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Some general properties of ideal fluids with regular critical levels

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SOME GENERAL PROPERTIES OF IDEAL FLUIDS
WITH REGULAR CRITICAL LEVELS

C.A. van Duin

CONTENTS

Abstract

The properties of reflection and transmission of internal gravity waves in planarly stratified, ideal fluids with a parallel shear flow are investigated. The shear flow and the Brunt-Väisälä frequency are modelled by arbitrary, smooth profiles. When the fluid has one critical level, explicit conditions for wave overreflection and for the existence of singular neutral modes can be derived by means of a Wronskian approach. The same applies when there are two or more critical levels. The theory not only applies to fluids with solid boundaries but also to unbounded flows.

In particular cases, when the profiles of the shear flow and the Brunt-Väisälä frequency are such that reduction to an equation of the Fuchsian type is possible, closed-form expressions for the neutral stability curves can be derived by application of a generalized theorem of Miles, derived in this work. For a given configuration of profiles the neutral stability curves determined in this manner prove to be the complete set of curves for these profiles.

1. Introduction

In this work the propagation and ducting of linear internal gravity waves in planarly stratified fluids with a parallel shear flow normal to the stratification is studied. The model is as follows.
The fluid is ideal and incompressible. The undisturbed density ρ_o and pressure p_o depend on the height z only; increasing z corresponds to increasing height. The background velocity or shear flow $\underline{U}$ depends on z only and the direction is independent of this coordinate. A Cartesian coordinate system is introduced. The x-axis of this system is taken parallel to the shear flow: $\underline{U} = (U(z),0,0)$. The gravitational acceleration has the components (0,0,-g). Rotation of the fluid is not taken into account.
The Brunt-Väisälä frequency N is defined by

$$N^2(z) = -g\,\rho_o^{-1}\,\rho_o' \,. \tag{1.1}$$

The prime denotes differentiation with respect to z. It is assumed that the undisturbed density decreases with height, i.e. the density distribution is statically stable. The Brunt-Väisälä frequency is real in this case.
The local Richardson number J is defined by

$$J(z) = N^2\,(U')^{-2}. \tag{1.2}$$

The perturbation quantities are time harmonic with frequency ω, and the horizontal phase velocity of the waves is parallel to the shear velocity, i.e. parallel to the x-axis. The perturbation quantities are independent of y:

$$q(x,z,t) = q(z)\exp\{i(kx-\omega t)\} = q(z)\exp\{ik(x-ct)\}, \tag{1.3}$$

where $c=\omega/k$ is the horizontal component of the phase velocity.

By linearizing the equations of motion and the continuity equation in the usual way, and remembering that the fluid is incompressible, we derive from the resulting equations

$$w'' + \rho_o^{-1} \rho_o' w' + \left\{ \frac{N^2}{(U-c)^2} - \rho_o^{-1} \rho_o' \frac{U'}{U-c} - \frac{U''}{U-c} - k^2 \right\} w = 0, \tag{1.4}$$

where $w(z)$ is the vertical component of the perturbed velocity. It will be convenient to transform equation (1.4) into the Helmholtz form

$$\phi'' + \left\{ \frac{N^2}{(U-c)^2} - \rho_o^{-1} \rho_o' \frac{U'}{U-c} - \frac{U''}{U-c} - k^2 + \right. \\ \left. - \tfrac{1}{4} (\rho_o^{-1} \rho_o')^2 - \tfrac{1}{2} (\rho_o^{-1} \rho_o')' \right\} \phi = 0, \tag{1.5}$$

where ϕ is related to w through the transformation

$$w(z) = \rho_o^{-\frac{1}{2}} \phi(z). \tag{1.6}$$

Within the so-called Boussinesq approximation equation (1.4) reduces to the Taylor-Goldstein equation

$$w'' + \left\{ \frac{N^2}{(U-c)^2} - \frac{U''}{U-c} - k^2 \right\} w = 0. \tag{1.7}$$

It is assumed that the fluid has at least one critical level, i.e. a level or height where the shear flow matches the horizontal phase velocity. We only consider regular critical levels, i.e. the zeroes z_k of $U(z)-c$ are simple zeroes. Then the singularities of equations (1.4) or (1.5) and (1.7) are poles of second order in $z-z_k$, i.e. regular singularities. For an irregular critical level, i.e. if $U(z_k)=c$ and $U'(z_k)=0$, the singularity at z_k is a pole of fourth order in $z-z_k$. Such a singularity is irregular.

Irregular critical levels may be relevant when jet-type flows are considered.

Since the classical paper by Booker and Bretherton[1] on propagation properties of internal gravity waves incident on a fluid with a shear flow containing one regular critical level many authors studied this subject. We recapitulate a number of known results from this literature. Booker and Bretherton[1] have shown that for $J_c > 1/4$, where J_c denotes the Richardson number at the place of the critical level, part of the energy of the incident wave is absorbed into the mean flow. For $J_c >> 1/4$ they have shown that the transmission coefficient is extremely small. The case $J_c \leq 1/4$ has not been treated by these authors. For sufficiently small values of J_c, overreflection can occur, which means that the amplitude of the reflected wave exceeds that of the incident one. This phenomenon has been discussed by various authors, e.g.[2-6]. Generally it is assumed that $J_c < 1/4$ is a necessary condition for overreflection. McKenzie[3] and Acheson[5] made use of the vortex-sheet profile. Such a profile allows the occurrence of resonant overreflection, which means that the shear flow, in the absence of an incident wave, spontaneously emits outgoing waves. The occurrence of this phenomenon has been discussed by e.g. McKenzie[3], Lindzen[7] and Grimshaw[8]. Lindzen and Grimshaw also considered the vortex-sheet profile. When the shear flow is modelled by a hyperbolic tangent profile, however, resonant overreflection does not occur[6]. In our opinion such a phenomenon cannot occur for smooth velocity profiles.

The so-called neutral stability curves are determined by solving equation (1.5) with boundary conditions $\phi \to 0$ for $z \to \pm \infty$ (or with finite boundaries, i.e. $\phi = 0$ at $z = z_1, z_2$) for real values of c. Any point of these curves corresponds to an eigenvalue of (1.5). In several cases a neutral stability curve proves to be a stability boundary[9-13]. This means that when such a curve is known, it is frequently possible to determine for what values of J_c and of other parameter values the shear flow is unstable. Howard[12] described a technique to test whether a neutral stability curve is a stability boundary or not.

However, there are cases known where Howard's technique fails[14,15]. A necessary condition for instability is that $J_c < 1/4$. Note that this condition is the same as that for overreflection. So when overreflection occurs, the shear flow may be unstable. This has been confirmed by Jones[2] and Van Duin and Kelder[6] for pariticular shear flow and buoyancy frequency profiles. The latter authors have also shown that overreflection cannot occur for values of J_c for which the shear flow is stable, i.e. overreflection implies instability. The reverse, however, is not necessarily true[6]. Acheson[5] considering the vortex-sheet profile, has shown that overreflection can occur even if the shear flow is stable. However, this model cannot be justified from a physical point of view[16]. Drazin et al.[17] and Teitelbaum and Kelder[18] studied the reflection properties of waves in fluids with two critical levels. The latter authors have shown that in such a fluid overreflection can also occur.

In this work we will derive some general properties of equation (1.5). These properties also apply to equation (1.7), i.e. the Boussinesq approximation to (1.5). We outline the essentials.

The Wronskian of the solutions ϕ (z) and ϕ^* (z) of equation (1.5), where the asterisk denotes the complex conjugate, is independent of z in regions without critical levels. The Wronskian is related to the Reynold stress. At the place of a critical level this stress is discontinuous[9]. In section 2 the Wronskian approach is introduced. The results derived are valid for arbitrary, smooth background velocity and Brunt-Väisälä frequency profiles. It is assumed, however, that the Brunt-Väisälä frequency does not vanish at one of the critical levels. Throughout we only consider unbounded flows.

In section 3 it will be shown that the Wronskian approach is suitable to derive results that are generalizations of results derived by Miles[9]. This author examined monotonic background velocities in a fluid with one critical level. When the fluid has two critical levels, the Wronskian approach also applies (section 4). In the sections 3 and 4 we will derive necessary conditions for the occurrence of overreflection and the existence of singular neutral modes, i.e. eigensolutions of (1.5) for real phase velocity c. These modes are related to neutral stability curves.

Section 5 is devoted to the determination of neutral stability curves for a particular class of profiles for the background velocity and the Brunt-Väisälä frequency. The class of profiles considered is such that equation (1.5) or (1.7) is reducible to an equation of the Fuchsian type. The latter equation is ordinary and linear, and every singular point in this equation, including the point at infinity, is a regular singular point[19]. By application of results derived in the previous sections, combined with the theory of Fuchsian equations, it will be shown that it is often possible to derive closed-form expressions for the neutral stability curves.

This work is an extension of chapter 6 of the author's thesis[20].

2. The Wronskian approach

The starting point of our treatment is the Helmholtz equation

$$\phi'' + k^2(z)\,\phi = 0, \qquad (2.1)$$

where $k^2(z)$ is real for real values of z. The prime denotes differentiation with respect to z.
We observe that if $\phi\,(z)$ is a solution of (2.1), the complex conjugate $\phi^{*}\,(z)$ is also a solution of this equation. We define

$$W(\phi, \phi^{*}) = i(\phi\,\phi^{*\prime} - \phi^{*}\phi'). \qquad (2.2)$$

The expression between brackets is known as the Wronskian of $\phi\,(z)$ and $\phi^{*}\,(z)$. When the profile $k^2(z)$ is without singularities, the Wronskian is independent of z. However, when $k^2(z)$ has a singularity at z=0, say, the value of the Wronskian in the region $z < 0$ is in general different from the value in the region $z > 0$. This is due to the fact that the point $z = 0$ is then a branch point of the solutions of equation (2.1). When equation (2.1) is related to an equation which governs the propagation of monochromatic waves in a planarly stratified medium with propagation properties that depend on the Cartesian coordinate z only, the assumption of real $k^2\,(z)$ means that dissipative effects are not taken into account.
Now we start from equation (1.5). Through the transformation (1.6) this equation is related to the equation governing the propagation of internal gravity waves in an incompressible, inviscid fluid, (eq. (1.4)).
The dependent variable w in (1.4) is the z-component of the perturbed velocity which is assumed to be proportional to the time factor $\exp(-i\omega t)$. Throughout we start from the presence of one or more critical levels, i.e. levels at which $U(z) = \omega/c$. At these levels equation (1.4), and the same applies to (1.7), has a singularity. One of the critical levels is situated at z=0, say.

Moreover we only consider regular critical levels, i.e. the singularities are characterized by poles of second order in the independent variable. The Richardson number at the place of the critical level situated at z=0 is denoted by J_c. The Brunt-Väisälä frequency N(z) and the shear flow U(z) are arbitrary, smooth functions of z. It is assumed that the Brunt-Väisälä frequency does not vanish at one of the critical levels.

For $J_c \neq 1/4$ the general solution of equation (1.5) in a neighbourhood of the point z=0 is a linear combination of the functions

$$\phi^{\pm}(z) = z^{\gamma^{\pm}}(1+a_1^{\pm} z+.....) = z^{\gamma^{\pm}} x^{\pm}(z), \qquad (2.3)$$

where

$$\gamma^{\pm} = \tfrac{1}{2} \pm i\,(J_c - \tfrac{1}{4})^{\frac{1}{2}}. \qquad (2.4)$$

For $J_c = \frac{1}{4}$ the general solution of (1.5) around z=0 is a linear combination of the functions

$$\phi^{(1)}(z) = z^{\frac{1}{2}} \sum_{n=0}^{\infty} b_n z^n, \qquad (2.5a)$$

$$\phi^{(2)}(z) = \phi^{(1)}(z) \log z + z^{\frac{3}{2}} \sum_{n=0}^{\infty} c_n z^n. \qquad (2.5b)$$

To make the solutions (2.3-5) one-valued around z=0, we have to introduce a branch cut in the complex z-plane. When the slope of the shear flow in z=0 is positive (negative), the branch cut should be taken upwards (downwards) [1].

We introduce the notations (cf. (2.4))

$$\gamma^{\pm} = \begin{cases} \tfrac{1}{2} \pm i\gamma_1 , & J_c > \tfrac{1}{4} , \\ \tfrac{1}{2} \pm \gamma_2 , & J_c < \tfrac{1}{4} , \end{cases} \qquad (2.6)$$

where γ_1 and γ_2 are taken positive. The exponents $\gamma^{\pm}$ are complex for $J_c > 1/4$, with $\gamma^+ = (\gamma^-)^*$, and real for $J_c < 1/4$.
The same is true for the functions $\chi^{\pm}(z)$ in (2.3): for $J_c > 1/4$ we have $\chi^+(z) = \{\chi^-(z)\}^*$; for $J_c < 1/4$ these functions are real. The coefficients b_n and c_n in (2.5) are real. It will be convenient to replace $\phi^+(z)$, $\phi^{(1)}(z)$ by $\phi_1(z)$ and $\phi^-(z)$, $\phi^{(2)}(z)$ by $\phi_2(z)$.
The total wave $\phi(z)$ is a linear combination of the functions $\phi_1(z)$ and $\phi_2(z)$:

$$\phi(z) = A_1 \phi_1(z) + A_2 \phi_2(z). \qquad (2.7)$$

In the rest of this section we assume the presence of one critical level, which is situated at z=0. From (2.2) and (2.7) it follows that, for $z \neq 0$

$$\begin{aligned} W(\phi,\phi^*) = {} & A_1A_1^*W(\phi_1, \phi_1^*) + A_2A_2^*W(\phi_2, \phi_2^*) \\ & + A_1A_2^*W(\phi_1, \phi_2^*) + A_2A_1^*W(\phi_2, \phi_1^*). \end{aligned} \qquad (2.8)$$

For $J_c < 1/4$ relation (2.8) reduces to

$$W(\phi,\phi^*) = \begin{cases} 4\gamma_2 \,\mathrm{Im}(A_1A_2^*), & z > 0, \\ -4\gamma_2 \,\mathrm{Im}\{A_1A_2^* \exp(\pm 2i\pi\gamma_2)\}, & z < 0, \end{cases} \qquad (2.9)$$

where Im denotes the imaginary part. The plus (minus) sign in the arguments of the exponential functions must be chosen accordingly as the slope of the shear flow at z=0 is negative (positive).

For $J_c = 1/4$ we obtain the relations

$$W(\phi,\phi^{*}) = \begin{cases} -2\,\mathrm{Im}\,(A_1A_2^{*}), & z > 0, \\ \pm 2\pi\, A_2A_2^{*} + 2\,\mathrm{Im}\,(A_1A_2^{*}), & z < 0. \end{cases} \qquad (2.10)$$

The meaning of the symbol $\pm$ in (2.10) is the same as that in (2.9).

For $J_c > 1/4$ relation (2.8) reduces to

$$W(\phi,\phi^{*}) = \begin{cases} 2\,\gamma_1\,(A_1A_1^{*} - A_2A_2^{*}), & z > 0, \\ 2\,\gamma_1\,[A_2A_2^{*}\exp(\pm 2\pi\gamma_1) - A_1A_1^{*}\exp\{-(\pm 2\pi\gamma_1)\}], & z < 0. \end{cases} \qquad (2.11)$$

When we start from the Taylor-Goldstein equation (1.7), the solutions of this equation in a sufficiently small neighbourhood of the point z=0 are also of the form (2.3) and (2.5) but with other values of the coefficients a_n, b_n, and c_n. Nevertheless, the same expressions (2.9 - 11) for the Wronskian (2.2) will be derived from (2.7). This is due to the fact that, for given N(z) and U(z), the solutions of equations (1.5) and (1.7) have the same behaviour in a small neighbourhood of the singular point z=0 since the term with $(U-c)^{-2}$ is predominant for $z \to 0$.
Next we write equation (1.5) in the Helmholtz form (2.1). The profile $k^2(z)$ has finite limits $k^2(\pm\infty)$ for $z \to \pm\infty$, with $k^2(\pm\infty) \neq 0$. If $k^2(-\infty) < 0$, $k^2(+\infty) > 0$ or $k^2(-\infty) > 0$, $k^2(+\infty) < 0$, we deal with a reflection problem. The regions $z \to -\infty$ resp. $z \to +\infty$ are then opaque while the regions $z \to +\infty$ resp. $z \to -\infty$ are transparent. If $k^2(-\infty) > 0$, $k^2(+\infty) > 0$, the incident wave is both reflected and transmitted for the two cases of incidence in the positive or negative z-direction since the regions $z \to \pm\infty$ are transparent in this case. If $k^2(-\infty) < 0$, $k^2(+\infty) < 0$, we have to do with an eigenvalue problem because we then should require that $\phi(z) \to 0$ for $z \to \pm\infty$ in order that the energy content of the wave is finite.

Let the asymptotic behaviour of the total wave $\phi(z)$ be of the form

$$\phi(z) \sim g^{\pm}(z), \qquad (z \to \pm \infty), \qquad (2.12)$$

We suppose that the profile $k^2(z)$ is such that

$$\frac{d\phi}{dz} \sim \frac{dg^{\pm}}{dz}, \qquad (z \to \pm \infty). \qquad (2.13)$$

So we suppose that differentiation of the asymptotic relation (2.12) is permissible.

It will be convenient to introduce the notations

$$k^{\pm} = \{ k^2(\pm \infty)\}^{\frac{1}{2}}, \qquad k^2(\pm \infty) > 0, \qquad (2.14a)$$

$$p^{\pm} = \{-k^2(\pm \infty)\}^{\frac{1}{2}}, \qquad k^2(\pm \infty) < 0. \qquad (2.14b)$$

Apart from multiplicative constants the symbols $k^{\pm}$ and $p^{\pm}$ determine the asymptotic behaviour of $\phi(z)$ and its derivative. When the region $z \to -\infty$ is transparent, the symbol k^{-} denotes the vertical wave number in that region. When the region $z \to -\infty$ is opaque, p^{-} determines the rate at which the decreasing part of $\phi(z)$ vanishes as $z \to -\infty$. When the region $z \to +\infty$ is considered, k^{+} resp. p^{+} denote vertical wave number resp. rate of vanishing in that region.

3. Fluids with one critical level

We consider a fluid with one critical level. This level is situated at $z=0$. The slope of the background velocity profile is positive at the place of the critical level, hence $U(z) < c$ for $z < 0$ and $U(z) > c$ for $z > 0$. The value of the Richardson number at the place of the critical level is denoted by J_c.
Neutral modes are solutions of equation (1.5) that for real frequencies satisfy the boundary conditions

$$\phi = 0, \qquad (z = z_1, z_2). \tag{3.1}$$

We only consider singular neutral modes, i.e. the critical level is situated within the interval (z_1, z_2). Consequently, $z_1 < 0 < z_2$. Instead of the solid boundaries z_1 and z_2 one also considers unbounded flows, i.e. boundaries at infinity:

$$\phi \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{3.2}$$

In this section the following theorems will be proven:

(i) For $J_c > 1/4$ overreflection cannot occur, i.e. the absolute value of the reflection coefficient is at most unity for these values of J_c. For $J_c = 1/4$ total reflection is, at least in principle, possible provided the region that is opposite to the region of incidence is opaque, i.e. there is no transmitted energy.

(ii) The existence of singular neutral modes requires that $J_c \leq 1/4$.

(iii) For $J_c < 1/4$ a singular neutral mode, if existing, is proportional to either $\phi_1(z)$ or $\phi_2(z)$. For $J_c = 1/4$ such a mode is proportional to (2.5a), i.e. proportional to the solution without a logarithmic singularity.

The theorems (ii) and (iii) will be derived under the assumption of unbounded flows, i.e. the boundary conditions (3.2) are considered.

However, when regarding the solid boundaries (3.1), these theorems remain valid. We remark that the theorems (ii) and (iii) are generalizations of previous results derived by Miles[9] for monotonic shear flow profiles.

Proof of theorem i.

We assume the incident wave to be propagating in the positive z-direction. Since the source is placed at the end point $z = -\infty$, this implies that $k^2(-\infty) > 0$. The amplitude of the incident wave is normalized to unity at $z = -\infty$. In the transparent region $z \to -\infty$, where $U(z) < c$, the z-component of the group velocity of the wave has the opposite sign of the z-component of the phase velocity.

We first consider the case $k^2(+\infty) > 0$. In the transparent region $z \to +\infty$, where $U(z) > c$, the vertical group velocity and phase velocity have the same direction. Then the asymptotic behaviour (2.12) of $\phi(z)$ should be prescribed according to (cf. (2.12))

$$g^-(z) = e^{-ik^-z} + Re^{ik^-z}, \qquad g^+(z) = Te^{ik^+z}, \tag{3.3}$$

where R and T are the reflection and transmission coefficients, respectively. From (2.2), (2.13) and (3.3) it follows that

$$W(\phi,\phi^*) = \begin{cases} 2k^+TT^*, & z > 0, \\ 2k^-(RR^*-1), & z < 0. \end{cases} \tag{3.4}$$

By equating (2.10), valid for $J_c = 1/4$, and (3.4) we obtain

$$RR^* + (k^+ / k^-)\, TT^* < 1, \tag{3.5}$$

hence $|R|<1$. By equating (2.11), valid for $J_c > 1/4$, and (3.4) we obtain an expression that is similar to (3.5). Note that the minus sign must be chosen in (2.10) and (2.11).

Next we consider the case $k^2(+\infty) < 0$, i.e. the region $z \rightarrow +\infty$ is opaque. Then we must require that $\phi(z)$ vanishes as $z \rightarrow +\infty$, i.e. the function $g^+(z)$ in (3.3) must be proportional to $\exp(-p^+z)$, where p^+ is given by (2.14b). Consequently, $W(\phi,\phi^*) = 0$ for $z > 0$. For $J_c > 1/4$ we find again that $|R| < 1$. For $J_c = 1/4$ total reflection is, at least in principle, possible. This can readily be seen from (2.10) under the assumption that the coefficient A_2 in (2.7) vanishes. If $A_1 = 0$, however, we find that $|R| < 1$ for $J_c = 1/4$.

When the incident wave propagates in the negative z-direction, the proof of the theorem is similar. When $k^2(-\infty) > 0$, the relation between $|R|$ and $|T|$ is of the form

$$RR^* + (k^- / k^+)\, TT^* \leq 1 \qquad (3.6)$$

When $k^2(-\infty) < 0$, we find that $|R| \leq 1$. Again total reflection is in principal possible for $J_c = 1/4$, and $|R| < 1$ for $J_c > 1/4$.

Proof of theorem ii.

The boundary conditions (3.2) can only be satisfied if $k^2(-\infty) < 0$ and $k^2(+\infty) < 0$, i.e. the regions $z \rightarrow \pm\infty$ must be opaque. Hence the functions $g^-(z)$ and $g^+(z)$ in (2.12) must be proportional to $\exp(p^-z)$ and $\exp(-p^+z)$, where p^- and p^+ are given by (2.14b). Thus singular neutral modes exist provided J_c, A_1, and A_2 are such that

$$W(\phi, \phi^*) = 0, \quad z \neq 0. \qquad (3.7)$$

Then it follows from (2.11) that for $J_c > 1/4$ relation (3.7) is satisfied if $\sinh(2\pi\gamma_1) = 0$. Since $\gamma_1 > 0$, we are led to a contradiction: for $J_c > 1/4$ singular neutral modes cannot exist.

Proof of theorem iii.

By equating (2.9) and (3.7) we obtain $A_1A_2^* \sin(2\pi\gamma_2) = 0$, from which it follows that $A_1 = 0$ or $A_2 = 0$ since $\gamma_2 > 0$.

For $J_c = 1/4$ we are led to the necessary condition $A_2 = 0$, cf. (2.10). This means that for $J_c = 1/4$ singular neutral modes, if existing, are proportional to (2.5a), cf. (2.7) where $\phi_1(z)$ is the counterpart of (2.5a).

We remark that the expressions (3.5) and (3.6), valid for $J_c \geq 1/4$, indicate that wave energy is lost. When there is no critical level, we must replace (3.5) and (3.6) by

$$RR^* + (k^+ / k^-)\, TT^* = 1 \text{ and } RR^* + (k^- / k^+)\, TT^* = 1, \qquad (3.8a,b)$$

respectively. These expressions indicate an energy flux balance.

It is well-known that overreflection is in principle possible for sufficiently small $J_c < 1/4$. The occurrence of this phenomenon, however, is not only determined by the value of J_c but also by other parameter values[6].
The Wronskian approach is not suitable to derive more explicit criteria for overreflection. This is due to the fact that the absolute value as well as the argument of the complex product $A_1 A_2^*$ are not known for $J_c < 1/4$ (A_1 and A_2 are the coefficients in relation (2.7)). For fixed γ_2 this product determines the values of the Wronskian (2.9), i.e. the absolute values of the reflection and transmission coefficients.
To be more specific, we consider an incident wave propagating in the positive z-direction. By equating (2.9) and (3.4) we find that $|R|$ and $|T|$ are uniquely determined by $A_1 A_2^*$ when the wave numbers at infinity, i.e. $k^{\pm}$, and γ_2 are fixed (γ_2 is determined by J_c, cf. (2.4) and (2.6)). When the region $z \to +\infty$ is opaque, $A_1 A_2^*$ is real. Then the sign of RR^*-1 is determined by the sign of $A_1 A_2^*$, i.e. $|R| < 1$ for $A_1 A_2^* < 0$ and $|R| > 1$ for $A_1 A_2^* > 0$. As has been mentioned before, however, $A_1 A_2^*$ is not known.

4. Fluids with two critical levels

The critical levels are situated at z=0 and z=d, where d is the distance between these levels. The symbols J_{c1} and J_{c2} denote the Richardson numbers at z=0 and z=d, respectively. The expressions (2.9 - 11) for the Wronskian (2.2) as determined from (2.3), (2.5) and (2.7), where γ_1 and γ_2 are defined by (2.4) and (2.6) with J_c replaced by J_{c1}, are now valid in the regions $z < 0$ and $0 < z < d$. We assume that the slope of the background velocity U(z) is positive at z=0. Hence $U(z) > c$ for $0 < z < d$ and $U(z) < c$ outside the interval [0,d]. To determine the value of the Wronskian in the region $z > 0$ we proceed as follows.
The solutions of equation (1.5) around the point z=d are denoted by $\phi_{1d}(z)$ and $\phi_{2d}(z)$; these solutions are taken as the counterparts of the solutions $\phi_1(z)$ and $\phi_2(z)$ as defined in section 2.
The total wave $\phi(z)$ is a linear combination of $\phi_{1d}(z)$, $\phi_{2d}(z)$:

$$\phi(z) = B_1\phi_{1d}(z) + B_2\phi_{2d}(z). \tag{4.1}$$

The expressions for the Wronskian (2.2) as determined from (4.1) are now given by (2.9 - 11) with A_k replaced by B_k; γ_1 and γ_2 are to be expressed in the parameter J_{c2}. The resulting expressions are valid for $z > 0$, $z \neq d$. Since the slope of U(z) is negative at z=d, the plus sign must be taken in (2.9 - 11). Note that the vertical components of the phase velocity and the group velocity of the wave have opposite signs in the regions $z \to \pm\infty$ when these regions are transparent. Consequently, when the incident wave propagates in the positive z-direction,and the region $z \to +\infty$ is transparent, the Wronskian is negative for $z > d$.
The existence of (singular) neutral modes requires that $W(\phi,\phi^*) = 0$ outside the interval [0,d]. Within this interval, however, the Wronskian does not necessarily vanish. There are two cases known where the Wronskian also vanishes in the interval [0,d]:

(a) The profiles U(z) and N(z) are such that the problem described is symmetric (section 5).

(b) The special case when $J_{c1} = J_{c2} = 1/4$, as will be shown in this section. Note that the "symmetric problem" includes this case.

The proof of the next theorems is rather straightforward. Therefore we content ourselves to citing them:

(iv) For $J_{c1} \geq 1/4$ and $J_{c2} \geq 1/4$ overreflection cannot occur. For $J_{c1} = J_{c2} = 1/4$ total reflection is, at least in principle, possible provided the region that is opposite the region of incidence is opaque, i.e. provided there is no transmitted energy.

(v) For $J_{c1} > 1/4$ and $J_{c2} > 1/4$ singular neutral modes cannot exist. The same is true when $J_{ci} = 1/4$ and $J_{ck} > 1/4$, where J_{ci} and J_{ck} denote the Richardson numbers at the critical levels.

(vi) For $J_{c1} = J_{c2} = 1/4$ a singular neutral mode, if existing, is totally represented by the solutions $\phi_1(z)$ and $\phi_{1d}(z)$, i.e. $A_2 = 0$ and $B_2 = 0$ in (2.7) and (4.1). Moreover, for $J_{c1} = J_{c2} = 1/4$, the Wronskian also vanishes in the region $0 < z < d$, i.e. there is no net energy flux from one critical level to another in this special case.

(vii) Under the assumption that the existence of singular neutral modes implies that the Wronskian also vanishes in the region $0 < z < d$, a necessary condition for such modes is that $J_{c1} \leq 1/4$ and $J_{c2} \leq 1/4$. In other words, singular neutral modes then cannot exist when the Richardson number at one of the critical levels exceeds the value 1/4.

Under the above assumption a singular neutral mode, if existing, is totally represented by $\phi_i(z)$ and $\phi_{kd}(z)$, where $i = 1$ or 2 and $k = 1$ or 2. The solution with a logarithmic term is excluded, i.e. the coefficient A_2 in (2.7) or B_2 in (4.1) vanishes when $J_{c1} = 1/4$ or $J_{c2} = 1/4$. For $J_{c1} = J_{c2} = 1/4$ see also theorem (vi).

The theorems (v) through (vii) apply to solid boundaries as well as to boundaries at infinity (unbounded flows).

Finally, we remark that the Wronskian approach is also suitable to determine reflection properties of fluids with more than two critical levels. The same applies to finding criteria for the existence of singular neutral modes. To analyse this problem is beyond the scope of this work.

5. Neutral stability curves

In this section the Brunt-Väisälä frequency and the background velocity will be modelled by profiles which are such that the governing equation, i.e. (1.5) or (1.7), is reducible to a differential equation of the Fuchsian type. The latter equation is an ordinary, linear, homogeneous equation in which every singular point is a regular singular point, including a possible singularity at infinity[19].
When the fluid has one critical level, the resulting Fuchsian equation has at least three singular points. When the fluid has two critical levels, the number of singular points amounts to at least four.
The number of singular points in the resulting Fuchsian equation depends on the number of critical levels and on the shape of the profiles of the background velocity and the Brunt-Väisälä frequency.
It will be shown that application of the results derived in the previous sections, in combination with the theory of Fuchsian equations, often enables us to derive closed-form expressions for the neutral stability curves. This is especially the case when it can be shown that the existence of singular neutral modes requires that the resulting Fuchsian equation has polynomial solutions. When this equation has no such solutions for physically meaningful values of the parameters, this means that singular neutral modes do not exist for the profiles of U(z) and N(z) considered. The theory will be worked out on the basis of a number of specific examples.
We first consider a fluid with one critical level. The profiles are of the form

$$\rho_o(z) = \rho_o(0)\exp(-\mu z), \quad U(z) = \frac{U_o}{2}\left(1 + \tanh\frac{z}{2d}\right). \tag{5.1}$$

For this density distribution the Brunt-Väisälä frequency is a constant: $N^2(z) = \mu g := N_o^2$.

Substitution of the transformation

$$\eta(z) = \tfrac{1}{2}\left(1 + \tanh\frac{z}{2d}\right) \tag{5.2}$$

into (1.4), with $\rho_o(z)$ and $U(z)$ given by (5.1), yields the equation

$$\frac{d^2w}{d\eta^2} + \frac{2\eta+\chi-1}{\eta(\eta-1)}\frac{dw}{d\eta} + \left\{\frac{J-p^2(\eta-a)^2}{\eta^2(\eta-a)^2(\eta-1)^2} + \frac{1-\chi-2\eta}{\eta(\eta-a)(\eta-1)}\right\} w=0, \qquad (5.3)$$

with the dimensionless parameters

$$a = c/U_o, \quad \chi = \mu d, \quad p = kd, \quad J = N_o^2 d^2/U_o^2. \qquad (5.4a,b,c,d)$$

Equation (5.3) is of the Fuchsian type, with singular points $\eta = 0, a, 1$, and ∞. It is a special form of Heun's equation[21,22].
The singular point $\eta = a$ corresponds to the place of the critical level. Hence $0 < a < 1$. The singular points $\eta = 0$ and $\eta = 1$ correspond to the end points $z = \pm\infty$ (cf. (5.2)). Note that the point at infinity has no physical significance.
The Richardson number at the place of the critical level is given by

$$J_c = J/a^2(1-a)^2. \qquad (5.5)$$

Now equation (5.3) is brought into standard form. This is done by means of the transformation

$$w = \eta^\alpha (1-\eta)^\beta (\eta-a)^\gamma \; v(\eta), \qquad (5.6)$$

where

$$\alpha^2 - \chi\alpha + (1-a)^2 J_c - p^2 = 0, \qquad (5.7a)$$

$$\beta^2 + \chi\beta + a^2 J_c - p^2 = 0, \qquad (5.7b)$$

$$\gamma^2 - \gamma + J_c = 0. \qquad (5.7c)$$

The standard form reads

$$\frac{d^2v}{d\eta^2} + \left(\frac{1+2\alpha-\chi}{\eta} + \frac{1+2\beta+\chi}{\eta-1} + \frac{2\gamma}{\eta-a}\right)\frac{dv}{d\eta} + \frac{A_o+A_1\eta}{\eta(\eta-a)(\eta-1)}\, v = 0, \tag{5.8}$$

where

$$A_o = (\alpha+\beta+\gamma-1)\chi-\alpha^2+\beta^2-\gamma^2+1-2\alpha\gamma$$
$$-a(\alpha^2+\beta^2-\gamma^2+\alpha+\beta+\gamma+2\alpha\beta), \tag{5.9a}$$

$$A_1 = (\alpha+\beta+\gamma+2)(\alpha+\beta+\gamma-1). \tag{5.9b}$$

To determine the neutral stability curve(s), we investigate the asymptotic behaviour of the solutions of (1.4) as $z \to \pm\infty$ and the behaviour of these solutions in a neighbourhood of the critical level.
The general solution of equation (5.3) in a neighbourhood of the point $\eta = 1$ is of the form

$$w = \eta^{\alpha}\ (\eta - a)^{\gamma}\{A_1\ (1 - \eta)^{\beta_1}v_1(\eta) + A_2\ (1 - \eta)^{\beta_2}v_2(\eta)\}\ .$$

The exponents β_1 and β_2 are the roots of (5.7b); $v_1(\eta)$ and $v_2(\eta)$ are power series in $1 - \eta$, with $\eta = \eta(z)$ given by (5.2). A_1 and A_2 are constants.
Since $\eta \to 1$ as $z \to +\infty$, the exponents β_1 and β_2 determine the asymptotic behaviour of w as $z \to +\infty$. From the requirement that w vanishes in this region, it follows that at least one of the exponents must be positive, i.e.

$$p^2 > a^2J_c, \tag{5.10}$$

cf. (5.7b). This implies that $\beta_1\beta_2 < 0$, i.e. one of the exponents, β_1 say, is positive and $\beta_2 < 0$. Consequently, the solution of (5.3) with the exponent β_1 has the required asymptotic behaviour. Since the solution with the exponent β_2 increases as $z \to +\infty$, the eigensolutions reduce to solutions of the form

$$w = \eta^{\alpha} (1 - \eta)^{\beta} (\eta - a)^{\gamma} v_1(\eta), \text{ with } \beta = \beta_1,$$

where $v_1(\eta)$ is a power-series solution of (5.8) in a neighbourhood of the point $\eta = 1$. This series, converging in the domain $|1-\eta| < 1-a$, defines an analytic solution of η in this domain.
To determine the asymptotic behaviour of w as $z \to -\infty$, we can proceed in a similar way as before. Since $\eta \to 0$ as $z \to -\infty$, we determine to that purpose the general solution of equation (5.3) in a neighbourhood of the point $\eta = 0$. The roots α_1 and α_2 of (5.7a) determine the behaviour of the solutions as $z \to -\infty$. These roots are real provided

$$p^2 + (\chi^2/4) - (1 - a)^2 J_c \geq 0. \qquad (5.11)$$

This condition is necessary for the existence of singular neutral modes. From theorem iii it follows that for $J_c < 1/4$ a singular neutral mode is represented by one of the solutions around η=a:

$$\eta^{\alpha} (1 - \eta)^{\beta} (\eta - a)^{\gamma_k} v_a^{(k)}(\eta), \quad k = 1 \text{ or } 2,$$

where γ_1 and γ_2 are the distinct roots of (5.7c). The solution $v_a^{(k)}(\eta)$ is a power-series solution of (5.8), with $\gamma = \gamma_k$, in the domain of convergence $|\eta| < \min(1,1-a)$, defining an analytic solution in this domain. In the case $J_c = 1/4$, when $\gamma_1 = \gamma_2 = 1/2$, the same conclusion can be drawn.
Now we first consider the parameter range $a \geq 1/2$. It follows from (5.7a) and (5.10) that $\alpha_1\alpha_2 < 0$. Then it is possible to take the parameter α in (5.6) positive. If this parameter is positive, however, the solution of equation (5.8) must also be analytic in a neighbourhood of the point $\eta = 0$. From the above considerations we conclude that for $\alpha > 0$ and $\beta > 0$, and by choosing the value of the parameter γ appropriately, the boundary conditions can only be satisfied for $a \geq 1/2$ if the solution of equation (5.8) is analytic in a neighbourhood of any of the finite singular points of this equation.

Consequently, this solution must be an entire function of η. Since (5.8) is a Heun equation, however, such a solution is necessarily a polynomial in η[11,20]. Substitution of a polynomial of degree n into equation (5.8) yields $\alpha + \beta + \gamma - 1 = - n$ or $\alpha + \beta + \gamma + 2 = - n$. Since α, β, and γ are positive, only $n = 0$ satisfies. Hence

$$\alpha + \beta + \gamma = 1. \tag{5.12}$$

So equation (5.8) must have a solution $v(\eta)$ = constant. This implies that $A_o = 0$ and $A_1 = 0$. Note that $A_1 = 0$ in view of (5.12). From (5.12) it follows that $A_o = 0$ provided

$$a = \beta \;/\; (\alpha + \beta). \tag{5.13}$$

Combining (5.7), (5.12), and (5.13) yields

$$J_c = r^2(1 - r^2), \qquad r^2 = 4p^2/(1 - \chi^2), \tag{5.14}$$

and

$$a = \tfrac{1}{2}\,(1 - \chi), \tag{5.15}$$

where $r^2 \leq 1$ and $\chi < 1$. Note that $a < 1/2$.

The result (5.15) implies that equation (5.8) has no polynomial solutions for $a \geq 1/2$. In other words, singular neutral modes cannot exist for this parameter range.

For real α, β, and γ the sets of equations (5.12 - 13) and (5.14 - 15) are equivalent. Hence equation (5.8) has indeed a solution $v(\eta)$ = constant for physically meaningful parameter values. In other words, singular neutral modes do exist if $a < 1/2$, where in view of (5.15) this parameter is determined by χ, with $\chi < 1$. Note that χ, as defined in (5.4b), is the ratio of the scale of the background velocity and of the undisturbed density.

The question remains, however, whether the neutral stability curve (5.14) is the only one since this curve is actually determined from the requirement that equation (5.8) has a polynomial solution. If $\alpha_1\alpha_2 < 0$ for all physically meaningful parameter values, the neutral stability curve (5.14) proves to be the only one since the solution of (5.8) must reduce to a polynomial in this case. However, in view of (5.7a) the product $\alpha_1\alpha_2$ is not necessarily negative. We do not pursue this any further.
Taking limits as $\chi \to 0$ in (5.14 - 15), the results derived previously by Drazin[23] and Miles[11] are recovered. The latter results apply to the Taylor-Goldstein equation (1.7) with the configuration (5.1).
In the second example we consider the configuration

$$N^2(z) \equiv N_0^2, \qquad U(z) = U_0 \operatorname{sech}^2 \frac{z}{2d} . \tag{5.16a,b}$$

Equation (1.7) with (5.16) is symmetric in the independent variable z. Consequently, the eigenfunctions of this equation can be distinguished in even and odd functions.
Substitution of the transformation

$$\eta = \tanh \frac{z}{2d} \tag{5.17}$$

into (1.7) yields an equation of the Fuchsian type with five singularities. The reduction of (1.7) to such a Fuchsian equation can be accomplished through various transformations of the independent variable.
The hyperbolic tangent transformation (5.17) is most advantageous for this purpose. Since this transformation is antisymmetric, the resulting equation also has solutions which are either symmetric or antisymmetric in the new independent variable η. An antisymmetric transformation preserves the symmetry properties of the problem.
By introducing the transformation

$$w = (\eta^2 - 1)^{\chi}(\eta^2 - b^2)^{\Upsilon}\psi(\eta), \tag{5.18}$$

where $\eta = \eta(z)$ is given by (5.17) and with

$$\chi^2 = k^2d^2 - b^2J_c, \qquad b^2 = (U_o - c)/U_o, \tag{5.19a,b}$$

$$\gamma^2 - \gamma + J_c = 0, \tag{5.19c}$$

equation (1.7) with (5.16) reduces to

$$(\eta^2 - 1)(\eta^2 - b^2)\frac{d\psi}{d\eta^2}$$
$$+ \left[\{4\gamma + 2(1 + 2\chi)\}\,\eta^3 - \{4\gamma + 2b^2(1 + 2\chi)\}\eta\right] \frac{d\psi}{d\eta} \tag{5.20}$$
$$+ \{(2\chi + 2\gamma + 3)(2\chi + 2\gamma - 2)\eta^2 + 2\gamma\,(1 - 2\gamma) - 2\chi b^2(1 + 2\chi) +$$
$$8b^2\gamma(\gamma - 1) + 2\}\ \psi = 0.$$

Equation (5.20) is of the Fuchsian type, with singular points $\eta = \pm 1, \pm b$, and ∞. Note that this equation has indeed even and odd solutions. It is also possible to reduce equation (1.7) with (5.16) to a Fuchsian equation by means of the transformation

$$w = (\eta^2 - 1)^{\chi}(\eta + b)^{\gamma_1}(\eta - b)^{\gamma_2}v(\eta) \tag{5.21}$$

instead of (5.18); where γ_1 and γ_2 are distinct roots of equation (5.19c). Due to the symmetry of the problem, however, and since $\eta(z)$ is an odd function of z, we must require that $\gamma_1 = \gamma_2 = \gamma$ in order to be able to distinguish even and odd eigensolutions.
Equation (5.20) can be further reduced to a Heun equation by means of a quadratic transformation of the independent variable. This may be of interest for the determination of reflection and transmission coefficients since Heun's equation has less singular points than (5.20). For the determination of the neutral stability curves, however, we start from equation (5.20).

Due to the symmetry of the problem the existence of singular neutral modes implies that the Wronskian of $w(z)$ and its complex conjugate not only vanishes in the regions outside the critical levels but also in the region between these levels. Physically, this means that there is no preferred direction of the z-component of the net energy flux. Consequently, there is no net energy flux from one critical level to another in this case. Thus the direction of this flux is normal to the z-axis.
The parameter χ in (5.18), where χ is a root of equation (5.19a), is taken positive.* Then the boundary conditions $w(\pm \infty) = 0$ can only be satisfied if the solution of equation (5.20) is analytic in neighbourhoods of the points $\eta = \pm 1$. From theorem vii it follows that the parameter Υ in (5.18), where Υ is a real root of equation (5.19c), can be chosen such that the solution of equation (5.20) must also be analytic in neighbourhoods of the points $\eta = \pm b$. Consequently, the solution of (5.20) must be an entire function of η. Since equation (5.20) is of the Fuchsian type, however, such a solution is necessarily a polynomial in η (Appendix of this work). From the requirement that equation (5.20) must have polynomial solutions, the neutral stability curves are uniquely determined.
Substitution of $\psi(\eta)$ = constant into (5.20) yields the relations

$$J_c = \frac{15p^2(1-p^2)}{(3+2p^2)^2}, a = \frac{6+4p^2}{15}, \quad p^2 \leq 1, \qquad (5.22a,b,c)$$

where a and p are given by (5.4a,c). Through (5.18) the solution $\psi(\eta)$ = constant corresponds to an even eigensolution.

*

The existence of singular neutral modes implies that the roots χ_1 and χ_2 of (5.19a) are real. For complex χ_1 and χ_2 the solution $w(z)$ is oscillating in the regions $z \to \pm \infty$.

Substitution of $\psi(\eta) \propto \eta$ into (5.20) yields the relations

$$J_c = \frac{1}{4} - \frac{64p^4}{(3+4p^2)^2}, a = \frac{(3+4p^2)^2}{3(3+20p^2)}, p^2 \leq \frac{1}{4}. \qquad (5.23a,b,c)$$

The solution $\psi(\eta) \propto \eta$ corresponds to an odd eigensolution. It can be seen by inspection that equation (5.20) has no polynomial solutions of degree 2,3,..... for physically meaningful parameter values. So there is one symmetric or sinuous mode, with neutral stability curve (5.22a), and one antisymmetric or varicose mode, with neutral stability curve (5.23a). The results (5.22) and (5.23) agree with those derived by Drazin and Howard[13]. The curves could be determined exactly since the governing equation is reducible to a Fuchsian equation and the symmetry of the problem implies that the Wronskian also vanishes in the region between the critical levels. This also enabled us to determine all possible neutral stability curves. The same is true for other symmetric profiles, e.g. the profiles

$$N^2(z) = N_o^2 \operatorname{sech}^4 \frac{z}{2d}, \quad U(z) = U_o \operatorname{sech}^2 \frac{z}{2d}. \qquad (5.24a,b)$$

For a = 0, i.e. when the horizontal phase velocity vanishes (cf. (5.4a)), equation (1.7) with (5.24) is reducible to a hypergeometric equation. Thorpe[24] found the curves

$$J_o = 4p^2, \qquad \text{(sinuous mode)} \qquad (5.25)$$

$$J_o = 3 + 4p^2, \qquad \text{(varicose mode)} \qquad (5.26)$$

where

$$J_o = 4\, d^2 N_o^2 \,/\, U_o^2$$

is a representative Richardson number.

For $a \neq 0$ equation (1.7) with (5.24) is reducible to a Fuchsian equation with five singular points by means of the transformation (5.17). Huppert[14] found the curve

$$J_o = \frac{2p(4p^2-10p+6)}{1+2p}, \quad a = \frac{2(2p-1)}{1+2p}, \quad \tfrac{1}{2} \leq p \leq 1, \qquad (5.27a,b,c)$$

for a sinuous mode, and the curve

$$J_o = \frac{p(4p^2-8p+3)}{1+p}, \quad a = \frac{4p^2+1}{2p^2+3p+1}, \quad 0 \leq p \leq \tfrac{1}{2}, \qquad (5.28a,b,c)$$

for a varicose mode.

Huppert remarked that the results (5.27 - 28) can be verified by direct substitution. He did not claim to have found the complete solution. With our methods, however, it can be shown that (5.25 - 26) and (5.27 - 28) represent the complete solution, i.e. there are no other neutral stability curves.

We conclude with the remark that when the governing wave equation is not symmetric in the independent variable, the Wronskian (2.2) not necessarily vanishes in the region between the critical levels. In that case it is not possible to determine the neutral stability curves by means of the methods described here.

Appendix

A differential equation of the Fuchsian type is an ordinary differential equation in which every singular point is a regular singular point, including a possible singularity at infinity[19]. In the next discussion we only consider Fuchsian equations of second order, with a singular point at infinity.

Heun's equation is an equation with four singular points[21,22]. If this equation has a solution that is an entire function of the independent variable, i.e. regular at any finite singular point, this solution necessarily reduces to a polynomial[11,20]. In this section it will be shown that the same is true for a Fuchsian equation with five singular points. The latter equation has been derived by Lamé[25] and has been studied in detail by Crowson[26,27].

Any Fuchsian equation with five singular points can be reduced to the standard form

$$\frac{d^2u}{d\zeta^2} + \left\{ \frac{\nu}{\zeta} + \frac{\delta}{\zeta - 1} + \frac{\varepsilon}{\zeta - a} + \frac{\mu}{\zeta - b} \right\} \frac{du}{d\zeta} + \frac{\alpha\beta\zeta^2 + \nu_1\zeta + \nu_2}{\zeta(\zeta - 1)(\zeta - a)(\zeta - b)} u = 0, \tag{A.1}$$

where

$$\gamma + \delta + \varepsilon + \mu - \alpha - \beta = 1. \tag{A.2}$$

The solutions of this equation may be characterized by the Riemann P - symbol

$$P \begin{bmatrix} 0 & 1 & a & b & \infty & \\ 0 & 0 & 0 & 0 & \alpha & \zeta \\ 1-\gamma & 1-\delta & 1-\varepsilon & 1-\mu & \beta & \end{bmatrix}, \tag{A.3}$$

which lists the location of the singular points and the exponents relative to them, as well as the argument.
The P - symbol does not characterize the solutions of equation (A.1) completely: there are two so-called accessory parameters, namely ν_1 and ν_2. The exponents in (A.3) relative to the singular points are not expressed in these parameters.
To obtain a solution of equation (A.1) in a neighbourhood of the singular point $\zeta = 0$, we apply the method of Frobenius, i.e. we substitute the series

$$u\ (\zeta) = \zeta^{\lambda} \sum_{n=0}^{\infty} c_n(\lambda)\zeta^n \qquad (A.4)$$

into (A.1), multiplied by $\zeta(\zeta-1)(\zeta-a)(\zeta-b)$. This leads to the indicial equation

$$\lambda(\lambda - 1) + \lambda\gamma = 0, \qquad (A.5)$$

with roots $\lambda_1=0$, $\lambda_2=1-\gamma$. Note that these roots, the exponents relative to the singular point $\zeta=0$, are indeed indicated in (A.3)
For the solutions in a neighbourhood of one of the other finite singular points one of the roots of the pertaining indicial equation also vanishes, cf. (A.3). Consequently, equation (A.1) may have a solution that is an entire function of ζ. Such a solution can be represented by the series (A.4), with $\lambda=0$. In general, however, the series (A.4) converges in the domain $|\zeta| < \min\ (1, |a|, |b|)$. Only for restricted combinations of parameter values, (A.4) converges in a larger domain. One of the singular points of equation (A.1) is located on the boundary of this domain, however.
In the previous section we treated examples of reduction of the Taylor-Goldstein equation (1.7) to a Fuchsian equation with five singular points. Reduction to such an equation was accomplished through the transformation (5.17). The extremes $\eta= \pm 1$ of this transformation belong to the singular points of the resulting Fuchsian equation. The other finite singular points lie in the interval (-1, 1). Therefore this equation is not of the form (A.1).

Reduction to this form is possible by means of the transformation $\zeta = \frac{1}{2}(1+\eta)$, with $\eta=\eta(z)$ given by (5.17). Since the original wave equation has two critical levels, the remaining finite singular points a and b of (A.1), with a and b real, and $a \neq b$ are then located in the interval (0,1).

Theorem.

If equation (A.1) has a solution that is an entire function of ζ, and the singular points a and b are located such that $|a| \neq 1$, $|b| \neq 1$, and $|a| \neq |b|$, this solution is necessarily a polynomial in ζ.

Before we give the proof, we remark that from the foregoing it will be evident that with the limitation to distinct values of the moduli of the singularities the theorem remains sufficiently general to our purpose.

Proof of the theorem.

The series (A.4), with $\lambda=0$,

$$u(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n, \qquad (A.6)$$

represents a solution of (A.1) that is analytic in a domain including the point $\zeta=0$. On the other hand, any analytic solution in this domain may be represented by (A.6). Therefore it suffices to show that when the series (A.6) is infinite, its radius of convergence is always finite. Consequently, if (A.6) defines a solution that is an entire function of ζ, this series necessarily reduces to a polynomial. The proof is based upon the theory of difference equations and runs as follows.

By multiplication of equation (A.1) with $\zeta(\zeta-1)(\zeta-a)(\zeta-b)$, and after substitution of (A.6) into the result, we find that the coefficients c_m satisfy a four-term recurrence relation of the form

$$\{abm^2 + f_1 m + f_2\}\, c_{m+1} - \{(a + b + ab)m^2 + f_3 m + f_4\}\, c_m$$

$$+ \{(a + b + 1)m^2 + f_5 m + f_6\} c_{m-1} - \{m^2 + f_7 m + f_8\}\, c_{m-2} = 0, \qquad (A.7)$$

$m = 2,3,4,\ldots\ldots$

The coefficients f_n only depend on the parameters of (A.1).
The recurrence relation (A.7) can also be understood as a linear, third-order difference equation of the form

$$c_{n+1} + g_n c_n + h_n c_{n-1} + k_n c_{n-2} = 0, \quad n = 2,3,4,\ldots.. \tag{A.8}$$

where the coefficients of this linear equation have finite limits

$$\lim_{n\to\infty} (g_n, h_n, k_n,) = (-\frac{a+b+ab}{ab}, \frac{a+b+1}{ab}, -\frac{1}{ab}). \tag{A.9}$$

The polynomial

$$\phi(t) = t^3 - (\frac{a+b+ab}{ab})t^2 + (\frac{a+b+1}{ab})t - \frac{1}{ab} \tag{A.10}$$

is called the characteristic polynomial of equation (A.8). This polynomial has the zeroes

$$t_1 = 1, \quad t_2 = 1/a, \quad t_3 = 1/b. \tag{A.11}$$

Now we assume that the series (A.6) is <u>infinite</u>. Since the sequence $\{c_n\}$ of the coefficients of this series is a solution of the difference equation (A.8), and the moduli of the zeroes of the characteristic polynomial (A.10) of (A.8) are distinct, the ratio c_{n+1}/c_n is defined for sufficiently large n and tends to one of the zeroes (A.11) as $n \to \infty$ (Poincaré's theorem[28]).
From (A.11) and d'Alembert's criterion for convergence it follows that the series (A.6) has a finite radius of convergence.
This completes the proof of the theorem.

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