f\psi + \varphi) + \nabla \cdot (\psi\nabla f) = 0. \quad (16)$$

By ignoring the second term on the left-hand side of this equation, we get

$$\nabla^2(-f\psi + \varphi) = 0, \quad (17)$$

which is equivalent to

$$-f\psi + \varphi = 0 \Leftrightarrow \varphi = f\psi, \quad (18)$$

putting the arbitrary constant of integration equal to zero. This is Daley's (1983) simplest form of the geostrophic relationship.

We recall that the basic assumption of Daley's (1983) simplest form of the geostrophic relationship is a small horizontal divergence. For large-scale atmospheric flow a small horizontal divergence is a valid approximation, even at the equator, as was argued by Charney (1963) on the basis of scale considerations and confirmed recently by Yano et al. (2009) on the basis of observational data. Furthermore, the condition of a small horizontal divergence can be applied globally, in contrast to either the theoretician's or the synoptician's geostrophy. It gives rise to the linear balance equation if the quadratic terms in the balance equation are ignored. The simplification of the linear balance equation that leads to Daley's (1983) simplest form of the geostrophic relationship [i.e., skipping the second term on the left-hand side of (16)] assumes that the meridional spatial scales are small compared to the scale of the Coriolis parameter. The approximation is expected to be the least accurate for the largest meridional scales involved. Both the linear balance equation and its simplification pose difficulties at the equator when ψ is to be calculated from φ . Indeed, Daley (1983) finds that the linear balance equation produces rather inaccurate results if applied within 20° from the equator. Calculating φ from ψ , on the other hand, poses no problems at the equator and it is only in this sense that the balance condition is used.

c. Balanced potential vorticity

We will now discuss how the potential vorticity equation [(11) and (12)] can be transformed approximately into a single equation in terms of the streamfunction ψ . We first write for the thickness field H

$$H = H_A + \eta - \eta_B = H_A \left(1 + \frac{\eta}{H_A} - \frac{\eta_B}{H_A} \right). \quad (19)$$

Assuming that the fractions η/H_A and η_B/H_A are small compared to one, as we already did in motivating a small horizontal divergence, we may linearize the expression of $1/H$:

$$\frac{1}{H} \approx \frac{1}{H_A} \left(1 - \frac{\eta}{H_A} + \frac{\eta_B}{H_A} \right). \quad (20)$$

This implies that we have

$$P \approx \frac{1}{H_A} \left(f + \zeta - f \frac{\eta}{H_A} - \zeta \frac{\eta}{H_A} + f \frac{\eta_B}{H_A} + \zeta \frac{\eta_B}{H_A} \right). \quad (21)$$

Ignoring, as a further linearization, the terms $\zeta\eta/H_A$ and $\zeta\eta_B/H_A$, we get

$$H_A P \approx q \equiv f + \zeta - f \frac{\eta}{H_A} + f \frac{\eta_B}{H_A}. \quad (22)$$

The field q , from now on, will be referred to as the potential vorticity. If we implement our balance condition we may use (13) to write

$$\zeta = \nabla^2 \psi. \quad (23)$$

Furthermore, recalling definition (2) of the geopotential and combining it with (18), we have

$$g\eta = f\psi. \quad (24)$$

Thus, expressing ζ and η in terms of ψ , we find for the potential vorticity q

$$q = f + \nabla^2 \psi - \frac{f^2}{gH_A} \psi + f \frac{\eta_B}{H_A}, \quad (25)$$

in which we recognize the equivalent barotropic potential vorticity. Note that the Cressman term is proportional to f^2 and that the contribution of the orography involves a factor f .

The evolution in time of q is given by the material conservation of q as implied by (11):

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = \frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad (26)$$

where the advection is by the balanced velocity [(13)] and which can be written using the Jacobian

$$J(\psi, q) = \mathbf{k} \times \nabla \psi \cdot \nabla q = \mathbf{k} \cdot \nabla \psi \times \nabla q. \quad (27)$$

The streamfunction ψ can be obtained from the potential vorticity q by solving

$$\left(\nabla^2 - \frac{f^2}{gH_A} \right) \psi = q - f - f \frac{\eta_B}{H_A}. \quad (28)$$

This is a second-order differential equation. The equation has a unique solution because the operator between brackets has a zero null space, as shown in the appendix. The result is the familiar equivalent barotropic vorticity equation in which the equivalent barotropic potential vorticity q , given by (25), is advected by the non-divergent velocity field [(13)] of which the streamfunction can be obtained from q by solving (28).

We conclude this section by considering the conservation of the mass \mathcal{M} and the energy \mathcal{E} . In the original shallow-water system the expressions in terms of η and \mathbf{v} are given by (5) and (6), respectively. There are equivalents of these quantities in the balanced system that we have derived; these equivalents are denoted by $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$, respectively. For the mass $\tilde{\mathcal{M}}$ we have

$$\tilde{\mathcal{M}} = \frac{\rho H_A}{\Omega} \int dS \frac{f^2}{gH_A} \psi. \quad (29)$$

This expression is obtained by substituting (24) into (5), multiplying the integrand with f/H_A , and putting the fraction H_A/Ω in front of the integral to restore dimensionality. The equivalent $\tilde{\mathcal{E}}$ of the energy in the balanced system is given by

$$\tilde{\mathcal{E}} = \rho H_A \int dS \left[\frac{1}{2} (\nabla\psi)^2 + \frac{1}{2} \frac{f^2}{gH_A} \psi^2 \right]. \quad (30)$$

This expression is obtained by replacing H by H_A in the original expression for \mathcal{E} and expressing \mathbf{v} and η in terms of ψ , in the same manner as above. In the appendix it is shown that for the balanced system governed by (25) and (26), the mass $\tilde{\mathcal{M}}$ and the energy $\tilde{\mathcal{E}}$ are conserved quantities.

4. Linear Rossby waves

In this section we will test the balanced system by studying the dynamics of linear Rossby waves. This is, of course, a rather limited context, but in this limited context the dynamics of the balanced system can be compared very stringently with the dynamics of the original shallow-water system. The properties of linear waves of the shallow-water system have been documented in much detail by Longuet-Higgins (1968) and were summarized concisely by Andrews et al. (1987).

a. Nondimensional equations

In the following discussion it is convenient to measure length in units of a and time in units of Ω^{-1} . Vorticity and potential vorticity are then measured in units of Ω and velocity in units of $a\Omega$. If the height variations η and η_B are expressed in terms of H_A , the shallow-water equations are characterized by a single nondimensional parameter γ , called Lamb's parameter, which is given by

$$\gamma = \frac{4\Omega^2 a^2}{gH_A}. \quad (31)$$

In the shallow-water Eqs. (1) and (3) the expressions of the geopotential [(2)] and thickness [(4)] become $\varphi = (4/\gamma)\eta$ and $H = 1 + \eta - \eta_B$. The balance relationship (24) assumes the form

$$\eta = \frac{\mu}{2} \gamma \psi, \quad (32)$$

where $\mu = \sin\phi$ and the streamfunction ψ is expressed in units of $a^2\Omega$. The nondimensional expression of the global balanced potential vorticity q reads

$$q = 2\mu + \nabla^2\psi - \gamma\mu^2\psi + 2\mu\eta_B. \quad (33)$$

In the balanced system, this field is advected by a non-divergent velocity field of which the zonal and meridional

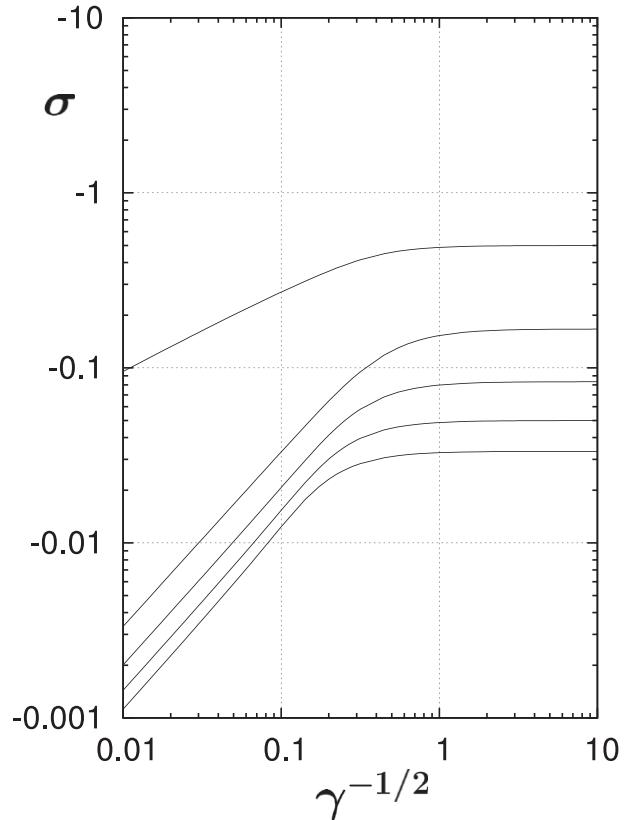


FIG. 1. The value of σ (frequency divided by two) as a function of $\gamma^{-1/2}$ for westward-propagating waves of the second class with $m = 1$ and $n = 1, 2, \dots, 5$, where the upper curve corresponds to $n = 1$ and the lower curve to $n = 5$. These results are identical to those given by Fig. 2b of Longuet-Higgins (1968).

components are given in terms of the streamfunction ψ by

$$u = -(1 - \mu^2)^{1/2} \frac{\partial\psi}{\partial\mu}, \quad v = (1 - \mu^2)^{-1/2} \frac{\partial\psi}{\partial\lambda}. \quad (34)$$

The nondimensional equation for the material conservation of potential vorticity is identical to (26), with the following nondimensional expression for the Jacobian:

$$J(\psi, q) = \frac{\partial\psi}{\partial\mu} \frac{\partial q}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda} \frac{\partial q}{\partial\mu}. \quad (35)$$

As a result of our choice of units, in the differential operators such as the Jacobian, Laplacian, and the gradient operator, the radius of the earth is to be replaced by 1.

b. Linearized balanced system

We will study the dynamics of linear Rossby waves in the absence of orography. In this case the potential vorticity is

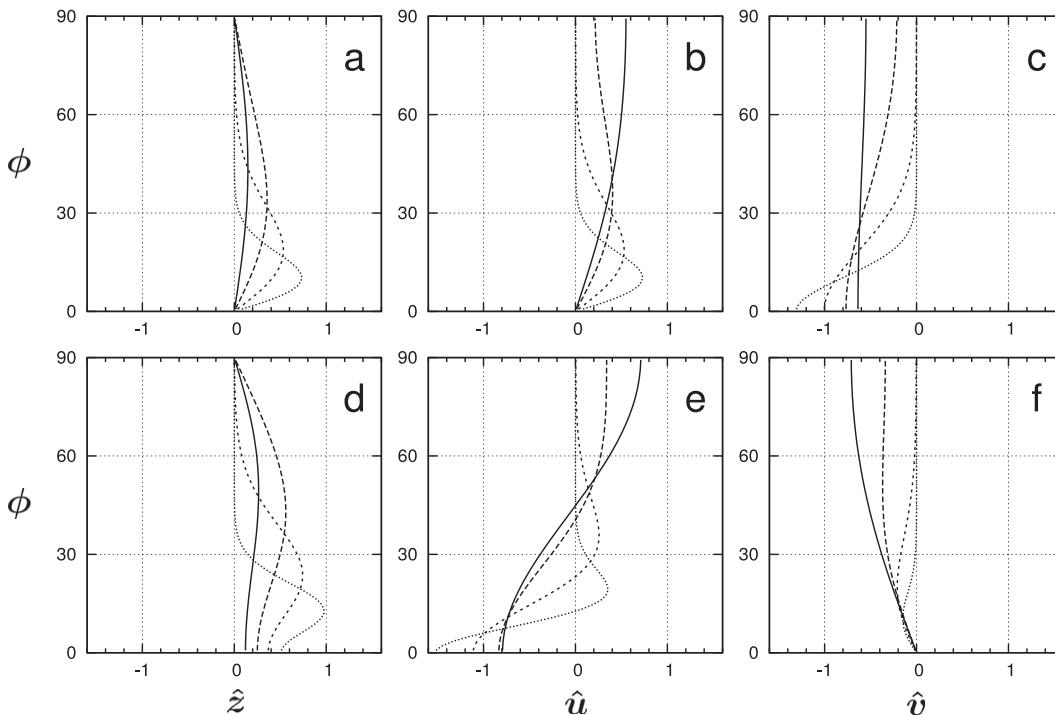


FIG. 2. (a)–(c) Meridional profiles of (a) \hat{z} , (b) \hat{u} , and (c) \hat{v} for a westward-traveling wave of the second class with $m = 1$ and $n = 1$. (d)–(f) As in (a)–(c) but with $m = 1$ and $n = 2$. Function values are on the horizontal axis and latitudes on the vertical axis. The different profiles in each plot refer to different values of γ , namely 1 (solid curve), 10 (long-dashed curve), 100 (short-dashed curve), and 1000 (dotted curve). Note that \hat{z} , \hat{u} , and \hat{v} can be identified with Z , U , and V , which, for these parameter values, are displayed in Fig. 9 of Longuet-Higgins (1968, p. 538).

$$q = 2\mu + \nabla^2\psi - \gamma\mu^2\psi, \tag{36}$$

and the equation governing linear waves reads

$$\frac{\partial}{\partial t}(\nabla^2\psi - \gamma\mu^2\psi) + 2\frac{\partial\psi}{\partial\lambda} = 0. \tag{37}$$

Searching for solutions in the form of the real part of a streamfunction ψ of which the longitude and time dependence is given by $\exp[i(m\lambda - 2\sigma t)]$, where m is the zonal wavenumber and 2σ is the frequency, this streamfunction should satisfy

$$\nabla^2\psi - \gamma\mu^2\psi = d_m\psi, \tag{38}$$

with

$$d_m = \frac{m}{\sigma}; \tag{39}$$

that is, it should be an eigenfunction of the operator $\nabla^2 - \gamma\mu^2$ with eigenvalue $d_m = m/\sigma$. This relation fixes the frequency 2σ of a wave in terms of the eigenvalue d_m . It follows from (38) that the meridional part of the streamfunction, denoted by $\hat{\psi}_m(\mu)$, has to satisfy

$$\left\{ \frac{d}{d\mu} \left[(1 - \mu^2) \frac{d\hat{\psi}_m}{d\mu} \right] - \frac{m^2}{1 - \mu^2} - \gamma\mu^2 \right\} \hat{\psi}_m = d_m \hat{\psi}_m. \tag{40}$$

This equation is known as the differential equation for the angular prolate spheroidal wave functions [see Abramowitz and Stegun 1965, Eq. (21.6.2)]. The eigenvalues and eigenvectors depend on γ as well as m and will be denoted by d_{mn} and S_{mn} , respectively, where n is an integer larger than or equal to $|m|$. We note that for $\gamma = 0$ the differential equation [(40)] is identical to the differential equation for associated Legendre functions P_{mn} . For these functions the eigenvalues d_{mn} are independent of m and given by $-n(n + 1)$.

The eigenvalue equation [(40)] can be solved numerically by writing the field $\hat{\psi}_m$ as a linear combination of associated Legendre functions P_{mn} , as explained in some detail in the appendix. By using recurrency relations one can express $\mu^2 P_{mn}$ in terms of the associated Legendre functions P_{mn-2} , P_{mn} , and P_{mn+2} and thereby transform the equation, for each m separately, into a tridiagonal matrix equation. By normalizing the associated Legendre functions according to Machenhauer (1979), the matrices are symmetric. Denoting by N the maximum value of n , we have for each m a symmetric

system with dimension $N - m + 1$. The eigenvalue problem is solved with $N = 105$, using a numerical routine from Press et al. (1986), based on the Jacobi method for symmetric eigenvalue problems.

Following Longuet-Higgins (1968) we normalize the eigenfunctions such that

$$\int dS(u^2 + v^2 + z^2) = \pi, \tag{41}$$

where z is defined by the following expression:

$$z = 2\gamma^{-1/2}\eta. \tag{42}$$

Expressing u and v in terms of the streamfunction ψ , using (32) and (42) as well as Gauss' theorem and the fact that ψ is an eigenfunction of (38) with eigenvalue d , we have

$$\int dS(u^2 + v^2 + z^2) = \int dS - \psi(\zeta - \gamma\mu^2\psi) = -d \int dS\psi^2, \tag{43}$$

so that ψ is normalized as

$$\int dS\psi^2 = -\frac{\pi}{d}. \tag{44}$$

Because the streamfunction—and all other variables—were taken to be the real part of a complex exponential in λ and t , we may write (choosing the phase of the wave such that at $t = 0$ the zero meridian is the axis of symmetry for ψ)

$$\psi(\lambda, \mu, t) = \hat{\psi}(\mu) \cos(m\lambda - 2\sigma t), \tag{45}$$

so that we have for u , v , and z

$$u(\lambda, \mu, t) = \hat{u}(\mu) \cos(m\lambda - 2\sigma t), \tag{46a}$$

$$v(\lambda, \mu, t) = \hat{v}(\mu) \sin(m\lambda - 2\sigma t), \quad \text{and} \tag{46b}$$

$$z(\lambda, \mu, t) = \hat{z}(\mu) \cos(m\lambda - 2\sigma t), \tag{46c}$$

where the meridional structure functions \hat{u} , \hat{v} , and \hat{z} are given by

$$\hat{u}(\mu) = -(1 - \mu^2)^{1/2} \frac{d\hat{\psi}(\mu)}{d\mu}, \tag{47a}$$

$$\hat{v}(\mu) = -(1 - \mu^2)^{-1/2} m\hat{\psi}(\mu), \quad \text{and} \tag{47b}$$

$$\hat{z}(\mu) = \gamma^{1/2} \mu \hat{\psi}(\mu). \tag{47c}$$

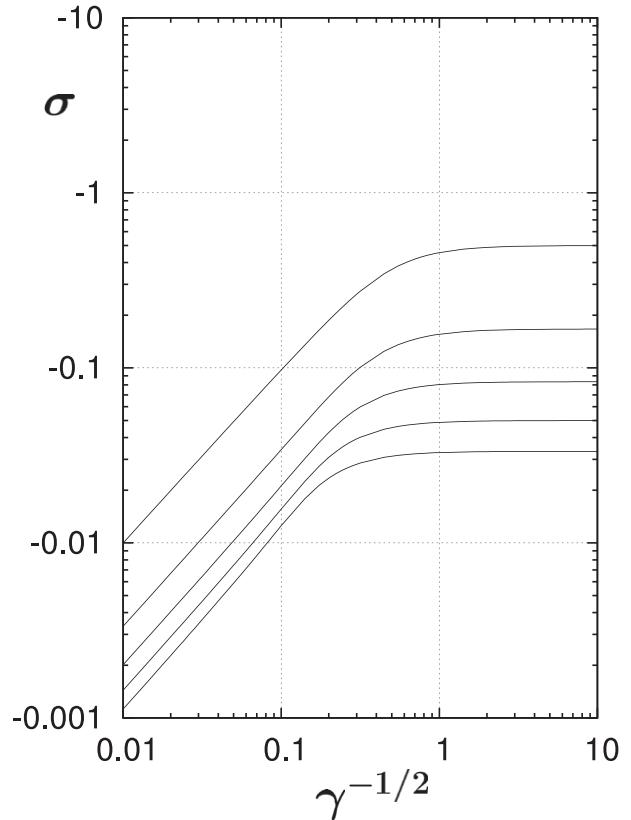


FIG. 3. As in Fig. 1, but for Rossby waves. The wave with $n = 1$ is to be identified with the mixed planetary–gravity wave.

c. Balanced Rossby waves

Before presenting the results of our analysis, we first give the frequency divided by two (σ) as a function of the inverse square root of Lamb's parameter ($\gamma^{-1/2}$) as well as a few meridional profiles for the westward-propagating waves of the second class, as calculated for the full shallow-water system by Longuet-Higgins (1968). The frequencies σ are shown in Fig. 1, for $m = 1$ and $n = 1, 2, \dots, 5$. The wave with $n = 1$ is the mixed planetary–gravity wave; the waves with $n > 1$ are the planetary waves. For $\gamma = 1, 10, 100$, and 1000 a few profiles \hat{z} , \hat{u} , and \hat{v} for $m = 1$ as functions of ϕ , where $\mu = \sin \phi$, are shown in Figs. 2a–c for $n = 1$ (mixed planetary–gravity wave) and in Figs. 2d–f for $n = 2$ (planetary wave). The results, obtained by a method that is similar to the method outlined above, can be checked to be identical to those of Longuet-Higgins (1968) by comparing our Fig. 1 with Fig. 2b from Longuet-Higgins (1968) or with Fig. 4.2b of Andrews et al. (1987). The profiles in Fig. 2 can be verified to be identical to those given by Longuet-Higgins (1968, his Fig. 9, p. 538) because the fields Z , U , and V that are shown in the latter figure can be identified with \hat{z} , \hat{u} , and

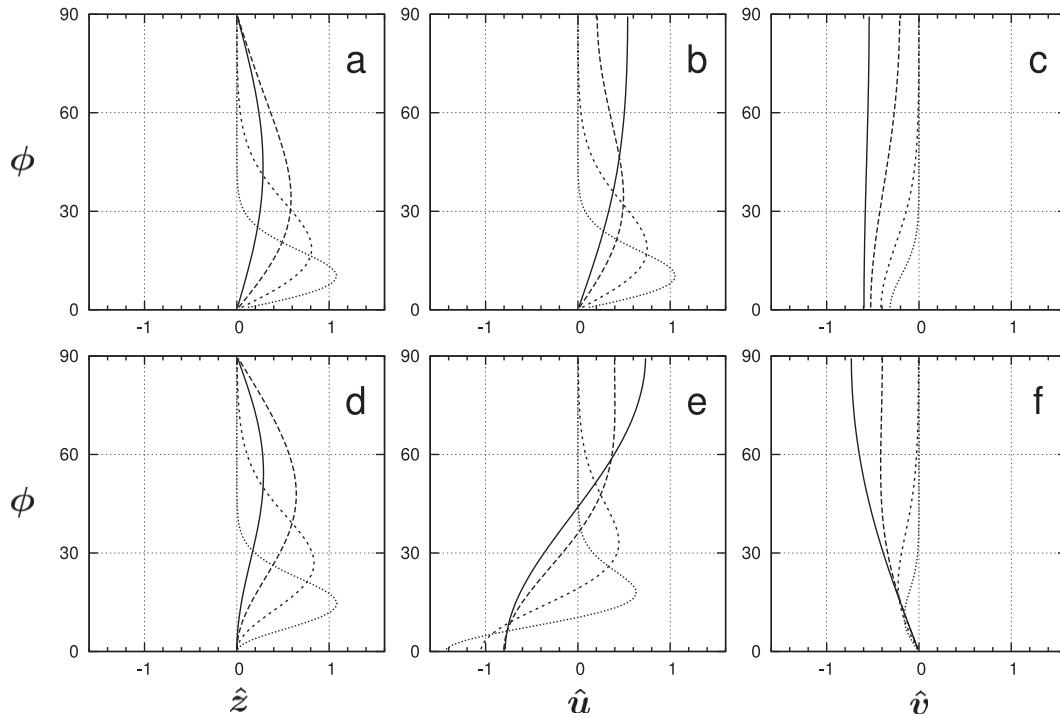


FIG. 4. As in Fig. 2, but for Rossby waves. The wave with $n = 1$ is to be identified with the mixed planetary-gravity wave.

\hat{v} , respectively. It should be kept in mind, of course, that in the calculations of Longuet-Higgins (1968) the meridional structure functions [(47a) and (47b)] also involve a velocity potential $\hat{\chi}$ and that \hat{z} is a separate field not related explicitly to the streamfunction $\hat{\psi}$ as in (47c).

For the balanced system we have calculated the eigenvalues d_{mn} and thereby the frequencies σ for several values of m and n . As an example, we show in Fig. 3 the result for $m = 1$ and $n = 1, 2, \dots, 5$ in the form of graphs that show σ as functions of $\gamma^{-1/2}$. When the graphs are compared with those of Fig. 1, we see that the equivalent barotropic model with variable f^2 reproduces quite well the frequencies of all waves, although the $n = 1$ wave, which corresponds to the mixed planetary-gravity wave, is less well reproduced. Also, we note that the meridional structures of the Rossby waves are rather accurate approximations of the structures of the westward-propagating waves of the second class. In particular, the concentration of amplitude near the equator for large values of Lamb's parameter is rather well reproduced. To illustrate this, we show in Fig. 4, for $\gamma = 1, 10, 100$, and 1000 , the meridional structure of two eigenfunctions with zonal wavenumber $m = 1$, namely for $n = 1$ (Figs. 4a–c, corresponding to the mixed planetary-gravity wave) and for $n = 2$ (Figs. 4d–f, corresponding to a planetary wave). Figure 4 can be compared directly with Fig. 2 [or with Fig. 9 of

Longuet-Higgins (1968, p. 538)]. We see that even for large values of γ the meridional structures are quite well reproduced. For the $n = 1$ case, however, the amplitude of \hat{u} is too large and the amplitude of \hat{v} is too small near the equator for large values of γ . For the $n = 2$ case both structure and amplitude are generally remarkably good, except for the fields \hat{z} close to the equator. This is also true, and increasingly so, for larger values of n . The same remarks can be made for the meridional structure functions for larger values of m , with the $n = |m|$ case (to be identified with the mixed planetary-gravity wave) being the least well represented (not shown). The fact that the planetary-gravity waves (which have the largest meridional scales for a given m) are least well reproduced by the balanced system is probably a consequence, at least partly, of the approximation that leads from the linear balance equation [(16)] to Daley's (1983) simplest form of the geostrophic relationship (18).

Concerning the eigenvalues d_{mn} for $\gamma \neq 0$, the numerical calculations revealed an interesting result, a result that can also be inferred from Eq. (21.7.5) of Abramowitz and Stegun (1965). From this equation it can be deduced that the eigenvalue d_{mn} approaches the value $-n(n+1) - \gamma/2$ if, for a given m , the value n goes to infinity so that m/n approaches zero. This means that for Rossby waves with n large compared to

m (which—as for associated Legendre functions—implies many wavelengths between pole and equator) the eigenvalues of the operator $\nabla^2 - \gamma\mu^2$ are the same as the eigenvalues of the operator $\nabla^2 - \gamma/2$. The latter operator corresponds to the case in which the factor μ^2 in (38) is approximated by $1/2$ or, equivalently, the factor f^2 in the Cressman term of (25) is approximated by $f_0^2 = 4\Omega^2 \sin^2 \phi_0$ with $\phi_0 = \pm 45^\circ$. The corresponding linear waves are the familiar Rossby–Haurwitz waves of which the meridional structure S_{mn} is given by the associated Legendre function P_{mn} . When normalized in the same way as before, but replacing μ^2 by $1/2$ in (43), we obtain for this approximated case

$$\hat{u}(\mu) = -\left[\frac{1}{2n(n+1) + \gamma}\right]^{1/2} (1 - \mu^2)^{1/2} \frac{dP_{mn}(\mu)}{d\mu}, \quad (48a)$$

$$\hat{v}(\mu) = -\left[\frac{1}{2n(n+1) + \gamma}\right]^{1/2} (1 - \mu^2)^{-1/2} mP_{mn}(\mu), \quad \text{and} \quad (48b)$$

$$\hat{z}(\mu) = \left[\frac{\gamma}{2n(n+1) + \gamma}\right]^{1/2} \mu P_{mn}(\mu). \quad (48c)$$

Here we recall that the associated Legendre functions P_{mn} are assumed to be normalized according to Machenhauer (1979) (see the appendix). In this normalization we have that $P_{11}(\mu) = (3/2)^{1/2}(1 - \mu^2)^{1/2}$ and $P_{12}(\mu) = (15/2)^{1/2}\mu(1 - \mu^2)^{1/2}$. The above expressions for the meridional structures show that when γ increases, there is only a change of amplitude, not of structure. This is different from the results with a varying μ^2 . Also, the frequencies deviate markedly from the previous case when γ increases. Only when γ is small (either g or H_A or both large) is the latter approximation accurate, even for small values of m/n . These features are illustrated by Figs. 5 and 6, respectively, which show the frequencies and meridional profiles for this simplified balanced model.

Figures 7 and 8 display the frequencies and profiles for the three models together and are meant to highlight the differences. Solid curves refer to the full shallow-water model, dashed curves to the balanced model with variable μ^2 , and dotted curves to the balanced model with μ^2 replaced by $1/2$. The upper curves in both figures refer to a mixed planetary–gravity wave with $m = 1$ and $n = 1$; the lower curves to a planetary wave with $m = 1$ and $n = 2$. In Fig. 8 we have chosen $\gamma = 100$.

We note that the differential equation for the angular prolate spheroidal wave functions, Eq. (40), also emerged from an analysis by Wunderer (2001) of a simplified form of the Hamiltonian balanced system discussed by the author (Verkley 2001). Wunderer (2001) uses the

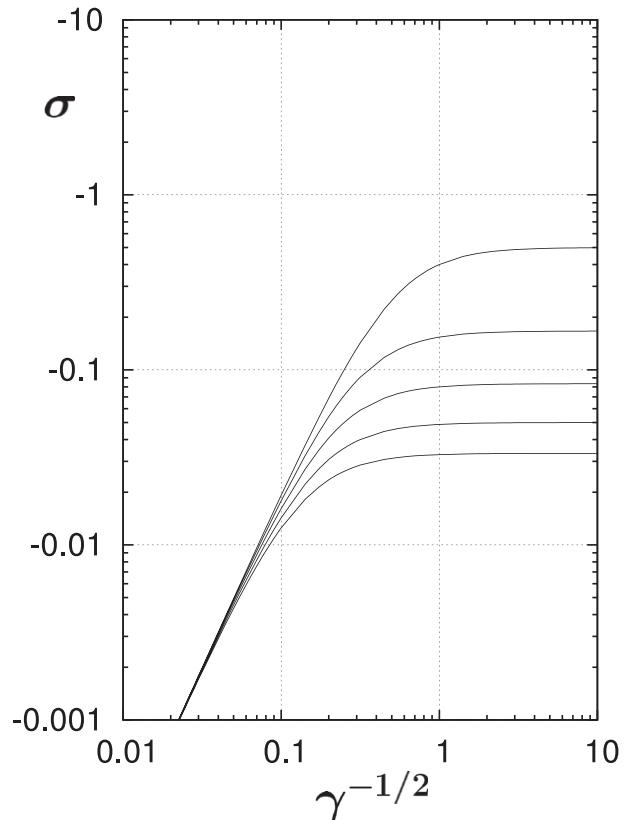


FIG. 5. As in Figs. 1 and 3, but with σ calculated on the assumption that μ^2 in the eigenvalue equation [(40)] can be replaced by $1/2$ so that the eigenvalues d_{mn} can be approximated by $-n(n + 1) - \gamma/2$.

same simplified form of balance as we do² but does not linearize the expression of the potential vorticity. For the analysis of linear waves this does not make a difference, so his Eq. (5.100) is identical to our Eq. (40) if his $1/\varepsilon^2$ is identified with our γ and his $-\omega$ with our σ . In his Table 5.2 Wunderer (2001) gives, for $\gamma = 1/\varepsilon^2 = 10$, numerically calculated frequencies $\omega = -\sigma$ for $m = 1, 2, \dots, 5$ and $n - m = 0, 1, \dots, 5$ as well as the percentage deviations from the full shallow-water results as reported by Longuet-Higgins (1968). These results fully agree with those presented here.

During the review stage of this article the author became aware of a manuscript by Schubert et al. (2009) in which a balanced system is proposed that is identical to (25) and (26)—apart from the orography term, which these authors do not include. The manuscript

² There is a slight difference in the sense that Wunderer (2001) subtracts the spatial average of $f\bar{\eta}$ from the right-hand side of the balance relationship (24) so as to guarantee that the spatial average of the height deviation is zero.

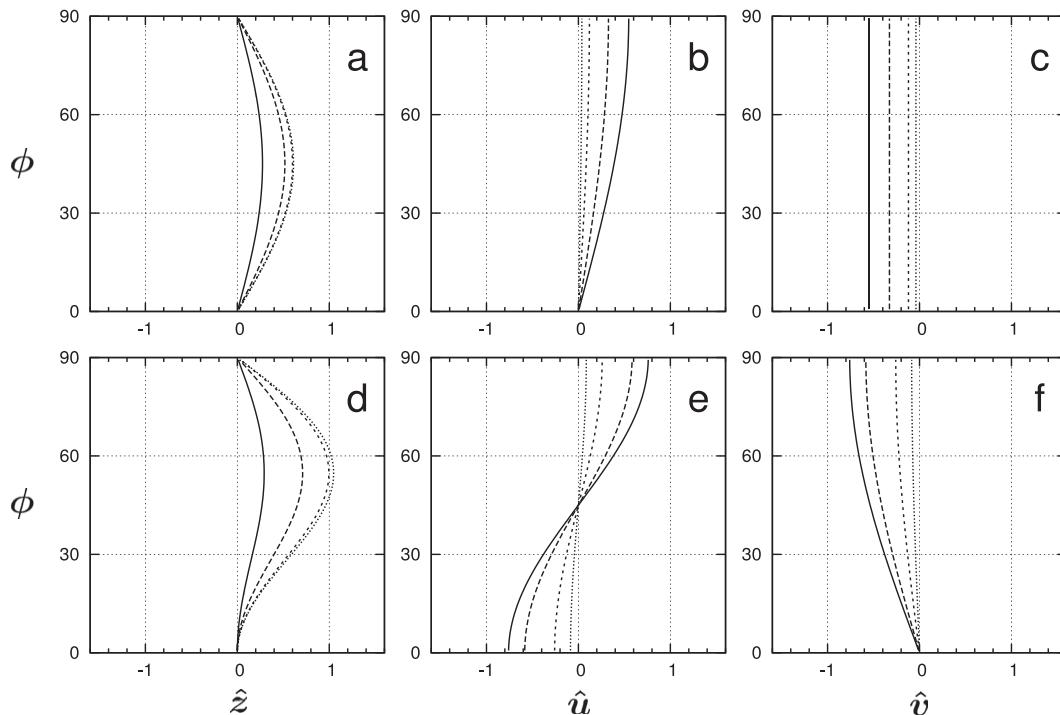


FIG. 6. As in Fig. 4, but for the case that μ^2 in the eigenvalue equation [(40)] is replaced by $1/2$. Here the profiles can be expressed directly in terms of associated Legendre functions and do not change shape when γ varies.

by Schubert et al. (2009) contains a short introduction and motivation of this system—sketched earlier in section 7 of Schubert and Masarik (2006)—and focuses on the eigenvalues and eigenfunctions of (38), the corresponding Rossby wave frequencies as implied by (39), and the implications for large-scale atmospheric turbulence. Although the present work and the work by Schubert et al. (2009) deal with the same balanced system, they emphasize different aspects of its dynamics.

5. Conclusions

We have seen how a globally valid form of the equivalent barotropic vorticity equation can be derived from a one-layer shallow-water model of the atmosphere. Our derivation follows the standard procedure in quasigeostrophic theory except for one crucial difference. We do not make use of either Blackburn's (1985) theoretician's geostrophy or his synoptician's geostrophy; instead, we use Daley's (1983) simplest form of the geostrophic relationship in which the balanced velocity field is nondivergent, $\mathbf{v} = \mathbf{k} \times \nabla\psi$, and where the deviation ϕ of the geopotential from a horizontally uniform reference field—corresponding to the state of rest—is given by $\phi = f\psi$, where f is the Coriolis parameter and ψ is the streamfunction. The basic assumption of a small horizontal divergence is reasonable

for large-scale atmospheric flow and can be applied globally.

The result of this study is an expression of the equivalent barotropic potential vorticity, Eq. (25), that is identical to the expression in its usual form except for the full variation of f^2 in the Cressman (stretching) term. The streamfunction of the balanced velocity can be obtained by inverting a linear differential equation, Eq. (28), which is shown to have unique solutions. The system has a mass as well as an energy invariant. The full variation of f^2 in the equivalent barotropic potential vorticity can be handled without problems. In fact, it gives a reasonably accurate approximation of both the frequencies and meridional structure of the westward-moving waves of the second class of the original shallow-water model, with the planetary waves being the best reproduced. It has been noticed that for waves with short meridional wavelengths the factor f^2 in Eq. (25) can be replaced by a constant value f_0^2 if f_0 is evaluated at $\pm 45^\circ$ latitude. This result legitimizes, to a certain degree, the use of a constant value f_0^2 in heuristic extensions of the beta-plane quasigeostrophic potential vorticity equation to a sphere (Marshall and Molteni 1993; Opsteegh et al. 1998) and provides, in addition, a motivated value of f_0^2 .

The derivation of Daley's (1983) simplest form of the geostrophic relationship from the basic assumption of a

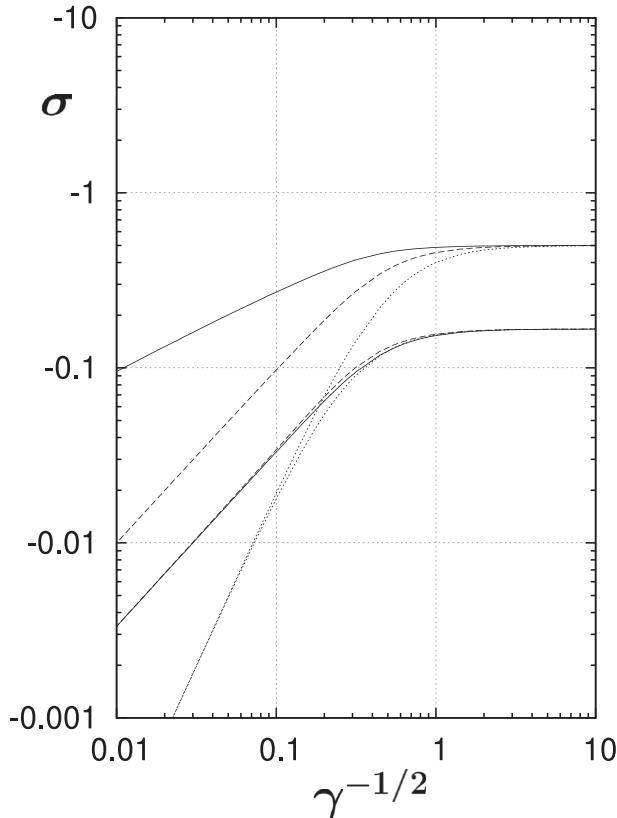


FIG. 7. The value of σ (frequency divided by two) as a function of $\gamma^{-1/2}$ for a Rossby wave (mixed planetary-gravity wave) with $m = 1$ and $n = 1$ (upper curves) and for a Rossby wave (planetary wave) with $m = 1$ and $n = 2$ (lower curves). The solid curves are calculated from the original shallow-water model, the dashed curves from the balanced model with variable μ^2 , and the dotted curves from the balanced model in which μ^2 is replaced by $1/2$.

small horizontal divergence requires two additional approximations. The first is the linearization of the balance equation, leading to the linear balance equation [(15)]. The second is the simplification of the linear balance equation, leading to Daley's (1983) simplest form of the geostrophic relationship [(18)]. To obtain our final result [(25)], we also need to linearize the potential vorticity. The linearization of the balance equation and the potential vorticity do not reveal their limitations in the context of linear wave solutions. This is different for the simplification of the linear balance equation. This simplification is likely to be responsible, at least in part, for the fact that Rossby waves with the largest meridional scales (for which $n = |m|$, corresponding to the mixed planetary-gravity waves) are the least well represented in the balanced system. For the other waves, the consequences of the latter approximation seem to be much less severe, as Fig. 7 clearly indicates.

We conclude that Daley's (1983) simplest form of the geostrophic relationship is a viable basis of a global

balanced approximation of the shallow-water equations on a sphere. It has played this role very satisfactorily in previous work by the author (Verkley 2001), but applications date from much earlier times. Kuo (1959) and Charney and Stern (1962) used this balance to derive a pressure and height coordinate version, respectively, of the continuously stratified equivalent of (25) and (26). In this context we also mention Dickinson (1968), who used it to simplify the linearized primitive equations in height coordinates. For a single vertical mode, the resulting equation is his Eq. (31), which is identical to our Eq. (40). Dickinson's (1968) analysis of this approximate equation is in accord with our results as can be seen from his Fig. 3. The simplified linear balance equation has also been used by Hollingsworth et al. (1976) in their study of momentum transports and by Simmons and Hoskins (1976) in their study of baroclinic instability, where for the linearized primitive equations in sigma coordinates it results in their Eq. (3.2). The same balance condition also forms an essential ingredient of the recent work by Schubert et al. (2009).

With hindsight, expression (25) of the equivalent barotropic potential vorticity could have been derived very quickly from the corresponding beta-plane expression by simply replacing in the latter expression the constant Coriolis parameter f_0 by the variable Coriolis parameter f . The mass and energy invariants (29) and (30) would follow immediately from the advection of the equivalent barotropic potential vorticity by a non-divergent horizontal velocity field, of which the streamfunction is ψ . The relationship between the streamfunction ψ and the surface elevation η could have been obtained by equating the potential energy contribution of (30) to the corresponding contribution of (6), resulting in Daley's (1983) simplest form of the geostrophic relationship (24). This derivation would be concise but would not give much insight into the physical foundation of the resulting system. That foundation should then come from the midlatitude beta-plane derivation of the quasigeostrophic potential vorticity equation and that derivation is based on the theoretician's geostrophy, a form of geostrophy that cannot be generalized to a sphere.

Acknowledgments. It is a pleasure to thank Dr. R. A. Pasmanter for pointing out to me that the solutions of (40) are known as the angular prolate spheroidal wave functions, which are discussed by Abramowitz and Stegun (1965). I am also grateful to Prof. P. Lynch for pointing out to me that Daley's (1983) simplest form of the geostrophic relationship had already found its way in previous studies of the linearized primitive equations.

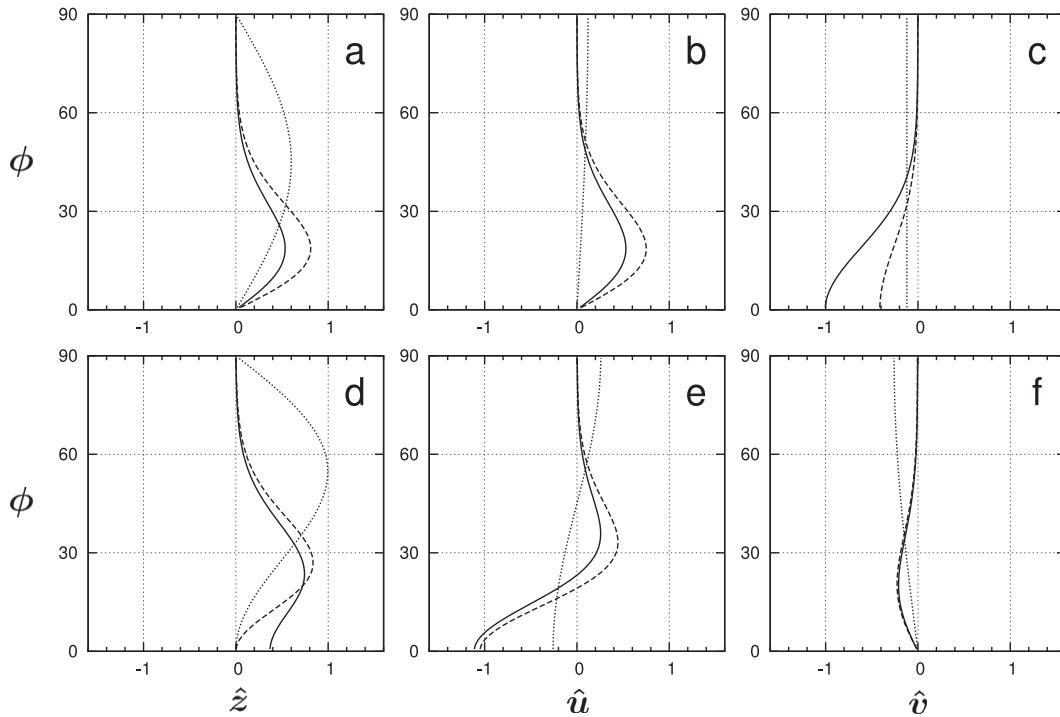


FIG. 8. (a)–(c) Meridional profiles of (a) \hat{z} , (b) \hat{u} , and (c) \hat{v} for a Rossby wave (mixed planetary–gravity wave) with $m = 1$ and $n = 1$. (d)–(f) As in (a)–(c), but for a Rossby wave (planetary wave) with $m = 1$ and $n = 2$. All profiles refer to the case $\gamma = 100$. The solid, dashed, and dotted curves refer to the different models, as in Fig. 7.

I would furthermore like to thank Drs. D. Crommelin, T. Gerkema, R. J. Haarsma, J. D. Opsteegh, G. van der Schrier, and F. M. Selten for their comments on earlier versions of the manuscript. The reviewers are acknowledged for scrutinizing the different steps of the derivation and pointing out the short-cut mentioned in the last paragraph of the conclusions section.

APPENDIX

Mathematical Details

a. Conservation laws of the shallow-water system

Adding the advective part of the material derivative to the term involving the horizontal divergence, the equation for the conservation of mass (3) can be written as

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (H\mathbf{v}) = 0. \tag{A1}$$

By integrating this equation over the whole sphere, it follows immediately that the global mass quantity (5) is constant in time. In a similar way it is possible to derive an energy equation. We have that

$$\frac{\partial}{\partial t} \left(\frac{1}{2} H\mathbf{v}^2 + \frac{1}{2} g\eta^2 \right) = \frac{\partial \eta}{\partial t} \left(\varphi + \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) + H\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t}, \tag{A2}$$

which, after substituting (A1) and (8) and noting that the Coriolis force is perpendicular to the velocity field, leads to

$$\frac{\partial}{\partial t} \left(\frac{1}{2} H\mathbf{v}^2 + \frac{1}{2} g\eta^2 \right) + \nabla \cdot \left[H\mathbf{v} \left(\varphi + \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) \right] = 0. \tag{A3}$$

This results immediately in the conservation of energy (6).

b. Potential vorticity inversion

We will show here that the streamfunction ψ is uniquely determined by the potential vorticity q . To this end we show that the null space of the corresponding operator in (28) is zero. Let us thus assume that ψ satisfies

$$\left(\nabla^2 - \frac{f^2}{gH_A} \right) \psi = 0. \tag{A4}$$

Multiplying this equation by ψ and integrating over the whole sphere S , it follows that

$$\int_S dS \left(\psi \nabla^2 \psi - \frac{f^2}{gH_A} \psi^2 \right) = 0. \tag{A5}$$

Using Gauss' theorem, this can be rewritten as

$$\int_S dS \left(\nabla \psi \cdot \nabla \psi + \frac{f^2}{gH_A} \psi^2 \right) = 0, \tag{A6}$$

from which it follows that $\psi = 0$, thereby proving the assertion.

c. Conservation laws of the balanced system

The balanced equivalents of the mass \mathcal{M} and energy \mathcal{E} are denoted by $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$ and are given by (29) and (30). Because the global integral of the Laplacian is zero, we may write

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\mathcal{M}} &= \frac{\rho H_A}{\Omega} \int dS \frac{\partial}{\partial t} \left(-\nabla^2 \psi + \frac{f^2}{gH_A} \psi \right) \\ &= -\frac{\rho H_A}{\Omega} \int dS \frac{\partial q}{\partial t} = \frac{\rho H_A}{\Omega} \int dS \nabla \cdot (\mathbf{v}q). \end{aligned} \tag{A7}$$

In the last equality we used (26) and the fact that the balanced velocity is nondivergent. Because the latter integral is zero, the result above proves that $\tilde{\mathcal{M}}$ is a conserved quantity in the balanced system. For the balanced equivalent $\tilde{\mathcal{E}}$ of the energy, we have

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\mathcal{E}} &= \rho H_A \int dS \psi \frac{\partial}{\partial t} \left(-\nabla^2 \psi + \frac{f^2}{gH_A} \psi \right) \\ &= -\rho H_A \int dS \psi \frac{\partial q}{\partial t} = \rho H_A \int dS \nabla \cdot (\psi \mathbf{v}q), \end{aligned} \tag{A8}$$

the last expression being zero. This proves that in the balanced system governed by (25) and (26) the energy $\tilde{\mathcal{E}}$ is also a conserved quantity.

d. Solving the eigenvalue equation

To solve (40) numerically, we first write quite generally

$$\hat{\psi}_m(\mu) = \sum_{n=|m|}^{\infty} \hat{\psi}_{mn} P_{mn}(\mu), \tag{A9}$$

where P_{mn} are the associated Legendre functions in the normalization of MACHENHAUER (1979):

$$P_{mn}(\mu) = (-1)^m \left[(2n+1) \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_n^m(\mu), \tag{A10}$$

with $P_n^m(\mu)$ defined by ABRAMOWITZ and STEGUN (1965). In this normalization we have for $\mu^2 P_{mn}(\mu)$

$$\begin{aligned} \mu^2 P_{mn}(\mu) &= \left\{ \frac{[(n-1)^2 - m^2](n^2 - m^2)}{(2n-3)(2n+1)} \right\}^{1/2} \frac{P_{m,n-2}(\mu)}{2n-1} \\ &+ \left[\frac{n^2 - m^2}{2n-1} + \frac{(n+1)^2 - m^2}{2n+3} \right] \frac{P_{mn}(\mu)}{2n+1} \\ &+ \left\{ \frac{[(n+1)^2 - m^2][(n+2)^2 - m^2]}{(2n+1)(2n+5)} \right\}^{1/2} \\ &\times \frac{P_{m,n+2}(\mu)}{2n+3}. \end{aligned} \tag{A11}$$

If we now use the fact that for $\gamma = 0$ the function P_{mn} is an eigenfunction of (40), the expression above can be used to transform the eigenvalue equation [(40)] into the following matrix equation:

$$\sum_{n'=|m|}^{\infty} a_{mnn'} \hat{\psi}_{mn'} = d_m \hat{\psi}_{mn}, \tag{A12}$$

where $a_{mnn'}$ is a tridiagonal matrix with the matrix elements

$$a_{mnn'} = -n(n+1)\delta_{nn'} - \gamma b_{mnn'} \tag{A13}$$

and $b_{mnn'}$ is given by

$$\begin{aligned} b_{mnn'} &= \left\{ \frac{[(n-1)^2 - m^2][n^2 - m^2]}{(2n-3)(2n+1)} \right\}^{1/2} \frac{\delta_{nn'+2}}{2n-1} \\ &+ \left[\frac{n^2 - m^2}{2n-1} + \frac{(n+1)^2 - m^2}{2n+3} \right] \frac{\delta_{nn'}}{2n+1} \\ &+ \left\{ \frac{[(n'-1)^2 - m^2][n'^2 - m^2]}{(2n'-3)(2n'+1)} \right\}^{1/2} \frac{\delta_{n+2n'}}{2n'-1}. \end{aligned} \tag{A14}$$

Note that m is a parameter; for each value of m we have a different eigenvalue problem.

As a result of the normalization (A10), the matrix of the eigenvalue problem is symmetric; it follows therefore that the eigenvalues d_{mm} and the eigenfunctions S_{mn} are real. The system (A12) is solved numerically by replacing the infinite sum by a finite sum with upper limit $N = 105$. The eigenvalue problem for a given value of m has $N - m + 1$ dimensions and can be solved by publicly available numerical routines. We used for this purpose a routine from PRESS et al. (1986) based on the Jacobi method. The eigenvalues are sorted in terms of increasing absolute value. When making plots of the meridional structure of the eigenfunctions, we calculate $P_{mn}(\sin \phi)$ as a function of ϕ by using recurrency relations.

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